Chapter 9
Linear Momentum and Collisions
This chapter is about interaction between TWO objects
Units of Chapter 9

- Linear Momentum
- Momentum and Newton’s Second Law
- Impulse
- Conservation of Linear Momentum
- Completely Inelastic Collisions
- Somewhat Inelastic Collisions
- Elastic Collisions
- Center of Mass
9-1 Linear Momentum

\[ K = \frac{1}{2} m v^2 \text{ only describes Energy of Motion (no indication of Direction).} \]

We need a new concept

**Definition of Linear Momentum, \( \vec{p} \)**

\[ \vec{p} = m \vec{v} \]

SI unit: \( \text{kg} \cdot \text{m/s} \)

Momentum is a vector; its direction is the same as the direction of the velocity.

Momentum is “velocity enhanced”, because it includes mass too.

Momentum (\( p=mv \)) is not Power (\( P=\text{Work}/\text{time}. \))
Change in momentum:

(a) \(mv\)?  
(b) \(2mv\)?

Initial momentum = \(-mv\)  
Final momentum = \(mv\)  
Change in momentum:  
\[ p_f - p_i = 2mv \]

Q: What can cause momentum to change?
9-2 Momentum and Newton’s Second Law

Forces can cause momentum to change.

If mass doesn’t change, change of momentum is

\[ \Delta \vec{p} = m \vec{v}_f - m \vec{v}_0 = m(\vec{v}_f - \vec{v}_0) = m \vec{a} \Delta t \]

How quickly does momentum change?

\[ \frac{\Delta \vec{p}}{\Delta t} = m \vec{a} = \vec{F}_{net} \]

Net force on an object determines how quickly its momentum changes and the momentum change will be in the net force direction.

\[ \Delta \vec{p} = \vec{F}_{net} \Delta t \]

This is also true when mass is changing. (Not required in this class.)
9-3 Impulse

\[ \Delta \vec{p} = \vec{F}_{\text{net}} \Delta t \]

To change momentum of an object, Force needs to be applied for a certain time.

**Definition of Impulse, \( \vec{I} \)**

\[ \vec{I} = \vec{F}_{\text{av}} \Delta t \]

SI unit: \( N \cdot s = kg \cdot m/s \)

Impulse is a vector, in the same direction as the average force.

It is like “force enhanced”, because it includes the amount of time the force is applied. It has the same unit as momentum \( p \) and momentum change \( \Delta p \).
We can rewrite
\[ \vec{F}_{av} = \frac{\Delta \vec{p}}{\Delta t} \]
as
\[ \vec{F}_{av} \Delta t = \Delta \vec{p} \]
So we see that
\[ \vec{I} = \vec{F}_{av} \Delta t = \Delta \vec{p} \]
The impulse is equal to the change in momentum.
The same change in momentum may be produced by a large force acting for a short time (fall and stop on a hard floor), or by a smaller force acting for a longer time (fall and stop on a soft cushion).
9-4 Conservation of Linear Momentum

The net force acting on an object is the rate of change of its momentum:

\[ \sum \vec{F} = \frac{\Delta \vec{p}}{\Delta t} \]

If the net force is zero, the momentum does not change:

Conservation of Momentum

If the net force acting on an object is zero, its momentum is conserved; that is, \( \vec{p}_f = \vec{p}_i \)

This is also true for a system containing more than one object.
Internal Versus External Forces:

Internal forces act between objects within the system.

As with all forces, they occur in action-reaction pairs. As all pairs act between objects in the system, the internal forces always sum to zero:

$$\sum \vec{F}_{\text{int}} = 0$$

Therefore, the net force acting on a system is the sum of the external forces acting on it.
9-4 Conservation of Linear Momentum

Furthermore, internal forces cannot change the momentum of a system.

**Conservation of Momentum for a System of Objects**

*Internal* forces have absolutely no effect on the net momentum of a system.

- If the *net external* force acting on a system is zero, its net momentum is conserved. That is,
  \[ \vec{p}_{1,f} + \vec{p}_{2,f} + \vec{p}_{3,f} + \ldots = \vec{p}_{1,i} + \vec{p}_{2,i} + \vec{p}_{3,i} + \ldots \]

However, the momenta of components of the system may change.
When is $\mathbf{p}$ useful to compute?

1. When $\mathbf{F}_{\text{net}} = 0$, you know $\Delta \mathbf{p} = 0$. 
   $\mathbf{P}_i = \mathbf{P}_f$. $P$ conserves. Very useful!

   (When $W_{\text{nc}} = 0$. $E_i = E_f$. $k + U$ conserves. Very useful.)

2. When you have more than one object and $\mathbf{F}_{\text{external on the system}} = 0$, you know that total $\mathbf{p}$ of the system do not change. $\Sigma \mathbf{P}_i = \Sigma \mathbf{P}_f$ 

   Momentum of the system conserves! Very useful.

Momentum of a system. Total $\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 + \ldots$

$\mathbf{P}_{\text{system}} = \Sigma \mathbf{P}$

Careful when you add momentums, they are vectors. Direction matters. Unlike energy & work.
9-4 Conservation of Linear Momentum

An example of internal forces moving components of a system: External force = zero.

Sample question:

Total momentum for the system before and after are both zero!

Since mass B > mass A, it makes sense that Speed B < Speed A
Demo* Two cart on frictionless track. Push off from rest.

Total \( \vec{P} \) before push = \( \overline{0} \)

Total \( \vec{P} \) after push = \( m_A \vec{V}_A + m_B \vec{V}_B \)

If \( M_A = M_B \), \( \vec{V}_A = -\vec{V}_B \)

If \( 2M_A = M_B \), \( \vec{V}_A = -\frac{1}{2} \vec{V}_B \)

"Think: If \( m_B > m_A \), will \( |\vec{V}_B| > \) or < \( |\vec{V}_A| \)?

If mass has a factor of 2, \( \vec{V} \) will have a factor of 2?

Demo*: Fan & Sail. Push off on track from rest.

Both got pushed away.

\( m_A \vec{V}_A = -m_B \vec{V}_B \)

Both speeds up ........
Why? Because the force that A acts on B and the force that B acts on A are within the system. Those forces change A and B’s momentum respectively, but do not change the momentum of the system.
9-5 Collisions two objects striking one another

If the two objects are not connected with other objects. Time of collision is short enough that external forces may be ignored.

Consider the two objects as a whole system:
For the whole system during collision
External force = zero.

The system’s total momentum doesn’t change.
This is true for all kinds of collisions.

\[
\pmb{\vec{p}}_{1,i} + \pmb{\vec{p}}_{2,i} = \pmb{\vec{p}}_{1,f} + \pmb{\vec{p}}_{2,f} \]

\[
\sum \pmb{\vec{p}}_i = \sum \pmb{\vec{p}}_f
\]
For any collisions:
The system’s total momentum doesn’t change.

\[ \vec{p}_{1,i} + \vec{p}_{2,i} = \vec{p}_{1,f} + \vec{p}_{2,f} \]

\[ m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i} = m_1 \vec{v}_{1f} + m_2 \vec{v}_{2f} \]

Attention: Momentum “ADDS” as VECTORS, direction matters.

Generally, \(m_1, m_2, v_{1i}, v_{2i}, v_{1f}, v_{2f}\). If we know any 5 of them, we can solve the unknown, for any collisions.

For special collisions, there are other constrains (we can set one more equation), so that to know 4 out of the six variables will be enough to solve the two unknowns.

**Completely inelastic collision**: objects stick together afterwards. \(v_{1f} = v_{2f} = v_f\)
Completely inelastic collision: objects stick together afterwards. \( V_{1f} = V_{2f} = V_f \)

Total \( p \) initial = Total \( p \) final

Example:

\( m_1 = 3 \text{kg}, \text{moving at } 6 \text{m/s} \)
\( m_2 = 2 \text{kg}, \text{moving toward } m_1 \text{ at } 4 \text{m/s}; \)

Stick together after collision

Find \( v_{1f}, v_{2f} \)

Equations:

\[
\vec{p}_{1,i} + \vec{p}_{2,i} = \vec{p}_{1,f} + \vec{p}_{2,f}
\]

\[
m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i} = m_1 \vec{v}_{1f} + m_2 \vec{v}_{2f}
\]

Total \( p \) before collision: \( 3 \text{kg} \times 6 \text{m/s} - 2 \text{kg} \times 4 \text{m/s} = 10 \text{kgm/s} \)

Total \( p \) after collision: \( (m_1+m_2)V_f \)

\( 3 \text{kg} \times 6 \text{m/s} - 2 \text{kg} \times 4 \text{m/s} = (m_1+m_2)V_f \)

\[
3 \times 6 - 2 \times 4 = (3+2)V_f \quad V_f = 2 \text{ m/s for both 1 and 2}
\]
Completely inelastic collision: objects stick together afterwards. $v_{1f} = v_{2f} = v_f$

Total $p$ initial = Total $p$ final

Is $p_{1f} = p_{1i}$? No.
Is $p_{2f} = p_{2i}$? No.
Momentum for both objects changed. The total $p$ stay unchanged.

Is total KE initial = total KE final? No. KE final < KE initial.
Where did some KE go?

$m_1 = 3$kg, moving at $6$ m/s
$m_2 = 2$kg, moving toward $m_1$ at $4$ m/s;

Equations:

\[
\vec{p}_{1,i} + \vec{p}_{2,i} = \vec{p}_{1,f} + \vec{p}_{2,f}
\]

$3 \times 6 - 2 \times 4 = (3 + 2)v_f$

$v_f = 2$ m/s
9-5 Completely Inelastic Collisions

**Stick together**

KE final < KE initial.

Don’t count on KE for calculation.

When they stick together, resistance force did negative work, $W_{nc} < 0$,

Some KE lost into heat and deformation of object….

Total

\[ p_i = m_1 v_{1,i} \pm m_2 v_{2,i} \]

Total

\[ p_f = (m_1 + m_2) v_f \]

\[ m_1 v_{1,i} \pm m_2 v_{2,i} = (m_1 + m_2) v_f \]

Be careful about the sign.
Somewhat Inelastic Collisions

Most real collisions are somewhat inelastic Collisions. Even if they don’t stick together, they deform and don’t recover shape, some KE is lost into heat, sound, wave, deformation……

Don’t consider KE for general collision problems. What we can always count on for all collision is that:

\[ \mathbf{p}_{1,i} + \mathbf{p}_{2,i} = \mathbf{p}_{1,f} + \mathbf{p}_{2,f} \]
9-5 2D Collisions

For general collisions in two dimensions, conservation of momentum is applied separately along each axis:

For any kind of collision in 2D it’s true:

\[ \vec{p}_{1,x} + \vec{p}_{2,x} = \vec{p}_{1,f,x} + \vec{p}_{2,f,x} \]

\[ \vec{p}_{1,y} + \vec{p}_{2,y} = \vec{p}_{1,f,y} + \vec{p}_{2,f,y} \]

Decompose velocities into x & y directions and set this equation for x and y direction separately:

\[ m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i} = m_1 \vec{v}_{1f} + m_2 \vec{v}_{2f} \]
Example: frictionless 2D collision:

Air hockey table

\[ \begin{align*}
V_{A1} &= 5 \text{ m/s} \\
V_{A1x} &= 3 \text{ m/s} \\
V_{A1y} &= 2 \text{ m/s} \\
V_{B1} &= 0
\end{align*} \]

\[ m_A = 1 \text{ kg} \]
\[ m_B = 2 \text{ kg} \]
\[ V_{Bf} = 2 \text{ m/s} \]
\[ \theta_2 = 45^\circ \]
\[ V_{Bf} = 45^\circ \text{ CW, to x direction} \]

Find \( \vec{v}_{Af} \) (\( \theta \), and \( v_{Af} \))

\[ \begin{align*}
V_{Af} &= V_{Af} \cdot \cos \theta, \\
V_{Af} &= V_{Af} \cdot \sin \theta,
\end{align*} \]

\[ \sum F_i = \sum F_f \]

\( x \) direction: \( \sum P_{ix} = \sum P_{fx} \)
\[ m_A V_{Aix} + 0 = m_A V_{AfX} + m_B V_{Bfx} \]
\[ m_A v_0 + 0 = m_A V_{AfX} + m_B V_{Bfx} \]

\[ m_A v_0 = V_{AfX} \cdot \cos \theta_1 + 2 \times V_{Bf} \cdot \cos \theta_2 \]

\( y \) direction: \( \sum P_{iy} = \sum P_{fy} \)
\[ m_A V_{Aiy} + m_B V_{Bfy} = m_A V_{AfY} - m_B V_{Bfy} \]
\[ 0 + 0 = m_A V_{AfY} \cdot \sin \theta_1 - m_B V_{Bfy} \cdot \sin \theta_2 \]

plug \( m_A, m_B, v_{Bf}, \theta_2 \), into 1 and 2, found:

\[ V_{Af} \cdot \cos \theta = 2.17 \text{ m/s} \]
\[ V_{Af} \cdot \sin \theta = 2.82 \text{ m/s} \]

\[ \theta = \tan^{-1} \left( \frac{2.82}{2.17} \right) = 52.2^\circ \]
9-6 Elastic Collisions

For all collisions, total momentum for the system is conserved. (External force on the system is zero).

We already know that in most collisions, system’s KE decreases and is partially converted to other energy form.

In an ideal situation, when the material is elastic like spring and rubber band or too hard to deform at all, it is possible to have elastic collision, in which kinetic energy of the system is also conserved.
The MATERIALS of the colliding objects determine which kind of collision it will be.

**Completely Inelastic**, stick together, lose a lot of KE into heat and deformation. Like play dough, bullet shot inside, etc.

**Most real collisions are Somewhat Inelastic**. Don’t stick together, and the system still lose some KE.

**Elastic**, Able to recover 100% from deformation

Or too hard to deform at all. KE is conserved.
9-6 Elastic Collisions

In elastic collisions, both kinetic energy and momentum are conserved. One-dimensional:

\[ m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i} = m_1 \vec{v}_{1f} + m_2 \vec{v}_{2f} \]

\[ \frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \]

Don’t assume it is Elastic unless clearly told so.

The first equation is true for all collisions. The second equation is only true for real Elastic collisions, which is very rare.
Elastic collision Example:

$m_1$ coming with $v_0$, $m_2$ was at rest. Elastic collision, find $v_{1f}$, $v_{2f}$.

\[ \sum \vec{p}_i = \sum \vec{p}_f \]
\[ \sum KE_i = \sum KE_f \]

\[ m_1 \vec{v}_0 = m_1 \vec{v}_{1f} + m_2 \vec{v}_{2f} \]
\[ \frac{1}{2} m_1 v_0^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \]

Solving for the final velocities:

We have two equations (conservation of momentum and conservation of kinetic energy) and two unknowns (the final velocities).
Elastic collision Example:

$m_1$ coming with $v_0$, $m_2$ was at rest. Elastic collision, find $v_{1f}$, $v_{2f}$.

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_0$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right)v_0$$

If, $m_1=m_2$, $v_{1f}=0$, $v_{2f}=v_0$, Cool demo…

If, $m_1>m_2$, $v_{1f}>0$, mass 1 keep moving forward, $v_{2f}>v_0$, Mass 2 is kicked fast.

(When a truck hits a small resting object)

If, $m_1<m_2$, $v_{1f}<0$, mass 1 bounced back, $v_{2f}<v_0$, Mass 2 is kicked a little.

(When a little guy hit a big resting truck)
Summary of Chapter 9

• Linear momentum: \[ \vec{p} = m\vec{v} \]

• Momentum is a vector

• Newton’s second law: \[ \sum \vec{F} = m\vec{a} \]

• Impulse: \[ \vec{I} = \vec{F}_{av}\Delta t = \Delta\vec{p} \]

• Impulse is a vector

• The impulse is equal to the change in momentum

• When you add momentum to get the total momentum of a whole system, direction matters.
• Momentum of the system is conserved if the net external force is zero

• Internal forces within a system always sum to zero. Internal force change individual object’s momentum, but doesn’t change the total momentum of the whole system.

• In any collisions, Total momentum of the whole system is conserved.

• Somewhat Inelastic collision: Kinetic energy is not conserved

• Completely inelastic collision: the objects stick together afterward, KE is partially lost. $v_{1f}=v_{1f}=v_f$

• A one-dimensional collision takes place along a line
• In two dimensions, conservation of momentum is applied separately to both x and y direction.

• Elastic collision: KE is also unchanged before and after collision. Rarely happen. Don’t assume.

<table>
<thead>
<tr>
<th>Kinds of Collisions</th>
<th>Total Momentum Conservation</th>
<th>Total KE Conservation</th>
<th>Stick Together?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Complete Inelastic</td>
<td>( \sum \vec{P}_i = \sum \vec{P}_f )</td>
<td>( \sum k_f \neq \sum k_i )</td>
<td>No</td>
</tr>
<tr>
<td>Some Inelastic</td>
<td>( \sum \vec{P}_i = \sum \vec{P}_f )</td>
<td>( \sum k_f = \sum k_i )</td>
<td>Yes</td>
</tr>
<tr>
<td>Elastic</td>
<td>( \sum \vec{P}_i = \sum \vec{P}_f )</td>
<td>( \sum k_f = \sum k_i )</td>
<td>No</td>
</tr>
</tbody>
</table>

Problem solving strategy: Never assume anything.
1. Always set equations for total momentum conservation first.
2. Apply \( v_{1f} = v_{2f} = v_f \) only for completely inelastic collisions (stick together).
3. Apply \( \sum KE_i = \sum KE_f \), only when it is clearly said to be Elastic collision, which rarely happens.
9-7 Center of Mass

The center of mass of a system is the point where the system can be balanced in a uniform gravitational field.
9-7 Center of Mass

For two objects:

\[ X_{\text{cm}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{m_1 x_1 + m_2 x_2}{M} \]

where \( x_1 \) and \( x_2 \) are location of the mass center of object 1 and object 2 respectively.

The center of mass is closer to the more massive object.

This is called weighted average, which ever has more mass speaks lauder about the mass center location.
The center of mass need not be within the object:
9-7 Center of Mass

X and Y Coordinate of the Center of Mass for a whole system of many parts:

\[ X_{cm} = \frac{m_1 x_1 + m_2 x_2 + \cdots}{m_1 + m_2 + \cdots} = \frac{\sum mx}{M} \]

\[ Y_{cm} = \frac{m_1 y_1 + m_2 y_2 + \cdots}{m_1 + m_2 + \cdots} = \frac{\sum my}{M} \]

Example:

\[ X_{com} = \frac{15 \times 0 + 5 \times 5 + 10 \times 2.5}{15 + 5 + 10} = 1.7 \text{ (m)} \]
9-7 Center of Mass

Motion of the center of mass:

**Velocity of the Center of Mass**

\[
\vec{V}_{cm} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2 + \ldots}{m_1 + m_2 + \ldots} = \frac{\sum m\vec{v}}{M}
\]

**Acceleration of the Center of Mass**

\[
\vec{A}_{cm} = \frac{m_1 \vec{a}_1 + m_2 \vec{a}_2 + \ldots}{m_1 + m_2 + \ldots} = \frac{\sum m\vec{a}}{M}
\]
• Center of mass:

\[ X_{\text{cm}} = \frac{m_1 x_1 + m_2 x_2 + \ldots}{m_1 + m_2 + \ldots} = \frac{\sum mx}{M} \]

\[ Y_{\text{cm}} = \frac{m_1 y_1 + m_2 y_2 + \ldots}{m_1 + m_2 + \ldots} = \frac{\sum my}{M} \]

\[ \vec{V}_{\text{cm}} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2 + \ldots}{m_1 + m_2 + \ldots} = \frac{\sum m\vec{v}}{M} \]

\[ \vec{A}_{\text{cm}} = \frac{m_1 \vec{a}_1 + m_2 \vec{a}_2 + \ldots}{m_1 + m_2 + \ldots} = \frac{\sum m\vec{a}}{M} \]
The total mass multiplied by the acceleration of the center of mass is equal to the net external force:

**Newton's Second Law for a System of Particles**

$$M\mathbf{A}_{cm} = \mathbf{F}_{\text{net,ext}}$$

The center of mass accelerates just as though it were a point particle of mass $M$ acted on by $\mathbf{F}_{\text{net,ext}}$.