Midterm, Physics 706, Spring 2014
This midterm is due at the beginning of class on April 15. There are three problems on this exam (two pages). The number of credit points is indicated for each problem. You are allowed to use your notes, homework solutions, and the textbook (Sakurai and Napolitano) only; no other books (older editions of Sakurai are ok), nor Mathematica or Matlab. Do not consult with anyone else. Show the details of your work: I have to be able to understand your reasoning from what you write on the exam! Good luck!

1. (10 points) Let $|\psi\rangle = \cos \left( \frac{\theta}{2} \right) |+\rangle + e^{i\phi} \sin \left( \frac{\theta}{2} \right) |\rangle + e^{i\phi} \sin \left( \frac{\theta}{2} \right) |-\rangle$ be the eigenstate of $\vec{S} \cdot \hat{n}$ with eigenvalue $\hbar/2$, where $\vec{S}$ is the spin operator for a system with spin $1/2$. $\hat{n}$ is the unit vector $(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$, with $\theta$ (angle with $z$-axis), and $\phi$ (angle in $xy$-plane with $x$-axis) the usual polar coordinate angles.

(a) (3 points) What are the expectation values of $S_x$ and $S_y$ in this state?
(b) (3 points) What are the dispersions of $S_x$ and $S_y$ in this state?
(c) (4 points) What directions of $\hat{n}$ minimize the dispersions of $S_x$ and $S_y$, respectively? What is the minimum value of the product of these dispersions?

2. (20 points) Consider two solutions $\psi_1(x)$ and $\psi_2(x)$ of the time-independent Schrödinger equation in one dimension, with the same energy. Assume that these solutions are normalized, so that
\[ \int_{-\infty}^{\infty} dx |\psi_i(x)|^2 = 1, \quad i = 1, 2. \]

For these integrals to be finite, we need that both solutions vanish at $x = \pm \infty$, and therefore these solutions correspond to bound states.

(a) (4 points) For an arbitrary potential $V(x)$ admitting these solutions, show that
\[ \frac{d}{dx} \left( \psi_2(x) \frac{d\psi_1(x)}{dx} - \psi_1(x) \frac{d\psi_2(x)}{dx} \right) = 0. \]

(b) (4 points) This implies that
\[ \psi_2(x) \frac{d\psi_1(x)}{dx} - \psi_1(x) \frac{d\psi_2(x)}{dx} = \text{constant}. \]

Show that this constant is, in fact, equal to zero. Then show that the result you obtained implies that
\[ \psi_2(x) = c\psi_1(x), \]
with $c$ a constant with absolute value equal to one.

(c) (4 points) Argue that what you just proved is that in one dimension there are no degenerate bound states. For the case $V(x) = 0$ (free particle), there are two solutions for each (positive) energy $E$, proportional to $e^{\pm ipx/\hbar}$, with $E = p^2/(2m)$ ($m$ is the mass of the particle). Why is this not conflict with the theorem you just proved?
(d) (4 points) The theorem you just proved is not true in more than one dimension. In order to see this, all one has to do is construct a counter example! Consider the two-dimensional harmonic oscillator, with hamiltonian

\[ H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2} m \omega^2(x^2 + y^2). \]

Show that the wavefunction

\[ \psi(x, y) = \langle x|1\rangle\langle y|0\rangle = Nx \exp \left[ -\frac{1}{2} \left( \frac{x}{x_0} \right)^2 + \left( \frac{y}{y_0} \right)^2 \right], \]

with \( x_0 = y_0 = \sqrt{\hbar/(m\omega)} \), and where \( N \) is a normalization factor that you do not have to calculate, is a solution of the time-independent Schrödinger equation, with energy \( E = 2\hbar\omega \).

(e) (4 points) Then, construct another energy eigenfunction that is different from the one above, but that has the same energy, thus providing a counter example to the theorem in two dimensions. Explain your construction. [Hint: for this last part you should not have to do any calculations!]

3. (20 points) Consider the state

\[ |\alpha\rangle = Ce^{\alpha a^\dagger}|0\rangle, \]

where \(|0\rangle\) is the ground state of the harmonic oscillator, \( a^\dagger \) is the creation operator of the harmonic oscillator, and \( \alpha \) and \( C \) are constants. (Such a state is called a “coherent” state.)

(a) (4 points) Show that \( a(a^\dagger)^k = (a^\dagger)^k a + k(a^\dagger)^{k-1} \).

(b) (4 points) Show that \( a|\alpha\rangle = \alpha|\alpha\rangle \).

(c) (4 points) Expand this state in a series of eigenstates of the number operator \( N = a^\dagger a \). Use this to find the probability that the state \( |\alpha\rangle \) contains \( n \) quanta of energy \( \hbar\omega \). You may assume that \( C \) is chosen such that the state is normalized.

(d) (4 points) From the result in part (c), find \( C \) such that \( |\alpha\rangle \) is normalized.

(e) (4 points) Show that \( |\alpha\rangle \) can be obtained by applying a finite translation over a distance \( l \) to the ground state, i.e., show that

\[ |\alpha\rangle = e^{-ipl/\hbar}|0\rangle, \]

with \( \alpha = l\sqrt{m\omega/2\hbar} \). You may use that

\[ e^{X+Y} = e^X e^Y e^{-[X,Y]/2} \]

for any two operators \( X \) and \( Y \) which both commute with \([X,Y]\).