Take-home Midterm, Physics 370, March 31, Spring 2005
DUE in class on Thursday April 6 at the beginning of class!

There are three problems on this exam (two pages). The number of credit points is indicated for each problem. You are allowed to use your notes, homework solutions, the textbook (Schroeder) and a calculator only. Do not consult with anyone else about this exam. Show the details of your work! Good luck!

1. \(20\) points A gas expands adiabatically and quasi-statically, from an initial state \(A\) in which the pressure is \(P_A = 32 \times 10^5\) N/m\(^2\) and the volume is \(V_A = 1\) liter, to a final state \(B\) in which the pressure is \(P_B = 10^5\) N/m\(^2\) and the volume is \(V_B = 8\) liters. During the process, the pressure of the gas is found to change with volume according to the relation

\[ P = \alpha V^{-5/3}. \]

(a) \(2\) points Draw a diagram of this process.
(b) \(3\) points Determine the coefficient \(\alpha\).
(c) \(4\) points Show by explicit calculation that the work done by the gas is equal to \(W_{\text{gas}} = 3600\) J.
(d) \(3\) points Show that the change in internal energy is equal to \(-3600\) J.

Now we let the gas go from state \(A\) to state \(B\), but via a different (still quasi-static) process. In the first step, we let the gas expand, but keep it at constant pressure, \(P = P_A = 32 \times 10^5\) N/m\(^2\), until it has volume \(V_B = 8\) liters. Then, in the second step, we decrease the pressure at constant volume, \(V = V_B = 8\) liters, to bring it to state \(B\).

(e) \(2\) points Draw this process in your diagram.
(f) \(3\) points Calculate the work done by the gas during this two-step process.
(g) \(3\) points What is the amount of heat added during this two-step process? [Hint: you should not have to use the ideal gas law, \(PV = NkT\), anywhere in this problem!]

2. \(15\) points A solid contains \(N\) magnetic iron atoms each having spin \(S_e\), implying that each atom’s spin can have \(2S_e + 1\) orientations. At very high temperatures, each spin is completely randomly oriented, i.e., equally likely to be in any of its \(2S_e + 1\) possible states. But, at low temperatures the interactions the magnetic interactions between the atoms causes them to exhibit ferromagnetic behavior, and for \(T \to 0\) all their spins become oriented long the same direction. At \(T = 0\) only one state is available to the whole system. This solid has a heat capacity \(C(T)\), which has a temperature dependence given by

\[
C(T) = C_1 \left( \frac{2T}{T_1} - 1 \right), \quad \text{for} \quad \frac{1}{2} T_1 < T < T_1, \\
= 0, \quad \text{otherwise.}
\]

The abrupt increase of \(C\) if \(T\) is lowered below \(T_1\) is due to the onset of ferromagnetic behavior at temperatures below \(T_1\). You may assume that \(C\), and the entropy \(S\), are functions of \(T\) only.
(a) (2 points) What is the entropy $S$ at $T = 0$?
(b) (4 points) Show that at very large temperatures $S = Nk \log (2S_c + 1)$.
(c) (5 points) Now use the relation between $S(T)$ and $C(T)$ to express the entropy at very large temperatures ($T > T_1$) in terms of the constant $C_1$.
(d) (4 points) Combine the results of (b) and (c) to express the constant $C_1$ in terms of $N$ and $S_c$.

3. (15 points) Consider one mole of a paramagnetic monatomic ideal gas. This is an ideal gas which can also be magnetized in an external magnetic field $B$. This gas is characterized in terms of any three of the variables $S$, $P$, $T$, $V$, $M$, $B$ (as long as we take one of them to be extensive). The basic equation is

$$dU = TdS - PdV + BdM,$$

where the last term represents the increase in energy of the molecules when their magnetization is increased by an amount $dM$ in an external field $B$ ($-BdM$ is the magnetic work done on the gas by a magnetic field $B$ when the magnetization changes by $dM$).

(a) (2 points) Consider the free energy $F = U - TS$, and give the expressions for $S$, $P$ and $B$ as partial derivatives of $F$. In each case, make clear which variables are being kept constant. For $S$ and $P$ you should find the relations on page 157 of Schroeder. (You do not have to worry about $N$, which is constant at 1 mole throughout this problem.)
(b) (3 points) Using the result of (a), show that

$$\left( \frac{\partial S}{\partial V} \right)_{T,M} = \left( \frac{\partial P}{\partial T} \right)_{V,M},$$

$$\left( \frac{\partial S}{\partial M} \right)_{T,V} = - \left( \frac{\partial B}{\partial T} \right)_{V,M}.$$

(c) (4 points) Suppose that our gas obeys the equations of state $PV = RT$ and $M = aB/T$, with $a$ another constant. Show that

$$\left( \frac{\partial U}{\partial V} \right)_{T,M} = 0, \quad \left( \frac{\partial U}{\partial M} \right)_{T,V} = 0.$$

(d) (2 points) Using that

$$dU = \left( \frac{\partial U}{\partial V} \right)_{T,M} dV + \left( \frac{\partial U}{\partial M} \right)_{T,V} dM + \left( \frac{\partial U}{\partial T} \right)_{M,V} dT,$$

show that

$$TdS = C_{V,M}dT + PdV - BdM,$$

in which $C_{V,M}$ is the heat capacity at constant $V$ and $M$.
(e) (4 points) The system starts at a temperature $T_0$, a volume $V_0$, and with a magnetization $M_0$. The gas now undergoes a quasi-static, adiabatic and isothermal change to a new volume $V$ and a new magnetization $M$. Find the new volume in terms of the new magnetization, $V_0$, $M_0$, $T_0$, $a$ and $R$.  

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