Finding $\vec{E}$ by Integration of Coulomb Law

- Review Question for last time
- If not enough symmetry to use Gauss’s Law, must integrate
- Read text “Interlude 2”, p. 248.
- Example: Short filament (length $l$); find electric field strength on bisector plane, a distance $y$ away.
  - Linear charge density $\lambda$ is uniform on filament, so $\lambda = Q/l$ where $Q$ is the total charge
Field at distance y above filament of length l
• Consider pairs of differential charge elements dQ=\lambda dx located at +x and -x (similar to previous case of pair of point charges), then integrate over x
\[ d\vec{E} = 2dE_{1,y} \hat{j} = 2dE_1 \cos \theta \hat{j} \]

\[ = 2 \frac{k dQ_1}{r^2} \frac{y}{r} \hat{j} = \frac{2ky\lambda dx}{r^3} \hat{j} \]

\[ \vec{E}[y] = \int_{x=0}^{l/2} \frac{2ky\lambda dx}{\left(x^2 + y^2\right)^{3/2}} \hat{j} \]

\[ = \frac{kQ}{y \sqrt{y^2 + (l^2 / 4)}} \hat{j} \]
Notes and Questions

• Q2 - When using integration to calculate $\mathbf{E}$
• Q3 - Contribution $dE$ from charge element $\lambda dx$
• Note that $\lambda$ may not be constant; may depend on position; for example $\lambda = \lambda_0 x/a$
  – Q4
Ex. 2 - Field on axis of circular filament

\[ dQ_1 dQ_2 \]

\[ dE_1 dE_2 \]
\[ d\tilde{E} = 2dE_1 z \hat{k} = 2dE_1 \cos \theta \hat{k} \]

\[ = 2 \frac{k dQ_1}{r^2} \frac{z}{r} \hat{k} = \frac{2kz (\lambda a d\phi)}{r^3} \hat{k} \]

\[ \tilde{E}[z] = \int_0^\pi \frac{2kz (\lambda a d\phi)}{\left(a^2 + z^2 \right)^{3/2}} \hat{k} \]

\[ = \frac{2kz \lambda a \pi}{\left(a^2 + z^2 \right)^{3/2}} \hat{k} = \frac{kQz}{\left(a^2 + z^2 \right)^{3/2}} \hat{k} \]
Surface Charge Distribution

- Surface charge density $\sigma$: $dQ = \sigma dA$; $Q = \int \sigma dA$
- If surface charge density is uniform, $\sigma = Q / A$
- Example: Infinite thin sheet of charge (Gauss’s law)
Note \( \vec{E} = E \hat{k} \) above; \( E = -\vec{E} \hat{k} \) below

\( \hat{n} = \hat{k} \) above; \( \hat{n} = -\hat{k} \) below; thus

\[
\oint \vec{E} \cdot \hat{n} dA = EA + EA = \frac{Q_{in}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}
\]

\[
E = \frac{\sigma}{2\varepsilon_0} \quad \& \quad \vec{E} = \frac{\sigma}{2\varepsilon_0} \hat{k} \quad (z > 0), \quad -\frac{\sigma}{2\varepsilon_0} \hat{k} \quad (z < 0)
\]

Field strength does not fall off as we move away
Ex. 2: Integration - Disk of charge

Circular filament (we already know the field at P from it!)

Field is along \( k \)

\[ \sigma = \frac{Q}{(\pi a^2)} \]

Integrate over set of rings, each of radius \( r \) and thickness \( dr \), with charge \( dQ = \sigma 2\pi r dr \)
\[ d\vec{E} = \frac{k(dq)z}{(z^2 + r^2)^{3/2}} \hat{k} = \frac{k(\sigma 2\pi r dr)z}{(z^2 + r^2)^{3/2}} \hat{k} \]

\[ \vec{E} = k\sigma\pi z k \int_{r=0}^{a} \frac{2r dr}{(z^2 + r^2)^{3/2}} \]

\[ \vec{E} = 2k\sigma\pi k \left( 1 - \frac{z}{\sqrt{z^2 + a^2}} \right) \]

\[ = \frac{2kQ}{a^2} \hat{k} \left( 1 - \frac{z}{\sqrt{z^2 + a^2}} \right) \]