Inductor Circuits: LR Circuit

- Decay of current in inductor-resistor (LR) circuit

Switch in position A for a long time. Current increases slowly; finally \( I = \frac{\varepsilon}{R} \) and \( \frac{dI}{dt} = 0 \). Also \( \Delta V_L = 0 \).

At \( t=0 \), move switch to position B. Current \( I(t) \) decreases over time.

Kirchhoff loop eqn for top loop

\[-IR - L \frac{dI}{dt} = 0\]

Rewrite loop eqn. to “separate variables”; integrate

\[\int_{I' = I_0}^{I'} \frac{dI'}{I'} = \int_{t'=0}^{t'} - \frac{R}{L} \ dt\]

\[\ln \left( \frac{I}{I_0} \right) = -\frac{Rt}{L}\]

\[I = I_0 e^{-\frac{Rt}{L}} = I_0 e^{-t/\tau}\]

Exponential decay; time constant \( L/R \)

- In one time constant, current decays to \( e^{-1} \) or 0.37 of original current. At \( t = 2\tau \), \( I = e^{-2}I_0 = 0.14 I_0 \)

- Quest.: Current in a 0.1H inductor is 10A. If a switch is turned at \( t = 0 \) to discharge the inductor through a 100 ohm resistor, when is the current 3.7A?

- Q2: What fraction of the original stored energy in the coil remains when the current is down to 2.7A?

“Charging” (Current Build-Up) in LR Circuit

- Close switch at \( t = 0 \), current \( I(t) \) increases over time

Similar to charging capacitor; get similar solution:

\[I(t) = I_0 \left( 1 - e^{-t/\tau} \right)\]

where \( \tau = L/R \) & \( I_0 = \varepsilon/R \)

Inductor-Capacitor (LC) Circuit

- Take capacitor \( C \) with charge \( Q \) and connect to inductor \( L \):

Kirchhoff loop Eqn.

\[\frac{Q}{C} - L \frac{dI}{dt} = 0\]

\[I = -\frac{dQ}{dt}\]

\[\frac{Q}{C} + L \frac{d}{dt} \left( \frac{dQ}{dt} \right) = 0\]

\[\frac{d^2Q}{dt^2} = -\frac{1}{LC} Q\]

Eqn. of Simple Harmonic Motion

- Mass \( m \) on spring \( k \) set in SHM; position \( x \) of mass:

\[x = A \cos(\omega t + \phi_0) \ldots \omega = \sqrt{\frac{k}{m}}\]
Oscillation of Charge and Current in LC Circuit

- For the LC circuit, the capacitor charge oscillates:
  \[ Q(t) = Q_0 \cos(\omega t + \phi_0) \]
- If we put this solution into the diff eqn., can relate \( \omega \) to the values of \( L \) and \( C \):
  \[
  \frac{d^2Q}{dt^2} = -\omega^2 Q_0 \cos(\omega t + \phi_0) = -\frac{Q_0}{LC} \cos(\omega t + \phi_0)
  \]
  \[
  \omega^2 = \frac{1}{LC} \Rightarrow \omega = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{LC}}
  \]
- The period \( T \) of the oscillation is \( 2\pi/\omega = 2\pi\sqrt{LC} \)
- Check dimensions: (L/R) has dim. of s; (RC) has dim. of s; (L/R)(RC)=LC has dim of s^2, (LC)^0.5 is s.

\[ Q_0 \] is the amplitude of the charge oscillation, and is equal to the original full charge on the capacitor
\( \omega \) is the “angular frequency” (in rad/s); \( f = \omega / 2\pi \) is the frequency in Hz
\( \phi_0 \) is the phase constant (in rad); determined by the part of the cycle (phase) which has \( t=0 \) (clock start)
  - For our case of \( Q=Q_0 \) at \( t=0 \), we have \( \phi_0 = 0 \).
  - If we picked \( t=0 \) when current at pos. max., \( \phi_0 = \pi/2 \)
  - For \( t=0 \) at \( Q = -Q_0 \), \( \phi_0 = \pi \)
  - For \( t=0 \) when current at neg. max., \( \phi_0 = 3\pi/2 \)

Q3: What is the oscillation frequency in Hz for \( L=1 \) mH and \( C=1 \) mF?

Current Oscillation

- Since \( I = -dQ/dt \):
  \[ I = \omega Q_0 \sin(\omega t + \phi_0) = I_0 \sin(\omega t + \phi_0) \]
- (for our case, \( \phi_0 = 0 \))

Energy Oscillation in LC Circuit

- Energy in the circuit oscillates between electric energy \( U_E=Q^2/(2C) \) & magnetic energy \( U_B=0.5LI^2 \)

Electric and Magnetic Energy in LC Circuit

- Plot of electric (\( Q^2/(2C) \)) and magnetic (0.5LI^2 energy)
- Similar to KE and PE for a mass on a spring

Q4: For \( Q_0 = 4C \), \( L=2H \), \( C=1F \), what is the peak current in the LC circuit?

Example: Circuit with \( L=0.1H \), \( C=1 \) mF has \( t=0 \) when capacitor charge is maximum. What fraction of the energy is magnetic at \( t=0.01s \)?
  - \( \omega = 1/\sqrt{LC} \Rightarrow 100 \) rad/s
  - \( I = I_0 \sin(100 \) rad/s \( (0.01s)) = 0.84I_0 \)
  - \( I^2 = (0.84)^2I_0^2 = 0.71I_0^2 \) so 71% of energy magnetic