Chapter 9 Stellar Atmospheres: Radiation fields

Let’s jump ahead just a bit and look at some definitions.

The light we see emerging from a star comes from the outer layers. The temperature, density and composition of the outer layers determines the features of the stellar spectrum.

\[ I_\lambda = \frac{\partial I}{\partial \lambda} = \frac{E_\lambda d\lambda}{d\lambda \ dt \ dA \cos \theta \ d\Omega} \]

**Specific intensity, \( I_\lambda \)** Energy passing through a solid angle from a point on the surface, at a given time, in a direction \( \vec{\ Omega} \), with wavelength between \( \lambda \) and \( \lambda + d\lambda \)

\( dA \) is a patch on the radiating surface of the star

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Blackbody radiation is isotropic. For blackbody radiation:

\[ \langle I_\lambda \rangle = B_\lambda \]

**Specific energy density** with wavelength between \( \lambda \) and \( \lambda + d\lambda \) is defined as:

\[ u_\lambda d\lambda = \frac{1}{c} \int I_\lambda d\lambda d\Omega \]

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For isotropic radiation:

\[ u_\lambda d\lambda = \frac{4\pi}{c} \langle I_\lambda \rangle d\lambda \]

For blackbody radiation:

\[ \langle I_\lambda \rangle = B_\lambda \]

\[ B_\lambda(T) = \frac{2hc^2}{\lambda^5} \left( \frac{1}{e^{hc/\lambda k T} - 1} \right) \]

Energy density in blackbody radiation for a characteristic wavelength:

\[ u_\lambda d\lambda = \frac{8hc}{\lambda^5} \left( \frac{1}{e^{hc/\lambda k T} - 1} \right) d\lambda \]
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Of course:

\[ u = \int_0^\infty u_\lambda d\lambda = \int_0^\infty u_\nu d\nu \]

For blackbody radiation,

\[ I_\lambda = B_\lambda \]

\[ u = \frac{4\pi}{c} \int_0^\infty B_\lambda(T) \ d\lambda = \frac{4\sigma T^4}{c} = aT^4 \]

\[ a = 4\sigma / c = 7.565767 \times 10^{-16} \text{ Jm}^{-3} \text{K}^{-4} \]

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Both the radiative flux and the specific intensity measure light received from a celestial source.

When you point a photometer at a light source, which of these are you measuring?

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Specific Radiative Flux:

\[ F_\lambda d\lambda = \int I_\lambda d\lambda \cos \theta \ d\theta \ d\Omega \]

\[ = \int_0^{2\pi} \int_0^{\pi} I_\lambda \cos \theta \sin \theta \ d\theta \ d\phi \]

This is the net energy with a wavelength between \( \lambda \) and \( \lambda + d\lambda \) that passes each second through a unit area in the direction of the z-axis.

Because of the factor \( \cos \theta \), oppositely directed rays can cancel!

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For a resolved source (e.g. observations of the Sun from an orbiting satellite) you are measuring specific intensity, \( I_{\text{spec}} \), the amount of energy passing through a solid angle \( \text{d} \Omega \text{min} \).

For an unresolved source (a distant star) it is the radiative flux that is being measured. The detector integrates the specific intensity over all solid angles. This is the definition of radiative flux. As the distance to the source increases, the amount of energy decreases as \( 1/r^2 \).
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**Radiation pressure**

Case 1: Photons incident on a perfectly reflecting surface

Momentum: \( p = E/c \)

\[
dp_x d\lambda = \left[ (p_x)_{\text{final}} - (p_x)_{\text{initial}} \right] d\lambda
\]

\[
= \left[ \frac{E_x \cos \theta}{c} - \frac{E_x \cos \theta}{c} \right] d\lambda
\]

\[
= \frac{2E_x \cos \theta}{c} d\lambda
\]

Recall:

\[
E_x = \int I_x d\lambda \cos \theta \cos \phi d\Omega
\]

\[
dp_x d\lambda = \frac{2}{c} I_x d\lambda dt dA \cos^2 \theta d\Omega
\]

\[
\frac{dp}{dt} \frac{dA}{dA} = F = \frac{2}{c} I_x d\lambda \cos^2 \theta d\Omega
\]

Force per unit Area = pressure

Case 1: Reflection

\[
P_{\text{rad}} d\lambda = \frac{2}{c} \int \text{hemisphere} I_x d\lambda \cos^2 \theta d\theta d\Omega
\]

\[
= \frac{2}{c} \int_{\phi=0}^{\pi} \int_{\theta=0}^{\pi/2} I_x d\lambda \cos^2 \theta \sin \theta d\theta d\phi
\]
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Case 2: Transmission - lose factor of 2 because no change in momentum on reflection

\[ P_{\text{rad}} d\lambda = \frac{1}{c} \int_{\text{sphere}} I_s d\lambda \cos \theta d\theta d\Omega \]
\[ = \frac{1}{c} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} I_s d\lambda \cos \theta \sin \theta d\theta d\phi \]
\[ = \frac{4\pi}{3c} I_s d\lambda \]

For blackbody radiation:

\[ P_{\text{rad}} = \frac{4\pi}{3c} \int_0^\infty B_\lambda(T) d\lambda = \frac{4\sigma T^4}{3c} = \frac{1}{3} a T^4 \]

\[ P_{\text{rad}} = \frac{1}{3} u \]

For comparison, the radiation pressure of an ideal monatomic gas is 2/3 its energy density.

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Effective temperature: from the Stefan-Boltzmann law characterizes the temperature at a particular depth in the star and is widely used as the global descriptive temperature

\[ L = 4\pi R^2 \sigma T^4 \]

Excitation temperature: defined by the Boltzmann equation

\[ \frac{N_b}{N_a} = \frac{g_b}{g_a} e^{-(E_b-E_a)/kT} \]

Ionization temperature: defined by the Saha equation

\[ \frac{N_{\text{ion}}}{N_i} = \left( \frac{2kT_{\text{ion}}}{P_e Z_i} \left( \frac{2\pi m_i kT}{h^2} \right)^{3/2} \right) e^{-\chi_{e}/kT} \]

Chapter 9: Stellar Atmospheres: Opacity

We approximate a stellar photosphere as a blackbody (e.g., dashed curve) but the radiation actually deviates substantially from this because absorption lines remove light from the “continuum” (a contraction of “continuous spectrum”).

Removal of flux from absorption lines is called line blanketing. In other parts of the spectrum (e.g., the ultraviolet) there may be contribution to flux at particular wavelengths from emission.

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Kinetic temperature: from the Maxwell-Boltzmann velocity distribution

\[ n_e dv = n \left( \frac{m}{2\pi kT} \right)^{3/2} e^{-m v^2 / 2kT} 4\pi v^2 dv \]

Color temperature: fitting a Planck function to the continuum
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Thermodynamic equilibrium
Every process (absorption of photons) is balanced by an inverse process (emission of photons).

Local thermodynamic equilibrium (LTE)
A star can hardly be in thermodynamic equilibrium. There is an outward flow of energy with temperature \( T = T(R) \). However, if the distance where temperature changes is large compared to the m.f.p. of particles and photons, then there is LTE in that region.

What is the temperature scale height?
How does it compare to the average distances for atoms between collisions?

Temperature Scale Height
According to model solar atmospheres (specifies temperature, density etc as a function of optical depth or radius) the temperature near the photosphere changes from 5580K to 5790K over a distance of 25 km. Scale height, \( H_T \), is given by:

\[
H_T = \frac{T}{dT/dr} = \frac{5685K}{(5790K - 5580K)/25.0km} = 677km
\]

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Distance between atomic collisions
According to model solar atmospheres the density of the photosphere in this same temperature region is about:

\[
\rho = 2.1 \times 10^{-4} \text{ km } m^{-3}
\]

and it consists primarily of neutral H atoms in the ground state (as you’d expect, now that you know about the Boltzmann eqna and Saha eqn!)

\[
n = \frac{\rho}{m_n} = 1.25 \times 10^{23} m^{-3}
\]
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Two atoms collide if their centers pass within two Bohr radii.

Equivalently: imagine a single atom with a radius of $2a_o$ moving with speed $v$ through a collection of points that represent the centers of other H atoms. The m.f.p. is the distance traveled divided by the number of atoms encountered (number density times the volume swept out).

\[
l_{\text{mfp}} = \frac{vt}{nV} = \frac{vt}{n\sigma v} = \frac{1}{n\sigma} = 2.27 \times 10^{-4} \text{ m}
\]

\[n = 1.25 \times 10^{23} \text{ m}^{-3}\]

\[
\sigma = \pi (2a_o)^2 = 3.52 \times 10^{-20} \text{ m}^2
\]

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Why do we only see photons from the photosphere? By definition, this is the layer of the Sun where photons can escape freely into space.

Processes that remove photons from a parallel beam of light are collectively termed absorption.
  - Atomic absorption
  - Compton scattering
  - Molecular transitions

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In the photosphere:

Temperature scale height: 677 km = 677,000 m
Distance between atomic collisions: $2.27 \times 10^4$ m

The temperature scale height exceeds the mfp of H atoms by factor of 3 billion. LTE seems like a reasonable assumption in the solar photosphere, but stay tuned...

The number of incoming photons that make it through a gas depends on...? distance traveled, density of the gas and cross-sectional area of intervening particles.
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Opacity, $\sigma$

cross-section for absorbing/removing photons of wavelength $\lambda$ per unit mass of stellar material.

$\sigma \lambda$ is also called the absorption coefficient and is wavelength dependent

The loss in photon intensity is proportional to $\sigma \lambda \rho I$, and $ds$:

$$dI_\lambda = -\kappa_\lambda \rho I \, ds$$

Greater the absorption coefficient, $\kappa_\lambda$, the more photons removed.

Greater the particle density, $\rho$, the more photons removed.

Greater the incoming intensity, the more photons removed.

Longer the path, $ds$, the more photons removed.

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Optical Depth, $\tau$

For scattered photons the characteristic distance, $\zeta$, is the m.f.p. of the photons.

$$I = \frac{1}{\kappa_\lambda \rho} = \frac{1}{n \sigma_\lambda}$$

Both $\sigma_\lambda$ and $n_\sigma_\lambda$ can be thought of as the fraction of photons scattered per meter.

$$d\tau_\lambda = -\kappa_\lambda \rho \, ds$$

Difference in optical depth along a path length, $ds$.

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The final intensity of light through a gas:

$$\int_{I_{\lambda,\text{final}}}^{I_{\lambda,\text{init}}} \frac{dl_\lambda}{I_\lambda} = -\int_0^s \kappa_\lambda \rho \, ds$$

Intensity decreases exponentially, falling by a factor of $e^\lambda$ over a characteristic distance, $\zeta = 1 / \kappa_\lambda$

In the solar atmosphere:

- $\kappa_{\text{init}} \approx 2.1 \times 10^4 \text{ kg m}^{-3}$
- $\kappa_{\text{opt}} \approx 0.03 \text{ m}^2 \text{ kg}^{-1}$

$\zeta \approx 160 \text{ km}$ the typical distance traveled by photons before being removed from the light beam.

This distance is comparable to the temperature scale height, showing that photons do not see a constant temperature and the LTE approximation is not strictly valid.

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$$\Delta \tau_\lambda = \tau_{\lambda,\text{final}} - \tau_{\lambda,\text{init}} = -\int_0^s \kappa_\lambda \rho \, ds$$

Outermost layers of star have $\kappa_{\lambda,\text{opt}} = 0$

After emerging from the star, the light travels unimpeded toward the observer.

Therefore, $\kappa_{\lambda,\text{opt}} = 0$ gives the initial optical depth of a ray of light that traveled a distance $s$ to reach the top of the photosphere.

$$\tau_\lambda = -\int_0^s \kappa_\lambda \rho \, ds$$

$$I_\lambda = I_{\lambda,\text{init}} e^{-\tau_\lambda \rho} = I_{\lambda,\text{init}} e^{-\kappa_\lambda \rho}$$
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\[ I_\lambda = I_{\lambda, \text{init}} e^{-\tau_\lambda} \]

If a packet of photons emerges from an optical depth, \( \tau = 1 \), the intensity of the ray is diminished by a factor of \( e^{-1} \).

The optical depth can be thought of as the number of mfp’s from the original position to the surface.

Typically, see no deeper into the star at a given wavelength than \( \tau \approx 1 \).

The corresponding physical depth in the star is wavelength dependent. Imagine shining a red light in the fog - move it deeper into the fog until it just starts to disappear. Then the light is at \( \tau = 1 \). Now, do the same for a blue light - you’ll find that the physical distance will be slightly different for \( \tau \approx 1 \) and \( \tau \approx 1 \) (Think about blue skies and red sunsets)

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The atmosphere of the Earth also has an optical depth that diminishes starlight.

\[ \tau_\lambda = \int_0^\infty \kappa_\lambda \rho \, ds = -\int_0^\infty \kappa_\lambda \rho \, dz \cos \theta \]

\[ = \sec \theta \int_0^k \kappa_\lambda \rho \, dz \]

\[ = \tau_{\lambda, 0} \sec \theta \]

The brightness of a star has to be corrected for atmospheric extinction (absolute photometry).

\[ I_\lambda = I_{\lambda, 0} e^{-\tau_{\lambda, 0} \sec \theta} \]

Intensity varies with position in the sky (hour angle) as a \( f(\theta) \)

Atmospheric dispersion corrector used to bring light of different wavelengths to a single focus

Optically thick: \( \tau_{\lambda, 0} \gg 1 \)

A light ray passing through a volume of gas loses most of its initial intensity

Optically thin: \( \tau_{\lambda, 0} \ll 1 \)

A light ray passing through a volume of gas emerges with most of its initial intensity

A gas can be optically thick at one wavelength and optically thin at other wavelengths! Earth’s atmosphere is optically thin at visible wavelengths, but optically thick at UV wavelengths.
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Sources of opacity

Set by interactions of photons with particles (atoms, ions, free electrons). Both scattering and absorption remove photons from a beam of light.

If opacity changes slowly with wavelength, it determines the shape of the continuum.

Rapid changes in opacity occur from atomic absorption and create spectral lines.

Bound-bound transitions: change in orbital level. Electron is initially bound and remains bound. Electron absorbs and re-emits in random direction, taking photon out of the solid angle light ray, or a cascade of photons is emitted (degraded) in random directions.

Bound-free transitions: photoionization

The cross section for photoionization is given by:

$$\sigma = 1.31 \times 10^{-19} \frac{1}{n^2} \left( \frac{\lambda}{500 \text{ nm}} \right)^3 \text{m}^2$$

Loosely bound electron

Free-free absorption: scattering

When and only when an electron is near an ion, the photon sees the pair essentially as an excited atom and the electron can absorb the photon and increase it’s kinetic energy.

Do you think this would contribute to line opacity (absorption line) or continuum opacity?

Right - this event can occur for a wide range of photon energies and contributes to continuum opacity
Free-free emission: Bremsstrahlung “braking radiation”
This is the inverse of ff absorption. When and only when an electron passes near an ion, it can loose kinetic energy and emit photons - again, this will contribute to the continuum.

**Compton scattering:**
A photon can be scattered when passing near an electron that is loosely bound to an atom. This occurs with high energy radiation where the wavelength of the photon is much smaller than the size of the atom.

The change in the energy of the photon is very small, but the direction is changed, reducing the number of photons in a beam.

**Electron scattering: Thompson scattering**
When a photon passes near an electron, the electron can absorb energy by oscillating in the E-M field of the photon.

\[
\sigma_T = \frac{1}{6\pi e_0} \left( \frac{e^2}{m_e c^2} \right)^2 = 6.65 \times 10^{-29} m^2
\]
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The primary source of continuum opacity in the atmospheres of stars later than F0V is....

\[ e^- \quad e^- \]

...H\(^+\) opacity

The binding energy of H\(^+\) is only 0.754 eV (1.6 \( \frac{\text{eV}}{\text{A}} \)) in the infrared, so any photon with more energy than this can be absorbed by this b-f process. At longer wavelengths, also get f-f absorption.

Rayleigh scattering:

When the photon wavelength is much larger than the size of the atom, the photons undergo Rayleigh scattering. The cross-section for Rayleigh scattering is much smaller than Thompson scattering and proportional to \( \frac{1}{\lambda^4} \), so decreases with increasing photon wavelength!

Blue light scatters more efficiently than red light. See blue sky when the Sun is overhead, but a greater optical depth is required to see scattered red light.

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The Balmer Jump!

Dramatic increase in opacity

\[ E^*_2 = \frac{13.2}{2^2} eV = -3.4 eV \]

\[ \lambda \leq \frac{hc}{3.4 eV} = 3647 \text{A} \]

So, photons with energy greater than this - i.e., with wavelengths shorter than 3647 Angstroms - will undergo b-f absorption.

What will the strength of the Balmer jump depend on?

Fraction of atoms in the n=2 level (Boltzmann-Saha eqns). So strongest for A0 stars with temperatures of about 9500K.

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In A and B-type stars, photoionization of hydrogen (b-f) and f-f absorption are main sources of opacity.

In O-type stars, most hydrogen is ionized, so electron scattering becomes more important and photoionization of helium contributes to the opacity.

The total opacity in a star is the sum of individual opacities, with different opacity terms being more or less important in different types of stars.

Depends not only on wavelength, but also on composition, density and temperature.
Rosseland Mean Opacity

- Averaged over all wavelengths
- Includes composition, density, temperature dependencies

\[
\langle \kappa \rangle = \frac{1}{\int_{0}^{\infty} \frac{1}{\kappa_v} \frac{\partial B_v(T)}{\partial T} dv} \int_{0}^{\infty} \frac{\partial B_v(T)}{\partial T} dv
\]

Weighting function depends on the rate that the bb spectrum changes with temperature

No analytical formulae for all of the contributions to b-b opacity by individual spectral lines, no Rosseland mean opacity for spectral lines.

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However, approximations have been developed for the average b-f and f-f opacities:

\[
\langle \kappa_{bf} \rangle = 4.34 \times 10^{-1} \frac{g_{bf}}{t} Z (1 + X) \frac{\rho}{T^{3.5}} \ m^2 k^{-1}
\]

\[
\langle \kappa_{ff} \rangle = 3.68 \times 10^{18} \frac{g_{ff}}{t} (1 - Z) (1 + X) \frac{\rho}{T^{3.5}} \ m^2 k^{-1}
\]

\[
\rho \quad \text{Density [km m}^{-3}] \quad T \quad \text{Temperature [K]} \quad X \quad \text{Hydrogen (mass) abundance} \quad Z \quad \text{Heavy metal (mass) abundance} \quad g \quad \text{Gaunt factor} \quad t \quad \text{“guillotine” factor - after atom is ionized, the opacity cuts off (has values of 1 - 100)}
\]

\[
\kappa_{bf} \rho / T^{3.5} \]

No analytical formulae for all of the contributions to b-b opacity by individual spectral lines, no Rosseland mean opacity for spectral lines.

---

Cross section for e- scattering is wavelength and temperature independent, so \( \kappa_{es} \) has simple form:

\[
\kappa_{es} = 0.02 (1 + X) \ m^2 k^{-1}
\]

Opacity from H- for Temp: 3000-6000K, \( \kappa_H \) between \( 10^{-7} - 10^{-2} \)

\[
\kappa_H \approx 7.9 \times 10^{-34} (Z/0.02) \rho^{1/2} T^9 \ m^2 k^{-1}
\]

Total Rosseland mean opacity:

\[
\kappa = \kappa_{bb} + \kappa_{bf} + \kappa_{ff} + \kappa_{es} + \kappa_H
\]

Opacity increases with increasing density at a given temperature.
At constant density it rises steeply from 5000K to 10,000K as the number of free electrons increases (ionization of H).
After the peak at 10000K, the decline in opacity follows Kramers law and is due to b-f and f-f absorption of photons.
HeII loses its second electron at about 40K - the increase in electrons causes a small bump in opacity.
Ionization of metals (Fe) cause the increase in opacity at about 10^5 K.
Flat floor: opacity due to electron scattering at high temperatures.
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First, a quiz!

What is the primary source of opacity in FGKM stars?
Name and describe sources of photon opacity.
What wavelengths typically undergo Compton scattering?
Rayleigh scattering?
What is the Rosseland mean opacity?
Does the Rosseland mean have a Kramers’ temperature dependency?

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At cooler temperatures, the opacity source is dominated by molecules.

Radiative Transfer in Stars

Radiative transfer in a star is relatively inefficient. For a random walk, the displacement is related to the size of the steps (mfp) and the number of steps:

\[ d = l \sqrt{N} \]

\[ d = \tau \ l = l \sqrt{N} \]

\[ \tau^2 = N \]

For \( \frac{r}{\sigma} >> 1 \), the average number of steps is roughly \( \frac{r}{\sigma} \). Near the surface, \( \frac{r}{\sigma} = 2/3 \), the average number of steps is 1. This is the definition of the photosphere.

The Opacity Project (OP) is an international collaboration formed in 1984 to calculate the extensive atomic data required to estimate stellar envelope opacities and to compute Rosseland mean opacities and other related quantities.

It involves research groups from France, Germany, the United Kingdom, the United States and Venezuela. The approach adopted by the OP to calculate opacities is based on a new formalism of the equation of state and on the computation by ab initio methods of accurate atomic properties such as energy levels, f-values and photoionization cross sections.
Looking into a star, we always see to an optical depth of \( \tau \sim 2/3 \).

Optical depth is proportional to the path length, so looking toward the limb of a star, we get to an optical depth, \( \tau \sim 2/3 \), at a cooler temperature than looking toward disk center.

The integrated spectrum of the Sun is spectral type G2V. However, the disk center of the Sun has the spectrum of a hotter F8V star!

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Pressure gradients: if you watch the fog roll in closely, you’ll see that individual aerosol particles are swirling in all directions but the mass of fog still has a net bulk motion because of a pressure gradient.

---

In the same way, the mfp of a photon near the center of the star is a fraction of a centimeter and a random-walk. But a gradient in the radiation pressure drives a slight net movement toward the surface of the star:

\[
\frac{dP_{\text{rad}}}{dr} = \frac{\kappa \rho}{c} F_{\text{rad}}
\]

Next, we’ll describe the emission and absorption processes that change the intensity of a ray of light with wavelength \( \lambda \) as it travels through a gas.

---

Emission coefficient

For pure emission (no absorption or scattering) the increase in intensity of a beam is proportional to the path length, \( ds \), and the density of the gas:

\[
dI_{\lambda} = j_{\lambda} \rho \ ds
\]

\( j_{\lambda} \) is the emission coefficient

The total change in intensity is:

\[
dI_{\lambda} = -\kappa_{\lambda} \rho \ I \ ds + j_{\lambda} \rho \ ds
\]

absorption

emission

These competing processes determine how rapidly the intensity of light changes.
- $\frac{dI_\lambda}{\kappa_\lambda \rho \, ds} = I_\lambda - \frac{j_\lambda}{\kappa_\lambda}$

Ratio of emission to absorption determines rate of intensity change.

Divide previous equation by $\rho \, ds$.

The source function is the ratio of emission coefficient to the absorption coefficient. It describes how photons traveling with beam are removed and replaced.

$S_\lambda = \frac{j_\lambda}{\kappa_\lambda}$

This is the "Radiative Transfer Equation".

The intensity of the beam evolves to match the intensity of the local source function.

Integrate:

$- \frac{dI_\lambda}{\kappa \rho \, ds} = I_\lambda - S_\lambda$

$\int \frac{dI_\lambda}{I_\lambda - S} = - \int \kappa \rho \, ds$

$\ln (I - S) - \ln (I_o - S) = -\kappa \rho \, s$

$\ln \left( \frac{I - S}{I_o - S} \right) = -\kappa \rho \, s$

$\frac{I - S}{I_o - S} = e^{-\kappa \rho \, s}$

$I - S = (I_o - S)e^{-\kappa \rho \, s}$

$I = I_o e^{-\kappa \rho \, s} + S(1 - e^{-\kappa \rho \, s})$

If the intensity of light does not vary, then $I_\lambda = S_\lambda$

If the intensity of light is greater than the source function, then $I_\lambda > S_\lambda$ and $dI/\rho \, ds$ is negative. The incoming (emission) photons can’t keep pace with the loss (absorption).

If the intensity of light is less than the source function, then $I_\lambda < S_\lambda$ and $dI/\rho \, ds$ is positive and there are more incoming photons.

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$I_\lambda = B_\lambda$

Radiation field can be described by Planck function.

$S_\lambda = B_\lambda$

Source field $(I_\lambda(0) \, S(0))$ can be described by Planck function.
$I = I_o e^{-k \rho \ s} + S(1 - e^{-k \rho \ s})$

For: $S = 2I_o$

$I = I_o(e^{-k \rho \ s} + 2 - e^{-k \rho \ s}) = 2I_o$

Incoming specific intensity tends toward the value of the source function!
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Gray atmosphere: \([\mathcal{K}]\)-independent opacity

\[
\cos \theta \frac{dl}{d\tau_{vert}} = I - S \\
\tau_{\lambda,v} \rightarrow \tau_{\lambda}
\]

Integrating over all solid angles:

The source function, \( S \), is independent of direction

\[
\frac{d}{d\tau_{vert}} \int I \cos \theta \, d\Omega = \int I \, d\Omega - S \int d\Omega
\]

\[
\frac{dF_{rad}}{d\tau_v} = 4\pi(\langle I \rangle - S)
\]

Gray, plane parallel atmosphere - what opacity case is this?

Because electron scattering is only relevant, gray opacity, not very realistic....

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Equilibrium: no net energy subtracted from or added to the radiation field.

The radiative flux must have the same value at every level of the atmosphere, including the surface.

\[
F_{rad} = \text{const} = F_{surface} = \sigma T^4 \\
\therefore \frac{dF_{rad}}{d\tau_v} = 0
\]

So, for the plane parallel gray atmosphere:

\[
\frac{dF_{rad}}{d\tau_v} = 4\pi(\langle I \rangle - S) \\
\langle I \rangle = S
\]

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A second important relation can be derived by multiplying through by \( \cos \theta \):

\[
\frac{d}{d\tau_{vert}} \int I \cos^2 \theta \, d\Omega = \int I \cos \theta \, d\Omega - S \int \cos \theta \, d\Omega
\]

\[
\int \cos \theta \, d\Omega = \left[ 2\pi \right]_{\theta=0}^{\pi} \cos \theta \sin \theta \, d\theta \, d\phi = 0
\]

In spherical coordinates:

\[
\frac{dP_{rad}}{d\tau_v} = \frac{1}{c} F_{rad}
\]

The radiative flux is driven by differences in the radiation pressure

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Integrating the equation that relates radiation pressure and radiative flux:

\[
\frac{dP_{rad}}{dr} = \frac{\kappa \rho}{c} F_{rad}
\]

\[
\int dP_{rad} = \frac{F_{rad}}{c} \int \kappa \rho \, dr
\]

\[
P_{rad} = \frac{1}{c} F_{rad} \tau_v + C
\]

Radiation pressure as a function of vertical optical depth
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The Eddington Approximation: at every point in the atmosphere, there is some intensity in and some intensity out.

Both $I_{\text{in}}$ and $I_{\text{out}}$ vary with depth in the atmosphere and $I_{\text{in}}=0$ at the top of the atmosphere.

\[
\langle I \rangle = \frac{1}{2} (I_{\text{out}} + I_{\text{in}})
\]

\[
F_{\text{rad}} = \pi (I_{\text{out}} - I_{\text{in}})
\]

\[
P_{\text{rad}} = \frac{2\pi}{3c} (I_{\text{out}} + I_{\text{in}}) = \frac{4\pi}{3c} \langle I \rangle = \frac{1}{c} F_{\text{rad}} \tau_v + \frac{2}{3c} F_{\text{rad}}
\]

Boundary condition: $I_{\text{in}}=0$

\[
T^4 = \frac{3}{4} T_{e}^4 \left( \tau_v + \frac{2}{3} \right)
\]

$T = T_e$ at $\langle I \rangle = 2/3$, not at $\langle I \rangle = 0$

Thus, the surface of a star is at $\langle I \rangle = 2/3$

\[
\frac{dI_{\lambda}}{d\tau_{\lambda}} = I_{\lambda} - S_{\lambda}
\]

Start with radiative transfer eqn

Multiply through by $e^{-t_{\lambda}}$

Separation of variables and integration by parts

Next, make $\langle I \rangle$ angle dependent....
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Include angle dependency for $I_\lambda$:

\[
I_\lambda(0) = I_\lambda e^{\sec \theta \tau_\lambda} - \int_0^{\sec \theta} S_\lambda e^{\sec \theta \tau_\lambda} d\tau_\lambda
\]

Parameterize the source function:

\[
S = a + b \cos \theta
\]

\[
I_\lambda(0) = a_\lambda + b_\lambda \cos \theta
\]

Limb-darkening revisited:

Parameterize $S = a + b$:

\[
a = \frac{\sigma}{2\pi T_e^4}
\]

\[
b = \frac{3\sigma}{4\pi T_e^4}
\]

\[
\frac{I(\theta)}{I(\theta = 0)} = \frac{a + b \cos \theta}{a + b} = \frac{2}{5} + \frac{3}{5} \cos \theta
\]

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Spectral line profiles:

\[
\frac{F_\lambda - F_c}{F_c} \quad \text{Line depth}
\]

One characterization of a line is equivalent width: the width of a box (zero to continuum) with the same area as the integrated area of the spectral line:

\[
W = \int \frac{F_\lambda - F_c}{F_c} d\lambda
\]

Graph of flux as a function of angle $\theta$:

Flux is plotted as fraction of continuum:

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Spectral line profiles:

Another way to characterize a spectral line is by the FWHM:
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Spectral line profiles

Spectral lines can be optically thin or optically thick. In the case shown here, the line is optically thin because the flux has not been completely blocked at any wavelength.

The opacity, $\mathcal{R}$, is greatest in the line core. If the opacity increases, you don’t see as deep into the star. Therefore, the line core forms in the higher (cooler) regions and the line wings form in deeper hotter regions.

At what optical depth does the continuum form?

$\mathcal{R} = 2/3$

Processes that broaden spectral lines

2. **Doppler broadening**: atoms in a gas will have a M-B velocity distribution

$$\Delta \lambda = \frac{\lambda}{c} \frac{v}{c} \quad \text{Non-relativistic Doppler equation}$$

$$v_{\text{wp}} = \sqrt{\frac{2kT}{m}} \quad \text{Most probably velocity from MB distribution}$$

$$\Delta \lambda = \frac{2 \lambda}{c} \sqrt{\frac{2kT}{m}} = 0.427 A$$

So, for H-atoms in the photosphere of the Sun (T=5777K), Doppler broadening is a factor of 1000 more important than natural broadening.

Processes that broaden spectral lines

1. **Natural broadening**: because of Heisenberg’s uncertainty principle, an orbit cannot have a precise wavelength

$$\Delta \lambda = \frac{\lambda^2}{2 \pi c} \left( \frac{1}{\Delta t_i} + \frac{1}{\Delta t_f} \right) \Delta t_i \quad \text{time in initial level}$$

For H-[$\mathcal{R}$=6563 A, this works out to be:

$$\Delta \lambda \approx 4.57 \times 10^{-14} m = 4.57 \times 10^{-4} A$$

Processes that broaden spectral lines

2. **Doppler broadening**: define the line width at half max

$$\langle \Delta \lambda \rangle_{1/2} = \frac{2 \lambda}{c} \frac{2kT \ln 2}{m} = 0.427 \times (0.83) A$$

Decreases exponentially moving away from line core because of fast falloff in MB distribution

Including larger scale turbulence:

$$\langle \Delta \lambda \rangle_{1/2} = \frac{2 \lambda}{c} \sqrt{\frac{2kT}{m} + v_{\text{turb}}^2} \ln 2$$

Turbulent term is important in atmospheres of giants and supergiants
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Processes that broaden spectral lines

3. **Pressure and collisional broadening**: perturb atomic orbitals during collisions or by electric fields of large numbers of close encounters with ions.

\[
\Delta \lambda = \frac{\lambda^2}{\pi} \frac{1}{c m} \quad \text{Similar format to natural broadening}
\]

\[
\Delta \lambda = \frac{1}{n \sigma} \frac{1}{2kT} \quad \text{Mean free path divided by } v_{\text{mp}}
\]

\[
\frac{\lambda^2}{n \sigma} \frac{2kT}{m} = \text{Line width is proportional to the number density of atoms: this forms the basis for luminosity classes.}
\]

\[
\frac{\lambda^2}{n \sigma} \frac{2kT}{m} \quad \text{The more luminous and tenuous giants have narrower lines than main sequence stars.}
\]

\[
\Delta \lambda = 2.36 \times 10^{-4} A \quad \text{For solar type main sequence stars, comparable to natural broadening.}
\]

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Case 1: thermal (Doppler) broadening

At $\tau = 0.1$, most photons escape before being scattered or absorbed.

Moving to greater optical depths, more photons are absorbed.

\[
\tau = -\kappa \rho \, ds
\]

So the optical depth could be increasing because the density of absorbers or scatterers is increasing or because the absorption coefficient is increasing.

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Calculation of a line profile described by a "curve of growth" and depends on:

- Temperature
- Density
- Opacity
- Element abundance

Quantum mechanical probabilities (oscillator strengths, or f-values)

A spectral line is broadened by composite of mechanisms. The total line profile is called a Voigt profile.

Doppler cores

Damping wings
For lines that are very optically thick, the increase in equivalent width is very slow because the only way to increase the line width is for the wings to grow.

The wings of the line are from Doppler shifted atoms.

Case 1: thermal (Doppler) broadening

\[ \nu = \frac{\Delta \lambda}{\lambda} c \]

At \( \Delta \nu = 0.5 \text{ A} \) and \( \lambda = 6563 \text{ A} \):

\[ \nu = 22800 \text{ m/s} \]

Case 2: pressure broadening

At large distances from the line center:

\[ \tau \propto \frac{1}{(\lambda - \lambda_c)} \]

No exponential, so the line grows more slowly than for thermal broadening and the lines have wider wings.
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Spectral line profiles

Case 2: pressure broadening
At large distances from the line center:

\[ \tau \propto \frac{1}{(\lambda - \lambda_0)^2} \]

No exponential, so the line grows more slowly than for thermal broadening and the lines have wider wings.

(Very large optical depths)

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Spectral line profiles

Compare thermally broadened lines (blue, purple) with collisionally broadened lines (green, red) at small optical depths.

Significantly different line profiles!

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Spectral line profiles

Compare thermally and collisionally broadened lines at large optical depths.

EW of Doppler-broadened lines shows a non-linear response once saturated.
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Spectral Parameters Of Cool Stars (SPOCS I.)

- with Jeff Valenti, STScI

Catalog of 1040 FGK stars from Lick, Keck, AAT

Spectral Synthesis Modeling (SME)

1) First use NSO atlas to tune log gf values, broadening coefficients, wavelengths.
2) Assume LTE, drive a radiative transfer code with Kurucz model atmospheres.
3) Marquardt fit to continuum and spectral lines in selected wavelength segments.

Spectral line profiles

Curve of growth

When the line saturates (when C > 5), so does the Doppler wing, because the line's shape will be smoothened near the line center.