Homework 2 Due 5:00PM 9/7

While I may have consulted with other students in the class regarding this homework, the solutions presented here are my own work. I understand that to get full credit, I have to show all the steps necessary to arrive at the answer, and unless it is obvious, explain my reasoning using diagrams and/or complete sentences.

Name Signature:

1. (50 points) A bead slides without friction on a rotating wire as shown in the figure below. The straight wire is oriented at constant polar angle \( \theta \), rotating about a vertical axis with constant angular velocity \( \Omega = \dot{\phi} \). The apparatus is in a uniform gravitational field \( g \) aligned with the vertical direction.

   (a) (20 points) Construct the Lagrangian using the radial distance \( r \) from the pivot \( (z = r \cos \theta) \) as the independent coordinate.

   (b) (20 points) Show that the condition for an equilibrium circular orbit is given by

   \[
   r = \frac{g \cos \theta}{(\Omega \sin \theta)^2}
   \]

   (c) (10 points) Discuss the stability of the orbit against small displacements by substituting \( r(t) = r_0 + \eta(t) \), where \( \eta \) is a small quantity compared to \( r_0 \).

2. (50 points) Show that if \( q_i = q_i(t) \) are solutions of the Euler-Lagrange equations for a given Lagrangian \( L(q_i, \dot{q}_i, t) \), then they are also solutions of the Euler-Lagrange equations for a new Lagrangian

\[
L'(q_i, \dot{q}_i, t) = L(q_i, \dot{q}_i, t) + \frac{df(q_i, t)}{dt}
\]

where \( f \) can be any arbitrary function of the generalized coordinates \( q_i \).

3. Mathematica Project: This problem is tracked separately from other homework for grading purposes. This problem concerns the double pendulum with massless rods of length \( \ell \) and equal masses \( m \).

   (a) Create a full, general Mathematica solution of the problem as represented in L&L problem (1), using the Lagrangian formalism. Your solution should start with the Lagrangian, and derive all equations of motions from it, inside Mathematica. Do not type the correct Lagrangian into Mathematica, but derive it correctly inside of Mathematica.

Please turn over
(b) Solve for the motion of the system using NDSolve. Together with (a) above, this will give you the tools to solve for the motion of any Lagrangian whatsoever.

(c) Use Mathematica or a similar program to plot the motion of the free end of the double pendulum. Plot the motion in the $(x_2, y_2)$ plane as well as in the phase space $(\theta_2, p_{\theta_2})$ plane. Your plots ought demonstrate the following set of initial conditions, all of them released from rest:

- Small initial displacement for both angles.
- Large initial displacement for the top pendulum, and small for the bottom.
- Large initial displacement for the bottom pendulum, and small for the top.
- Large initial displacement for both of the pendula.

For the final case, try to see what happens when you change your starting angle on the top pendulum slightly.

The attached examples ought to get you going. They show the motion for the simple harmonic oscillator, and also the following system of equations with initial conditions $x = 1, y = 1, \dot{x} = 0, \dot{y} = 0$:

$$\ddot{x} = \sin^5 y; \ddot{y} = \cos^5 x$$

N.B. the above equation is just an example and does not actually correspond to the correct set of equations for this problem.
(Define system of 2nd order differential equations)
eq1 = x''[t] == Sin[y[t]]^5
eq2 = y''[t] == Cos[x[t]]^5
(*Solve them*)
sol = NDSolve[{eq1, eq2, x[0] == 1, y[0] == 1, x'[0] == 0, y'[0] == 0},
{x[t], y[t]}, {t, 0, 50}][[1]];
(*This plots x vs.y*)
ParametricPlot[{x[t], y[t]} /. sol, {t, 0, 50}, AspectRatio -> 1]
(*Calculate the derivative with respect to time*)
deriv = D[sol, t];
(*This plots x vs.x ot*)
ParametricPlot[{x[t] /. sol, x'[t] /. deriv}, {t, 0, 50}, AspectRatio -> 1]
x''[t] = Sin[y[t]]^5
y''[t] = Cos[x[t]]^5