Astronaut Dr. Shannon Lucid is working in the Neutral Buoyancy Simulator, a tank full of water. Even though her helmet and suit are bulky, her head still looks too small. This effect is explained by the refraction of light at the curved air/water boundary ($18.4.2$).

The books are something like our books, only the words go the wrong way.

Lewis Carroll

**Chapter 18**

Geometrical Optics

**Concepts**
- Optical surface
- Real image
- Virtual image
- Virtual object
- Ray diagram
- Focus
- Lens
- Microscope
- Telescope
- Aberration

**Goals**
- Be able to:
  - Locate an image formed by a plane mirror, spherical mirror, or spherical refracting surface.
  - Determine the character of an image.
  - Use ray diagrams to construct the image of a finite object.
  - Compute the behavior of a compound optical system.
  - Give qualitative descriptions of the common aberrations in optical systems.
rays are all the information your eyes have about the location of the light source. Your eyes interpret the rays as if they came directly from an image of the original source. An ordinary plane mirror illustrates these ideas.

### 18.1.1 Images in a Plane Mirror

The image that stares back at you each morning from the bathroom mirror looks like a real person, about as far behind the mirror as you are in front of it (Figure 18.4a). To understand how this image is formed, look at the tip of your nose. Rays emerging from your nose reflect from the mirror into your eye (Figure 18.4b). It's usually easier to find the position of an image with rays chosen for that purpose, rather than with rays that actually enter the eye (Figure 18.4c). A ray perpendicular to the mirror is reflected directly backward at B. A ray striking the mirror at A reflects so that the angle of incidence \( \angle OAN = \theta \) equals the angle of reflection \( \angle NAE \). The two reflected rays (lines BO and AE) diverge from the image point I. The image is as far behind the mirror as the object is in front.

---

**Figure 18.4**

(a) Mirrors have been used for centuries. The mirror image of a face is a familiar sight. (b) Rays from the tip of your nose reflect from the mirror to your eye. They appear to diverge from the image of your nose, behind the mirror.

(c) Since all the reflected rays diverge from the image, specially selected rays show the position of the image most clearly. Here a ray perpendicular to the mirror is chosen, as well as one that reflects at an arbitrary point A. The angle of incidence is \( \angle OAN \), here called \( \theta \). It equals the angle of reflection \( \angle NAE \). Triangles OBA and IBA are congruent, so OB = IB. This conclusion does not depend on the actual value of \( \angle OAN \), and so holds for any ray from O. All reflected rays from your nose appear to come from the image at I.
Figure 18.6
Definition of real and virtual images and objects. (a) Light rays converge to, pass through, and diverge from a real image. A screen may also be placed at I. (b) Rays appear to diverge from a virtual image. Virtual images may be produced by reflected or refracted rays. (c) Rays diverge from a real object. (d) Rays converge toward a virtual object. (The effect of the optical surface on the incident rays is not shown here.)

Image 1 in Example 18.1 is a real object for the second mirror, according to this definition. A real object does not have to be an actual tangible thing. The important point is that an optical surface is associated with light approaching a surface, while an image is associated with light leaving an optical surface.

Suppose a mirror is placed in front of a movie screen. Light from the projector converges toward an image on the screen, but the mirror intercepts the light before it can form the real image. For such cases, we need the bizarre notion of a virtual object (Figure 18.6d):

A virtual object is one toward which light converges.

Virtual objects are used for convenience in analyzing compound optical systems; they occur only when light does not actually arrive at the object location.

Plane light waves are a special case. The rays are parallel lines and the wave fronts are parallel planes. We may think of such waves as diverging from a real object at an infinite distance.

In geometrical optics, we are interested in relationships between objects and images, and we need a way to describe their positions. There are three rules (Figure 18.7):

1. Object and image coordinates are measured from the optical surface.
2. For objects, the positive direction of the coordinate \( x \) is backward along light rays approaching the surface.
3. For images, the positive direction of the coordinate \( x \) is forward along light rays leaving the surface.

These conventions guarantee that real objects and images have positive coordinates while virtual objects and images have negative coordinates (Table 18.1). With this notation, our result for images in a plane mirror is written:

\[
\text{Image in a plane mirror: } x_i = -x_o. \tag{18.1}
\]

If the nose in Figure 18.4 is 10 cm in front of the mirror, then \( x_o = +10 \text{ cm} \) and \( x_i = -10 \text{ cm} \). The minus sign indicates that the image is virtual and behind the mirror.

<table>
<thead>
<tr>
<th>Table 18.1 Types of Objects and Images</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Object</strong></td>
</tr>
<tr>
<td>(light actually present)</td>
</tr>
<tr>
<td>Rays converge from it.</td>
</tr>
</tbody>
</table>

Section 18.1 • Images Formed by Plane Surfaces
18.2 Images Formed by Curved Surfaces

8.2.1 Spherical Mirrors

Curved mirrors have important applications in telescopes, automobile headlights, rearview mirrors, and the eyes of nocturnal animals. Figure 18.9 illustrates how the behavior of a concave mirror depends on object distance. When the object is close to the surface the mirror produces a virtual image, as a plane mirror would. The mirror forms a real image of a distant object. To study these images, we apply the law of reflection to selected rays striking the mirror surface.

Directions and positions in an optical system are referred to a line called the optical axis. The optical axis of a simple system is its symmetry axis. Optical surfaces are often spherical because they are the easiest curved surfaces to manufacture. They are also the easiest to analyze mathematically. Any line through the center of curvature \( C \) of a spherical mirror is a line of symmetry, so we define the optical axis as the line passing through the object and \( C \) (Figure 18.10).

Spherical mirrors form a sharp image so long as the rays used to view it make very small angles with the optical axis.

Paraxial rays are those that remain close to and make very small angles \((<<1\text{ rad})\) with the optical axis.

Rays at large angles to the axis produce a fuzzy blur around the paraxial image. The paraxial image itself is described by compact expressions that provide an excellent first approximation to the behavior of real systems. It is impractical to draw a diagram using only paraxial rays; it would not be legible! So we draw construction rays, which form large, clear diagrams and whose behavior is easy to analyze. Then we make the paraxial ray assumption during the algebraic calculation by assuming the angles are small.

First, we locate the image of an object placed at \( O \), close to a concave spherical mirror of radius \( r \) (Figure 18.10). The object coordinate \( x_o \) is positive and less than \( r/2 \). The normal to the mirror at any point passes through the center of curvature \( C \). A ray leaving \( O \) along the optical axis and striking the mirror at point \( A \) is normal to the mirror and is reflected back through \( C \). A ray striking the mirror at \( P \) has angle of incidence \( \theta \), so the reflected ray lies along \( PQ \). The rays diverge from the virtual image \( I \) behind the mirror.

We determine the distance \( AI = |x_i| \), using the paraxial ray approximation. At point \( P \), \( \theta + \angle OPI = \pi \) rad. Also, from triangle \( OPI \), \( \alpha + \beta + \angle OPI = \pi \) rad. Thus: \( 2\theta = \alpha + \beta \).

Similarly, in triangle \( CPI \), angle \( \theta \) equals the sum of the two internal angles \( \gamma \) and \( \beta \):

\[ \theta = \gamma + \beta. \]

**Figure 18.9**
Images in a concave mirror. When the object is close to the mirror, the image is virtual. When the object is far away, the image is real.

**Figure 18.10**
(a) Rays that remain close to the optical axis \( COI \) of a spherical mirror form a sharp image. These are paraxial rays. (b) Selected rays from an object \( O \) allow us to find the position of the image \( I \). The optical axis is \( COA \). Ray \( OA \) is perpendicular to the mirror and reflects back along the same line through \( O \) and \( C \). Ray \( OPQ \) has an angle of incidence of \( \theta \). Both rays diverge from the image at \( I \). True paraxial rays would not be distinguishable from the axis \( COA \). The angles are drawn large for clarity.
The focal point \( F \) of a mirror is the location of the image when parallel light falls on the mirror.

The focal length \( |f| \) is the distance from the mirror to the focal point; for spherical mirrors, it equals one half the radius of curvature.

When assigning coordinates to the focal point and the center of curvature, follow the rule for images.

In Figure 18.12b, light leaves the mirror to the left. Both \( F \) and \( C \) also lie to the left of the mirror, so their coordinates are positive.

The focusing property of mirrors is widely used: automobile headlight bulbs are placed at the focus of a mirror; emergency mirrors are used to concentrate sunlight and start a fire.

We may express eqn. (18.3) in terms of \( f \) rather than \( r \):

\[
\frac{1}{x_0} + \frac{1}{x_i} = \frac{1}{f}.
\]  

(18.4)

For spherical mirrors:

\[
\frac{1}{f} = \frac{1}{2r}.
\]  

(18.5)

A plane mirror may be regarded as a special case of a curved mirror with radius \( r = \infty \). Then \( 2/r = 0 \) and the mirror formula reduces to:

\[
\frac{1}{x_0} + \frac{1}{x_i} = \frac{1}{r} \Rightarrow x_i = -x_0.
\]

The mirror eqns. (18.4) and (18.5) apply to all spherical mirrors, including the special case of the plane mirror. We shall find that eqn. (18.4) (but not 18.5) remains valid for thin lenses as well.

Figure 18.13 illustrates the behavior of a convex mirror. Parallel rays incident on the mirror from an object at infinity (Figure 18.13a) produce a virtual image at the focal point. In fact, the mirror produces a virtual image of a real object at any distance from the mirror (Figure 18.13b). To locate the image, we determine the path of paraxial rays like \( OPQ \) (Figure 18.14). The image is at \( I \) and the image coordinate \( x_i = -AI \). The center of curvature is at \( C \) and, using the image rule, its coordinate is \( r = -AC \).

Figure 18.13
(a) Parallel light reflected from a convex mirror appears to diverge from a focal point behind the mirror. According to our sign conventions, both \( f \) and \( r \) are negative. (b) The image formed by a convex mirror is virtual.

Figure 18.14
We select one ray normal to the mirror and another with an angle of incidence \( \theta \) to find the image position. The angle \( \alpha \) is exaggerated.
18.2.2 Spherical Refracting Surfaces

Light entering our eyes is refracted at a series of curved surfaces so as to form a real image on the retina. Camera lenses and other precision optical instruments employ similarly intricate sequences of refracting surfaces. We begin the study of such systems by investigating the image produced by paraxial rays refracted at a spherical surface.

Light from a source at \( O \) travels through a material of refractive index \( n_1 \) and is refracted into a material with index \( n_2 > n_1 \) (Figure 18.17). A ray traveling along the optical axis is normal to the surface at \( A \) and passes undeflected through \( C \). A different ray reaching the surface at \( P \) is refracted and intersects the first ray at the image \( I \). The radius \( CP \) is normal to the surface at \( P \), so \( \theta_1 \) and \( \theta_2 \) are the angles of incidence and refraction, respectively. To satisfy the paraxial condition, all angles are small, and Snell's law becomes:

\[
n_1 \theta_1 = n_2 \theta_2.
\]

We use the ray paths to determine the distance \( AI \) from the surface to the image. Since \( \theta_1 \) is an external angle of triangle \( OPC \):

\[
\theta_1 = \alpha + \gamma. \tag{ii}
\]

Similarly, the external angle of triangle \( OPI \) is \( \theta_1 - \theta_2 \):

\[
\theta_1 - \theta_2 = \alpha + \beta. \tag{iii}
\]

Using eqn. (i) to eliminate \( \theta_2 \) and eqn. (ii) to eliminate \( \theta_1 \), eqn. (iii) becomes:

\[
\alpha n_1 + \beta n_2 = \gamma (n_2 - n_1). \tag{iv}
\]

Since the angles are small, they are approximately equal to their tangents and the distance \( AS \) is negligible, so eqn. (iv) becomes:

\[
n_1/OA + n_2/MA = (n_2 - n_1)/AC.
\]

After meeting the surface, the light passes through and travels to the right. The image \( I \) and the center of curvature \( C \) are both to the right of the surface—that is, on the side of outgoing light—and so have positive coordinates, \( x_i = AI \) and \( r = AC \). The real object also has a positive coordinate, \( x_o = OA \), so:

\[
\frac{n_1}{x_o} + \frac{n_2}{x_i} = \frac{n_2 - n_1}{r}. \tag{18.6}
\]

When using this formula, recall that the object lies in the medium whose refractive index is \( n_1 \) and the observer (but not necessarily the image) is in the medium with refractive index \( n_2 \).

**EXAMPLE 18.4** The radius of curvature of the cornea is about 0.78 cm. The interior of the eye has an average index of refraction of about 1.38 (similar to water). Where would the image of a match flame 0.52 m from the eye be formed if the eye were just a single refracting surface?

**MODEL** We model the cornea as a spherical refracting surface, and we use the paraxial ray approximation.

> See §18.5.1 and Problem 109 for further discussion of the eye.
A lens consists of two refracting surfaces in series. When light from an object is incident on the lens, the rays are bent at the first surface toward a real image at \( i_1 \). At the second surface, the rays are bent again to the real image at \( i_2 \). The image \( i_1 \) is a virtual object for \( S_2 \).

The object and the center of curvature of the surface \((x_o, r)\), are positive. The light rays converge toward the image \( i_1 \) (Figure 18.19b), and

\[
\frac{n_s}{x_o} + \frac{n}{x_{c,1}} = \frac{n - n_s}{r_1} \Rightarrow \frac{n}{x_{c,1}} = \frac{(n - n_s)x_o - n_sr_1}{x_o r_1}.
\]

Thus:

\[
x_{c,1} = \frac{r_1 nx_o}{n x_o - (x_o + r_1)n}.
\]

The image is real, as shown in the figure, if \( x_{c,1} \) is positive—that is, if \( n x_o > (x_o + r_1)n \). Otherwise, the image is virtual and to the left of the surface.

After completing the calculation of the first surface and move on to the second. In Figure 18.19c, the light rays reach the second surface before converging on the second surface which would be the image \( I_1 \). Here we have our first example of a virtual object: image \( I_1 \) forms a virtual object for surface \( S_2 \). To proceed with the calculation, we need its object coordinate \( x_{c,2} \).

The image \( I_1 \) has become the object \( O_2 \), but the image coordinate \( x_{c,1} \) is measured from the first surface, and the object coordinate \( x_{c,2} \) is measured from the second surface. The distance \( d \) of \( I_1 \) from \( S_2 \) is \( x_{c,2} = \ell \), where \( \ell \) is the distance separating the two surfaces, and so the coordinate of the virtual object is:

\[
x_{o,2} = -(x_{c,1} - \ell) = \ell - x_{c,1}.
\]

**EXERCISE 18.7**: Show that the same expression for \( x_{o,2} \) holds if \( I_1 \) is formed to the left of the second surface.

The next step is to use this result in eqn. (18.6) to locate the image formed by \( S_2 \). It's usually easier to do this numerically than to obtain an algebraic expression that works in all cases (see Problems 53 and 54, for example.)

See Figure 18.6 for the definition of a virtual object.
18.3.2 Thin Lenses

Thick-lens problems always require a sequential, surface-by-surface approach (§18.3.1). However, when the separation $\ell$ of the two surfaces is small compared with all other lengths in the problem—the radii of curvature and the distances of all objects and images from the lens surfaces—the lenses may be treated as thin. We may thus simplify the analysis by ignoring any difference between the distances of object or image from the front and back surfaces of the lens. So, we ignore $\ell$ in eqn. (18.7) and substitute $x_{35} = -x_{13}$ in eqn. (18.6).

$$\frac{n_1}{n_2} \left( \frac{x_1 - (x_{13} + r_1)n_1}{r_1 n_1} \right) + \frac{n_3}{n_2} = \frac{n_2 - n}{r_2}.$$  

We simplify the first part of this expression by canceling a factor of $nx_1$ in the first two terms and a factor $nr_1$ in the third:

$$\frac{n_2 - n}{r_1} + \frac{n_1}{x_{13}} + \frac{n_3}{x_3} = \frac{n_2 - n}{r_2}.$$  

For thin lenses, we no longer need to distinguish the surfaces from which we measure object and image distances, so we replace $x_{13}$ with $x_1$. Then:

$$\frac{1}{x_1} = \frac{1}{x_0} - \frac{n - n_2}{n_2} \left( \frac{1}{r_1} - \frac{1}{r_2} \right).$$  

With eqn. (18.8) we may compute the coordinate of the final image formed by the lens, $x_1$, given the coordinate $x_0$ of the object and the properties $n$, $n_2$, $r_1$, and $r_2$ of the lens.

**EXAMPLE 18.5** An old house has an ornate window made of glass 2.0 cm thick with a refractive index of $n = 1.50$. A portion of the window has curved surfaces of radii 1.0 and 0.50 m, respectively (Figure 18.20). A light bulb inside the house is 0.50 m from the window. Where is the image of the light bulb formed by the window?

**MODEL** Since the window thickness (2.0 cm) is only 4% of the next smallest length in the problem (50 cm), the thin-lens approximation should serve fairly well.

**SETUP** First, we evaluate the right-hand side of eqn (18.8):

$$\frac{n - n_2}{n_2} \left( \frac{1}{r_1} - \frac{1}{r_2} \right) = \frac{1.50 - 1.00}{1.00} \left( \frac{1}{1.00} - \frac{1}{0.50} \right) = 0.50 /\text{m}.$$  

**SOLVE** The object coordinate, measured from the center of the lens, is $x_0 = 0.51$ m. Using eqn. (18.8):  

$$\frac{1}{0.51} + \frac{1}{x_1} = 0.50 /\text{m} \Rightarrow \frac{1}{x_1} = -1.46 /\text{m}$$  

or  

$$x_1 = -0.68 \text{ m}.$$  

**ANALYZE** The image is virtual and 68 cm from the center of the lens. This answer differs from a more exact calculation (Problem 53) by approximately 3%.

The right-hand side of eqn. (18.8) gives the reciprocal of the lens’ focal coordinates, a relation known as the lens maker’s equation.
To prove this claim, let \( x_0 = f \) in eqn. (18.8). Then \( x = \infty \) and \( f \) is the coordinate of the first focal point. Letting \( x = \infty \) gives \( 1/x = 1/f \), so that \( f \) is also the coordinate of the second focal point. The two focal coordinates of a thin lens are equal, but the focal points are on opposite sides of the lens (Figure 18.21). If the direction of light passing through the lens is reversed, then the focal points exchange names while remaining in the same positions. The focal coordinate \( f \) may be positive or negative. If \( f \) is positive, the lens is said to be converging, while if \( f \) is negative, the lens is diverging (Figure 18.22). The absolute value of \( f \), \(|f|\), is the focal length of the lens.

Usually a lens is used in air, so we may replace the refractive index \( n \) of the ambient medium by 1.

\[
\frac{1}{f} = (n - 1) \left( \frac{1}{r_1} - \frac{1}{r_2} \right) \quad \text{(lens used in air).}
\]

**(Example 18.6)** A spectacle lens is made of glass with a refractive index of 1.50 and has radii of curvature of 4.0 cm and 6.0 cm (Figure 18.23). What is the focal length of the lens? Is it a converging or diverging lens?

**MODEL**: We model the lens as thin and apply eqn. (18.10).

**SOLVE**: If we suppose that light is incident from the left, then using the image rules, the appropriate sign for each radius is negative: \( r_1 = -4.0 \) cm and \( r_2 = -6.0 \) cm.

\[
\frac{1}{f} = (1.50 - 1.00) \left( \frac{1}{-4.0 \, \text{cm}} - \frac{1}{-6.0 \, \text{cm}} \right) = (0.50) \left( \frac{-2.0 + 2.0}{12.0 \, \text{cm}} \right) = 0.25 \text{ cm}^{-1}
\]

\[ f = 4 \text{ cm}. \]

The lens is diverging with a focal length of 4 cm.

**ANALYSE**: An optician describes a lens in terms of its dioptric power, \( 1/f \), measured in units of diopters, where 1 diopter = 1/m. This lens has a power of \(-2.5\) diopters.

A lens maker designing a lens with a specified focal coordinate \( f \) has considerable freedom in choosing the radii \( r_1 \) and \( r_2 \). Only the combination \( r_1^{-1} - \frac{1}{f} \) determines \( f \). Quite often \( r_2 = -r_1 \) and the lens is said to be symmetric. If one face is plane, \( r_2 = \infty \), the lens is called **plane-convex** \((r_1 > 0)\) or **plane-concave** \((r_1 < 0)\).