1. a) Write the Feynman propagator as
\[ D(x-y) = -i \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} \left( \theta(x^0 - y^0)e^{-i\omega_k(x^0-y^0)} + \theta(y^0 - x^0)e^{i\omega_k(x^0-y^0)} \right) \]
\[ \equiv -i\theta(x^0 - y^0)D_+(x-y) - i\theta(y^0 - x^0)D_-(x-y). \]
Prove that
\[ D_{\pm}(x) = \frac{1}{(2\pi)^3} \int d^4k \delta(k^2 - m^2)\theta(\pm k^0)e^{-i k_x}. \]

b) Argue that this implies that \( D_{\pm}(x) \) is Lorentz invariant, i.e., prove that \( D_{\pm}(x) = D_{\pm}(\Lambda x) \), with \( \Lambda \) a Lorentz transformation. Do this by using that each of the elements, \( \delta(k^2 - m^2) \), \( \theta(\pm k^0) \), \( kx \) and \( d^4k \) are invariant. [Hint: use that \( kx = (\Lambda^{-1}k)(\Lambda^{-1}x) \) for any \( \Lambda \).]

2. In class, we derived the following expression for the energy between two static sources, located at \( \vec{x}_1 \) and \( \vec{x}_2 \):
\[ E = -\int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k} \cdot (\vec{x}_1 - \vec{x}_2)}}{k^2 + m^2}. \]
In the massless limit, this resulted in the inverse-square force law. Repeat the same calculation, but now assuming that we are in two, rather than three, spatial dimensions. First, write down the two-dimensional equivalent of the expression above. This will now contain a measure \( d^2k = dk_x dk_y \). Perform first the integral over \( k_y \). In the remaining \( k_x \) integral, use the substitution \( k_x = m \sinh u \), and show that the result can be written as
\[ E(r) = \frac{1}{4\pi} \int_{-\infty}^{\infty} du \, e^{-mr \cosh u}, \]
where \( r = |\vec{x}_1 - \vec{x}_2| \). (It is a little easier to take \( \vec{x}_2 = 0, \vec{x}_1 = \vec{x} \), so that \( r = \sqrt{x^2 + y^2}. \) Be very careful in this substitution; in particular, pay attention to the integration path! Look this integral up in an integral table, and give an expression for the force between these two static sources in the limit that \( m \to 0 \). Does your result agree with what you would expect from Gauss’ law in two dimensions? Why?

3. In order to find the propagator for a massive spin-1 field, we need to invert the matrix \( -(k^2 - m^2)g^{\mu\nu} + k^\mu k^\nu. \) The inverse matrix has to be built from the four vector \( k \) (and the mass of the spin-1 field). Because of covariance under Lorentz transformations, the general form of the inverse has to look like
\[ A(k^2)g^{\mu\nu} + B(k^2)k_\mu k_\nu. \]
Use this to find the inverse matrix, i.e., determine the functions \( A \) and \( B. \)