Physics 726, Problem set 3, Fall 2007

1. Consider a boost in the direction of the z-axis,

$$z' = \gamma(z - vt) \ ,$$
$$t' = \gamma(t - vz) \ ,$$

with v the boost parameter, i.e. the relative velocity between the two inertial frames; we have set $c = 1$. In 2×2 matrix form:

$$\begin{pmatrix} z' \\ t' \end{pmatrix} = \begin{pmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{pmatrix} \begin{pmatrix} z \\ t \end{pmatrix} \ .$$

(Since $x' = x$ and $y' = y$ do not transform, we can simplify things by just considering 2×2 matrices.)

a) Consider two successive boosts, one between the unprimed frame with coordinates x, y, z, t and the primed frame x', y', z', t' with boost parameter v, and then one from the primed frame to a double-primed frame x'', y'', z'', t'' with boost parameter v'. It is clear that the resulting transformation from the unprimed to the double-primed frame is again a boost in the z direction, with a boost parameter v''. Express v'' in terms of v and v', using the matrix form for a boost in the z-direction given above. We will call the boost that results from doing these two successive boosts the "product" of the boosts with parameters v and v'. (But note that v'' is definitely not the product of v and v'!)

b) Show that this set of transformations forms a group. In part a) you have shown that the product of two elements of this group is again an element of this group. So what remains is to show that there exists a unit element, that each element has an inverse, and that the associative law for the product defined in part a) holds.

c) The group we just studied is a subgroup of the full Lorentz group. Argue that it is sensible to call this subgroup SO(1,1). Show that this subgroup is abelian. (The full Lorentz group SO(3,1) is not!)

2. In class, we derived the following expression for the energy between two static sources, located at $\vec{x}_1$ and $\vec{x}_2$:

$$E = -\int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{x}_1 - \vec{x}_2)}}{\vec{k}^2 + m^2} \ .$$

In the massless limit, this resulted in the inverse-square force law. Repeat the same calculation, but now assuming that we are in two, rather than three, spatial dimensions. First, write down the two-dimensional equivalent of the expression above. This will now contain a measure $d^2k = dk_x dk_y$. Perform first the integral over k_y. In the remaining k_x integral, use the substitution $k_x = m \sinh u$, and show that the result can be written as

$$E(r) = \frac{1}{4\pi} \int_{-\infty}^{\infty} du \, e^{-mr\cosh u} \ ,$$

where $r = |\vec{x}_1 - \vec{x}_2|$. (It is a little easier to take $\vec{x}_2 = 0$, $\vec{x}_1 = \vec{x}$, so that $r = \sqrt{x^2 + y^2}$.) Be very careful in this substitution; in particular, pay attention to the integration path! Look this integral up in an integral table, and give an expression for the force between these two static sources in the limit that $m \to 0$. Does your result agree with what you would expect from Gauss' law in two dimensions? Why?