Problem set 9

1) Kruskal–Szekeres coordinates and the Kruskal diagram.
Starting with the Schwarzschild metric, and choosing units in which Newton’s constant $G = 1$, let us perform the following coordinate transformation:

$$U = \left( \frac{r}{2M} - 1 \right)^{1/2} e^{r/4M} \cosh \left( \frac{t}{4M} \right), \quad r > 2M,$$

$$V = \left( \frac{r}{2M} - 1 \right)^{1/2} e^{r/4M} \sinh \left( \frac{t}{4M} \right),$$

$$U = \left( 1 - \frac{r}{2M} \right)^{1/2} e^{r/4M} \sinh \left( \frac{t}{4M} \right), \quad r < 2M,$$

$$V = \left( 1 - \frac{r}{2M} \right)^{1/2} e^{r/4M} \cosh \left( \frac{t}{4M} \right),$$

while we leave $\theta$ and $\phi$ the same.

a) Show that the metric in terms of the new coordinates is equal to

$$(d\tau)^2 = \frac{32M^3}{r} e^{-r/2M} \left( (dV)^2 - (dU)^2 \right) - r^2 \left( (d\theta)^2 + \sin^2 \theta (d\phi)^2 \right),$$

with $r$ a function of $U^2 - V^2$ implicitly defined by

$$\left( \frac{r}{2M} - 1 \right) e^{r/2M} = U^2 - V^2, \quad \text{all } r > 0.$$

b) We will be drawing a space-time diagram, with $U$ on the horizontal axis and $V$ on the vertical axis (suppressing $\theta$ and $\phi$). Draw the curves corresponding to $r = 2M$, using the second equation of part (a). What are the values of $t$ on these curves?

c) Locate $r = 0$ in your diagram.

d) Draw a curve in your diagram corresponding to a stationary observer at $r = r_0$ with $r_0 > 2M$.

e) Indicate in the figure the region in which we live, if we are outside a black hole with mass $M$ at the origin. Also indicate the region inside the horizon. [Note that outside the horizon, $U \geq V$ and $U > 0$, while inside the horizon $V \geq U$ and $V > 0$.]

f) At any point in these two regions, what does the lightcone look like? Explain your argument, and draw a few examples in your diagram. In particular, indicate what the future lightcone is. Explain why nothing can escape the region inside the horizon, and why everything inside the horizon will eventually crash into the singularity. [The simple form lightcones take in Kruskal–Szekeres coordinates is one motivation for using these coordinates.]
g) Draw the worldline of an object falling into the black hole, starting at some time from $r = r_0$. Show that on the clock of the stationary observer staying behind (say, at $r = r_0$) it takes an infinite amount of time for the object to reach the horizon, but that on a clock falling in with the object, it takes a finite amount of time. Argue on the basis of the diagram, don’t do any calculations.

h) What does the world line of a lightray look like in this diagram? Indicate the point (if it exists) on the curve of the stationary observer from which light can still reach the object before it crashes into the singularity.