problem set 3

1) In this and the following problem we’ll study some problems in non-relativistic mechanics using our new formalism. We’ll restrict ourselves to motion in a plane, but we’ll treat time $t$ explicitly as one of our coordinates. For classical mechanics, the appropriate $\eta$ tensor has $\eta_{00} = \eta_{11} = \eta_{22} = 1$, all other elements are equal to zero. Here $x^0 = t$, $x^1 = x$ and $x^2 = y$. Now we want to know what the space-time interval looks like in polar coordinates, so we consider the coordinate transformation

$$
\begin{align*}
x^0 &= t' = t, \\
x^1 &= r = \sqrt{(x^2 + y^2)}, \\
x^2 &= \phi = \arctan \frac{y}{x}.
\end{align*}
$$

Find the metric tensor $g_{\mu\nu}$ and the space-time interval for this coordinate system, and also calculate the inverse $g^{\mu\nu}$.

2) Now consider a new coordinate system $x'^{\mu}$, which rotates relative to the previous inertial system with angular speed $\omega$. We thus have $x'^{0} = x^{0}$, $x'^{1} = x^{1}$, but $x'^{2} = x^{2} - \omega t = \phi - \omega t$. Find the metric for this coordinate system. Calculate the affine connection $\Gamma^\lambda_{\mu\nu}$, and give the geodesic equations for $x'^{\mu}$, which you may now re-label $t$, $r$ and $\phi$. Show that we can take $t = \tau$, and explain what the terms containing $\omega$ in the geodesic equations for $r$ and $\phi$ mean. [If you don’t remember what they mean, look it up in your classical-mechanics textbook!]

3) We call a second-rank tensor traceless if

$$
g^{\mu\nu}T_{\mu\nu} = T^\mu_{\mu} = 0. $$

Given an arbitrary second-rank tensor $T_{\mu\nu}$, show that

$$
T_{\mu\nu} - \frac{1}{4} (T_\kappa^{\kappa}) g_{\mu\nu}
$$

is traceless. From this, show that an arbitrary second-rank tensor may be written as the sum of an antisymmetric tensor, a symmetric traceless tensor, and a multiple of the metric tensor.

4) Find the metric of euclidean three-dimensional space in polar coordinates $r$, $\theta$ and $\phi$. Restrict this metric to that of a two-dimensional spherical surface by taking $r$ fixed, i.e. $dr = 0$. The surface of the sphere is thus described by the two coordinates $\theta$ and $\phi$. Give the metric tensor on the surface of the sphere in terms of these two coordinates. Calculate all components of the affine connection $\Gamma$. In general (i.e., not specifically for the metric in this problem), how many independent components does $\Gamma$ have in two dimensions?

5) What are the geodesic equations for the surface of a sphere? [Use the results of the previous problem.] What we mean by geodesic equations is the following. We are just
considering the surface of a sphere, with no time coordinate involved at all. Still any curve
on a sphere can be described as a function of for instance its arc length $s$, which replaces
proper time in the geodesic equation (which we will not call equation of motion in this
problem because $s$ is not a time). “Geodesics” on a curved manifold (here our sphere) are
by definition solutions of the geodesic equation. Show that the equator and longitude lines
on the surface of the earth are geodesics, but that latitude lines (except the equator) are
not.