

**Final, Physics 706, Spring 2009**

**This exam is due at the beginning of class on May 14.** There are three problems on this exam (two pages). The number of credit points is indicated for each problem. You are allowed to use your notes, homework solutions, and the textbook (Sakurai) *only*, no other books or Mathematica/Matlab, no help from anyone else. Show the details of your work! Good luck!

1. (15 points) Suppose an electron is in a state described by the wave function

$$\psi = \frac{1}{\sqrt{4\pi}} (e^{i\phi} \sin \theta + \cos \theta) g(r) ,$$

where

$$\int_0^\infty |g(r)|^2 r^2 dr = 1 ,$$

and  $\phi$  and  $\theta$  are the azimuthal and polar angles, respectively. In parts (a)–(c), ignore the spin of the electron.

- (a) (3 points) What are the possible results of a measurement of the  $z$ -component  $L_z$  of the angular momentum of the electron in this state?
- (b) (3 points) What is the probability of obtaining each of the possible results you found in part (a)?
- (c) (3 points) What is the expectation value of  $L_z$ ? What is the expectation value of  $\vec{L}^2$ ? Is the electron in an eigenstate of either one of these two operators?
- (d) (3 points) If we now also consider the spin of the electron, what values of  $J_z$  and  $\vec{J}^2$  can be outcomes of a measurement of these quantities, if  $\vec{J} = \vec{L} + \vec{S}$  is the total angular momentum of the electron?
- (e) (3 points) Show that this wave function is normalized.

2. (20 points) Consider the one-dimensional potential

$$V = -\frac{\hbar^2 a^2}{m} \frac{1}{\cosh^2(ax)} ,$$

with  $a$  a positive constant.

- (a) (5 points) Sketch this potential, and show that

$$\psi_0(x) = \frac{A}{\cosh(ax)}$$

is an energy eigenstate; find its energy, and normalize the wave function (*i.e.*, find  $A$ ).

- (b) (2 points) Show that  $\psi_0$  is a bound state.

(c) (4 points) Now we will apply the WKB method to see whether we can find more bound states. Show that this amounts to solving the equation

$$\left(n + \frac{1}{2}\right) \pi \hbar = 2\sqrt{2}\hbar a \int_0^{x_0} \sqrt{\frac{1}{\cosh^2(ax)} + \frac{mE}{\hbar^2 a^2}} dx ,$$

for  $E$  as a function of  $n = 0, 1, 2, \dots$ . Explain how you get this equation from Eq. (2.4.43) of Sakurai, and explain what  $x_0$  is in this equation.

(d) (5 points) Calculate this integral, by substituting  $z = 1/\cosh^2(ax)$ , and using that (for  $b > 0$ )

$$\int_b^1 \frac{1}{x} \sqrt{\frac{(x-b)}{(1-x)}} dx = \pi(1 - \sqrt{b}) .$$

(e) (4 points) Show that, according to the WKB solution to this problem, there is only one bound state. Compare the approximate energy of this bound state you found with the WKB method with the exact energy you found in part (a).

3. (15 points) Consider the one-dimensional harmonic oscillator, with angular frequency  $\omega$  and mass  $m$ . Introduce the dimensionless coordinate  $y = x(m\omega/\hbar)^{1/2}$ .

(a) (4 points) Express  $y$  in terms of the raising and lowering operators  $a^\dagger$  and  $a$ , and use this to prove that

$$\langle m|y|n\rangle = \sqrt{\frac{n}{2}}\delta_{m,n-1} + \sqrt{\frac{m}{2}}\delta_{m,n+1} ,$$

where  $|n\rangle$ ,  $n = 0, 1, 2, \dots$ , denote the energy eigenkets.

(b) (6 points)

Consider now the matrix elements

$$\langle m|y^3|0\rangle , \quad m = 0, 1, 2, \dots ,$$

Find all values of  $m$  for which these matrix elements do not vanish, and find the value of the nonvanishing matrix elements.

(c) (5 points) Now we perturb the oscillator by an additional potential  $\delta V = \lambda\hbar\omega y^3$ . Find the correction to the ground state energy to lowest nonvanishing order. (If you did not complete part (b), work on this part as much as you can.)