Midterm Exam, PHYS 701, Fall 2009, Tuesday October 27

This midterm is due in class on Tuesday November 3.

There are three problems on this exam (three pages). The number of credit points is indicated for each problem. You are allowed to use your own notes, homework solutions, and the textbook, nothing else. Show the details of your work. Good luck!

Problem 1. (*16 points*) Two weights with masses m_1 and m_2 are connected by a rope of length ℓ which hangs over a fixed pulley. The weights can only move vertically, and we will ignore any frictional forces. We take as coordinates x_1 and x_2 of the two weights their distances below the pulley, so that at any time $x_1 + x_2 = \ell$ (we ignore the size of the pulley). We will study this set up using the lagrangian theory of constraints.

a) (*4 points*) Draw a picture of the set up. Introducing a Lagrange multiplier τ, show that the lagrangian L for this system can be written as

$$L = \frac{1}{2} m_1 \dot{x_1}^2 + \frac{1}{2} m_2 \dot{x_2}^2 + m_1 g x_1 + m_2 g x_2 - \tau(x_1 + x_2) \ . \qquad (1)$$

Explain each term carefully (g is the acceleration due to gravity).

b) (*4 points*) Find the lagrangian equations of motion for x_1 and x_2. What is the physical meaning of the Lagrange multiplier τ?

c) (*4 points*) Solve for τ, *i.e.*, find an expression for τ in terms of m_1, m_2 and g.

d) (*4 points*) Solve for x_1 and x_2 as a function of time, assuming that they were both initially at rest, and suspended at an equal height.

Problem 2. (*18 points*) In this problem, we will consider the shortest path between two points on the surface of a sphere. We will take the sphere to be centered at the origin, and to have radius R.

a) (*3 points*) In three-dimensional space, the distance between two points with coordinates (x, y, z) and $(x+dx, y+dy, z+dz)$ is $ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$. Express ds in polar coordinates, and argue that the distance between two points on the surface of the sphere with coordinates (R, θ, ϕ) and $(R, \theta + d\theta, \phi + d\phi)$ is given by

$$ds = R\sqrt{(d\theta)^2 + \sin^2\theta(d\phi)^2}\ . \qquad (2)$$

Here dx, dy, dz, $d\theta$ and $d\phi$ are infinitesimal.

b) (*4 points*) Taking ϕ as the parameter parametrizing a curve between two points 1 and 2 on the sphere, respectively with angular coordinates θ_1, ϕ_1 and θ_2, ϕ_2, argue that the length of this curve is given by

$$s = R\int_{\phi_1}^{\phi_2} d\phi \sqrt{\left(\frac{d\theta}{d\phi}\right)^2 + \sin^2\theta}\ . \qquad (3)$$

Show that the differential equation for the function $\theta(\phi)$ that minimizes the distance between the two points 1 and 2 can be written as

$$\frac{d}{d\phi}\left(\frac{\dot{\theta}}{\sqrt{\dot{\theta}^2 + \sin^2\theta}}\right) - \frac{\sin\theta\cos\theta}{\sqrt{\dot{\theta}^2 + \sin^2\theta}} = 0\ , \qquad (4)$$

in which $\dot{\theta} \equiv d\theta/d\phi$.

c) (*6 points*) Show that the function $\theta(\phi)$ defined by

$$C\sin\theta\cos(\phi - \alpha) = \cos\theta \qquad (5)$$

solves the differential equation you found in part (b). C and α are two arbitrary constants.

d) (*5 points*) Translate equation (5) back into cartesian coordinates, and interpret the result. Explain your reasoning carefully. [Hint: Show that equation (5), when expressed in terms of cartesian coordinates, defines a plane through the origin.]

Problem 3. (*16 points*) Consider a cylinder of mass M rolling down an incline without slipping. Let the angle of the incline with the horizontal plane be α, and take the x-axis along the incline. The moment of inertia for a cylinder rotating around its axis is $\frac{1}{2}MR^2$, where R is the radius of the cylinder. You may take the gravitational acceleration g to be constant. We will treat this problem using the lagrangian method.

a) (*5 points*) What is the lagrangian for this system, and what, if any, are the constraints?

b) (*6 points*) Derive the Euler–Lagrange equations, and find the linear acceleration down the incline, as well as the angular acceleration of the cylinder, while it is rolling down.

c) (*5 points*) Calculate the frictional force that prevents the cylinder from slipping while it is rolling down the incline.