Final Exam, PHYS 701, Fall 2007, Tuesday December 11

This final is due in my campus mailbox no later than 6 pm on Tuesday December 18. (You may also slip it under my office door.)

There are three problems on this exam (4 pages). The number of credit points is indicated for each problem. You are allowed to use your notes and the textbook, nothing else. Show the details of your work. Good luck!

Problem 1. (*20 points*) Consider the double oscillator shown in the figure (see next page), consisting of two masses m_1 and m_2, fastened to fixed supports by springs with spring constants k_1 and k_2, and connected by a third spring with spring constant k_3. All springs are assumed to be relaxed when $x_1 = 0$ and $x_2 = 0$.

a) (*3 points*) Argue that the lagrangian for this problem is given by

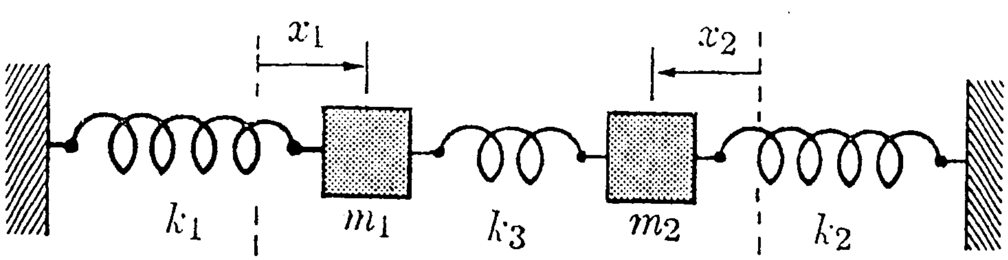
$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}k_1x_1^2 - \frac{1}{2}k_2x_2^2 - \frac{1}{2}k_3(x_1 + x_2)^2 \ . \tag{1}$$

b) (*5 points*) Treat this problem as a "small-oscillation" problem. Find the normal-mode frequencies, in terms of the constants $m_{1,2}$ and $k_{1,2,3}$. Show that both normal-mode oscillations are stable.

c) (*4 points*) For simplicity, take $m_1 = m_2 = m$, and $k_1 = k_2 = k$. For this case, show that the normal-mode frequencies are $\omega_1 = \sqrt{(k + 2k_3)/m}$ and $\omega_2 = \sqrt{k/m}$. Find the relation between the coordinates $x_{1,2}$ and the normal coordinates $\rho_{1,2}$. Describe the motion of the system when initial conditions are chosen such that only $\rho_2(t) = 0$, as well as for the case that they have been chosen such that $\rho_1(t) = 0$.

d) (*5 points*) Find $x_1(t)$ and $x_2(t)$ when initially $x_1 = a$, $x_2 = 0$ and $\dot{x}_1 = 0$, $\dot{x}_2 = 0$.

e) (*3 points*) Show that the solution you found is an exact solution, and argue why this is so.


x_1
x_2
k_1
m_1
k_3
m_2
k_2

Problem 2. (*20 points*) In this problem we consider the stability of force-free rotation of a rigid body around its principal axes. (This has applications, for instance, to the stability of a spinning satellite.) Stability means that if a small perturbation is applied to the rotating system, this perturbation will not grow over time.

a) (*4 points*) Consider a rotating rigid body, and choose a body system of which the axes coincide with the principal axes. Assume that the three moments of inertia I'_1, I'_2 and I'_3 relative to these principal axes are all unequal to each other. Suppose that the body is rotating with angular velocity $\vec{\omega} = \omega'_1 \hat{e}'_1$, with $\hat{e}'_i$ the unit vector along the ith principal axis. Now suppose that a perturbation produces a small rotation around the other two axes as well, so that $\vec{\omega} = \omega'_1\hat{e}'_1 + \omega'_2\hat{e}'_2 + \omega'_3\hat{e}'_3$, with $\omega'_2 \ll \omega'_1$ and $\omega'_3 \ll \omega'_1$. Write down the force-free Euler equations, ignoring any terms that contain $\omega'_2\omega'_3$, and explain why such terms can be ignored relative to the other terms in these equations. Because we will be working only with this body system, we will drop the primes on all these quantities in the rest of the problem.

b) (*5 points*) Show that one of these equations implies (with the approximation we made of dropping terms of order $\omega_2\omega_3$) that ω_1 is constant. Use the other two equations to show that ω_2 satisfies the approximate equation of motion

$$\ddot{\omega}_2 + K\omega_2 = 0 \;, \tag{2}$$

$$K = (I_1 - I_3)(I_1 - I_2)\omega_1^2/(I_2 I_3) \;. \tag{3}$$

Show that ω_3 satisfies the same equation of motion.

c) (*5 points*) Argue that $K > 0$ implies stable motion (*i.e.*, if ω_2 and ω_3 are initially small, they will stay small), whereas $K < 0$ implies that this is not the case. Explain both cases.

d) (*3 points*) Explain that this implies that the rotation is stable when I_1 is the largest of all moments of inertia, as well as when it is the smallest of all moments of inertia, but not if it is "in between," with $I_2 < I_1 < I_3$, or $I_3 < I_1 < I_2$.

e) (*3 points*) Discuss the cases $I_2 = I_3$ and $I_1 = I_2$.

Problem 3. (*10 points*) Consider a system of N point masses with a potential function

$$U(\vec{r}_1, \ldots, \vec{r}_N) = \sum_{n=1}^{N} \sum_{n'=1}^{n-1} f(\vec{r}_n - \vec{r}_{n'}) \ . \tag{4}$$

a) (*2 points*) What is the lagrangian for this system? (Label the masses of the particles by m_n, $n = 1, \ldots, N$.)

b) (*4 points*) Show that the transformation

$$\vec{r}\,'_n = \vec{r}_n + \vec{a} \ , \qquad n = 1, \ldots, N \tag{5}$$

with $\vec{a}$ a constant vector is a symmetry of the system, *i.e.*, that the lagrangian written in terms of the primed coordinates and velocities has exactly the same form as written in terms of unprimed coordinates and velocities.

c) (*4 points*) Use Noether's theorem to prove that the total momentum $\vec{P}$ is a constant of the motion.