Periodic Motion Characteristics

**Period** $T$: time required for one cycle of periodic motion (unit: s)

**Frequency** $f$: number of cycles per unit time

$$f = \frac{1}{T}$$

SI unit: cycle/second = 1/s = s$^{-1}$

The frequency unit is called a hertz (Hz): 1 Hz = 1 cycle/second

**Amplitude** $A$: maximum distance object moves from equilibrium (unit: m)

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**Frequency and Period**

$$f = \frac{1}{T} \quad \text{and} \quad T = \frac{1}{f}$$

$T$ is the period (units: s)

$f$ is the frequency

(units: Hz $\equiv$ oscillations per second)

<table>
<thead>
<tr>
<th>TABLE</th>
<th>Units of frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>Period</td>
</tr>
<tr>
<td>$10^3$ Hz = 1 kilohertz = 1 kHz</td>
<td>1 ms</td>
</tr>
<tr>
<td>$10^6$ Hz = 1 megahertz = 1 MHz</td>
<td>1 $\mu$s</td>
</tr>
<tr>
<td>$10^9$ Hz = 1 gigahertz = 1 GHz</td>
<td>1 ns</td>
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</tbody>
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**Example: Frequency and Period**

What is the frequency of a pendulum swing that takes 2 seconds to complete a cycle?

$$f = \frac{1}{T} = \frac{1}{2\text{s}} = 0.5 \text{ s}^{-1} = 0.5 \text{ Hz}$$

Note that 1 Hz = 1/s
**Simple Harmonic Motion (SHM)**

SHM - special kind of periodic motion that results when a mass is acted on by force proportional to mass's displacement from equilibrium.

A spring exerts a restoring force that is proportional to the displacement from equilibrium: \[ F_{\text{sp}} = -kx \]

The minus sign indicates that \( F_{\text{spring}} \) is a restoring force – it tries to restore mass to equilibrium position.

Look at mass attached to spring of spring constant \( k \):

We assume that surface is frictionless. There is a point where spring is neither stretched nor compressed; this is the equilibrium position. We measure displacement from that point \( (x = 0) \). The force exerted by the spring depends on the displacement:

\[ F = -kx \]

**Simple Harmonic Motion**

A mass on a spring has a displacement as a function of time that is a sine or cosine curve:

Here, \( A \) is called the amplitude of the motion.
Position vs. time in SHM

If we call the period of the motion $T$ (this is the time to complete one full cycle) we can write the position as a function of time as:

$$x = A \cos\left(\frac{2\pi}{T} t\right)$$

For this position function, the position at time $t + T$ is the same as the position at time $t$ (one period earlier), as we would expect.

Uniform Circular Motion vs. SHM

Consider particle rotating ccw, with the angle $\phi$ increasing linearly with time:

$$x = A \cos \phi$$

$$\omega = \frac{\Delta \phi}{\Delta t} \quad \text{so} \quad \phi = \omega t \quad \text{if} \quad \phi = 0 \quad \text{at} \quad t = 0.$$  

$$x(t) = A \cos \omega t$$

Connections between Uniform Circular Motion and Simple Harmonic Motion

Here, the object in circular motion has an angular speed of

$$\omega = \frac{2\pi}{T}$$

where $T$ is the period of motion of the object in simple harmonic motion.
Summary: Position Function for SHM
The position as a function of time for object in SHM started at $x=A$ at time $t=0$:

$$x = A \cos (\omega t) = A \cos (2\pi ft) = A \cos (2\pi t/T)$$

Example: Finding the Time
A mass, oscillating in simple harmonic motion, starts at $x=A$ and has period $T$.
At what time, as a fraction of $T$, does mass first pass through $x = \frac{1}{2}A$?

$$x = \frac{1}{2} A = A \cos \left( \frac{2\pi t}{T} \right)$$
$$t = \frac{T}{2\pi} \cos^{-1}\left( \frac{1}{2} \right) = \frac{T \pi}{3} = \frac{1}{6} T$$

Example: A System in SHM
An air-track glider is attached to spring, pulled 20 cm to the right, and released at $t=0$. It makes 15 complete oscillations in 10 s.

a. What is the frequency of oscillation?
b. What is object's maximum speed?
c. What is its velocity at $t=0.50$ s?

$$f = 15 \text{ oscillations} / 10s = 1.5 \text{ Hz}$$
$$v_{\text{max}} = \omega A = 2\pi f A = 2(3.14)(1.5/s)(0.2m) = 1.89 \text{ m/s}$$
$$v = -\omega A \sin(\omega (0.8s))$$
$$= -(1.89 \text{ m/s}) \sin[2\pi (1.5/s)(0.8s)] = -0.59 \text{ m/s}$$
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**Acceleration in SHM**

The acceleration as a function of time for an object in SHM started at $x=A$ at time $t=0$:

$$a = -A\omega^2 \cos(\omega t)$$

**Example: Acceleration in SHM**

- A can in a paint can shaker is in SHM with an amplitude of 0.5m and $\omega=3$ rad/s. What is the maximum acceleration experienced by the can?
- $a_{\text{max}} = \omega^2 A = (3 \text{ rad/s})^2(0.5\text{m}) = 4.5 \text{ m/s}^2$

**The Period of a Mass on a Spring**

Since the force on a mass $m$ on a spring is proportional to the displacement $x$ from equilibrium, Newton’s 2nd law gives:

$$ma = -kx$$

Substituting the time dependencies of $a$ and $x$ gives:

$$m[-A\omega^2 \cos(\omega t)] = -k[A \cos(\omega t)]$$

$$\omega^2 = \frac{k}{m}$$

$$\omega = \sqrt{\frac{k}{m}}$$
The Period of a Mass on a Spring
Therefore, the period is:

Two Oscillating Systems
Two identical masses attached to two identical springs rest on a horizontal frictionless surface. Spring 1 is stretched to 5 cm, spring 2 is stretched to 10 cm, and the masses are released at the same time.

Which mass reaches the equilibrium position first?

Two Oscillating Systems
Because $k$ and $m$ are the same, the systems have the same period, so they must return to equilibrium at the same time.

The frequency and period of SHM are independent of amplitude.

Question
Shown are two mass + spring systems. The blocks have the same mass.

When set into oscillation, what is the relation between the oscillation periods $T_1$, $T_2$ of the two systems?

(a) $T_1>T_2$  (b) $T_1= T_2$  (c) $T_1< T_2$
(d) Need to know $m$ and $k$ to answer
Example: Block on Spring

2.00 kg block is attached to spring as shown. Force constant of the spring is \( k = 196 \text{ N/m} \). Block is held distance of 5.00 cm from equilibrium and released at \( t = 0 \).

(a) Find angular frequency \( \omega \), frequency \( f \), and period \( T \).

(b) Write an equation for \( x \) vs. time.

\[
\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{196 \text{ N/m}}{2 \text{.00 kg}}} = 9.90 \text{ rad/s}
\]

\[
T = \frac{1}{f} = 0.635 \text{ s} = \frac{\omega}{2\pi} = \frac{9.90 \text{ rad/s}}{2\pi} = 1.58 \text{ Hz}
\]

\[
x = (5.00 \text{ cm}) \cos \left[ (9.90 \text{ rad/s}) t \right]
\]

Energy Conservation in Oscillatory Motion

In an ideal system with no nonconservative forces, total mechanical energy is conserved. For a mass on a spring:

\[
E = K + U = \frac{1}{2} m v^2 + \frac{1}{2} k x^2
\]

Since we know the position and velocity as functions of time, we can find the maximum kinetic and potential energies:

\[
U_{\max} = \frac{1}{2} k A^2
\]

\[
K_{\max} = \frac{1}{2} m A^2 \omega^2 = \frac{1}{2} m A^2 (k/m) = \frac{1}{2} k A^2
\]

Energy Conservation in Oscillations

As a function of time,

\[
E = U + K = \frac{1}{2} k A^2 \cos^2(\omega t) + \frac{1}{2} k A^2 \sin^2(\omega t)
\]

\[
= \frac{1}{2} k A^2 [\cos^2(\omega t) + \sin^2(\omega t)] = \frac{1}{2} k A^2
\]

Total energy is constant; as kinetic energy increases, potential energy decreases, & vice versa.

\[
E = \frac{1}{2} m v_x^2 + \frac{1}{2} k x^2 = \frac{1}{2} k A^2 = \frac{1}{2} m v_{\text{max}}^2 = \text{constant}
\]

- anywhere
- at \( x = \pm A \)
- center

End of Lecture 24

- For Wednesday, read Walker, 13.5-6, 15.1-2.
- Homework Assignment 13a is due at 11:00 PM on Thursday, Nov. 5.
- Quiz Wednesday-Chaps. 9, 10, 11 (but not 11.8-9)