Rolling Motion

If a round object rolls without slipping, there is a fixed relationship between the translational and rotational speeds:

\[ v = \frac{2\pi r}{T} \quad \Rightarrow \quad v = r\omega = v_f \]

Rolling Motion

We can consider rolling motion to be a combination of pure rotational and pure translational motion:

Rolling Motion

We may also consider rolling motion at any given instant to be a pure rotation at rate \( \omega \) about the point of contact of the rolling object.
Example: A Rolling Tire

Car with tires of radius 32 cm drives at speed of 55 mph.

What is angular speed \( \omega \) of the tires?

\[
\omega = \frac{v}{r} = \frac{(55 \text{ mph})(0.447 \text{ m/s/mph})}{(0.320 \text{ m})} = 77 \text{ rad/s}
\]

Rotational Kinetic Energy & Moment of Inertia (I)

For this point mass \( m \),

\[
K = \frac{1}{2}mv^2 = \frac{1}{2}m(\omega r)^2 = \frac{1}{2}(mr^2)\omega^2
\]

Define the moment of inertia \( I \) of a point mass \( m \) located at distance \( r \) from rotation axis:

\[
I = mr^2
\]

Example: A Dumbbell

Use definition of moment of inertia to calculate that of a dumbbell-shaped object with two point masses \( m \) separated by distance of \( 2r \) and rotating about a perpendicular axis through their center of symmetry.

\[
I = \sum m_i r_i^2 = \sum m_i r_i^2 + m_2 r_2^2 = 2mr^2
\]
**Example: Grindstone**

Grindstone of radius \( r = 0.610 \text{ m} \) being used to sharpen an axe.

If linear speed of stone is 1.50 \( \text{m/s} \) and stone's rotational kinetic energy is 13.0 \( \text{J} \), what is its moment of inertia.

\[
\omega = \frac{v}{r} = \frac{1.50 \text{ m/s}}{0.610 \text{ m}} = 2.46 \text{ rad/s}
\]

\[
K = \frac{1}{2} I \omega^2 \quad \Rightarrow \quad I = \frac{2K}{\omega^2} = \frac{2(13.0 \text{ J})}{(2.46 \text{ rad/s})^2} = 4.30 \text{ kg m}^2
\]

**Moment of Inertia of a Hoop**

\[
I = \sum m_i r_i^2
\]

All the mass of hoop is at same distance \( R \) from center of rotation, so its moment of inertia is same as that of point mass rotated at the same distance.

**Moments of Inertia**

- Hoop or cylindrical shell \( I = MR^2 \)
- Disk or solid cylinder \( I = \frac{1}{2} MR^2 \)

- Hollow sphere \( I = \frac{2}{3} MR^2 \)
- Solid sphere \( I = \frac{2}{5} MR^2 \)

**Question**

The T-shaped object is rotated about each axis shown. For which axis does it have the largest moment of inertia?
The Parallel-Axis Theorem*

If you know the moment of inertia \( I_{cm} \) of an object about its center of mass, you can get \( I \) about any other parallel axis by adding \( Md^2 \), where \( M \) is object's mass and \( d \) is separation of the axes.

\[
I = I_{cm} + Md^2
\]

Kinetic Energy of Rolling Object

Total kinetic energy of a rolling object is the sum of its linear and rotational kinetic energies:

\[
K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2
\]

\[
= \frac{1}{2}mv^2 + \frac{1}{2}I\left(\frac{v}{r}\right)^2 = \frac{1}{2}mv^2\left(1 + \frac{I}{mr^2}\right)
\]

The second equation makes it clear that the kinetic energy of a rolling object is a multiple of the kinetic energy of translation.

Example: Rolling Disk

1.20 kg disk with radius of 10.0 cm rolls without slipping. The linear speed of disk is \( v = 1.41 \) m/s.

(a) Find the translational kinetic energy.
(b) Find the rotational kinetic energy.
(c) Find the total kinetic energy.

\[
K_T = \frac{1}{2}mv^2 = \frac{1}{2}(1.20 \text{ kg})(1.41 \text{ m/s})^2 = 1.19 \text{ J}
\]

\[
K_r = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{2}mr^2\right)(v/r)^2 = \frac{1}{4}(1.20 \text{ kg})(1.41 \text{ m/s})^2 = 0.595 \text{ J}
\]

\[
K_T = K_T + K_r = (1.19 \text{ J}) + (0.595 \text{ J}) = 1.79 \text{ J}
\]

Question

A solid sphere and a hollow sphere of the same mass and radius roll forward without slipping at the same speed.

How do their kinetic energies compare?

(a) \( K_{\text{solid}} > K_{\text{hollow}} \)
(b) \( K_{\text{solid}} = K_{\text{hollow}} \)
(c) \( K_{\text{solid}} < K_{\text{hollow}} \)
(d) Not enough information to tell
**Rolling Down an Incline**

Conservation of mechanical energy.

\[
K_f + U_i = K_f' + U_i'
\]

\[
U_i = mgh
\]

\[
K_f = \frac{1}{2} mv^2 (1 + I / mr^2)
\]

\[
mgh = \frac{1}{2} mv^2 (1 + I / mr^2)
\]

\[
v = \sqrt{\frac{2gh}{(1 + I / mr^2)}}
\]

---

**Question**

Which of these two objects, of the same mass and radius, if released simultaneously, will reach the bottom first? Or, is it a tie? Assume rolling without slipping.

(a) Hoop:  
(b) Disk:  
(c) Tie:  
(d) Need to know mass and radius.

---

**The Great Downhill Race**

A sphere, a cylinder, and a hoop, all of mass \(M\) and radius \(R\), are released from rest and roll down a ramp of height \(h\) and slope \(\theta\). They are joined by a particle of mass \(M\) that slides down the ramp without friction. Who wins the race? Who is the big loser?

Hollow Cylinder:  \(I = mr^2; \ v = \sqrt{gh}\)

Solid Cylinder:  \(I = \frac{1}{2} mr^2; \ v = \sqrt{\frac{4}{3}gh}\)

Hollow Sphere:  \(I = \frac{2}{3} mr^2; \ v = \sqrt{\frac{6}{5}gh}\)

Solid Sphere:  \(I = \frac{2}{5} mr^2; \ v = \sqrt{\frac{10}{7}gh}\)
Lecture 21

The Winners

- Particle
- Solid sphere
- Solid cylinder
- Circular hoop

Example: Spinning Wheel

Block of mass \( m \) attached to string wrapped around circumference of wheel of radius \( R \) and moment of inertia \( I \), initially rotating with angular velocity \( \omega \). The block rises with speed \( v = r\omega \). The wheel rotates freely about its axis and the string does not slip. To what height \( h \) does the block rise?

\[
E_i = E_f \\
E_f = mgh \\
E_i = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv^2 + \frac{1}{2}I(v/R)^2 = \frac{1}{2}mv^2(1 + I/mR^2) \\
h = \left(\frac{v^2}{2g}\right)\left(1 + \frac{I}{mR^2}\right)
\]

Example: Flywheel-Powered Car

Your vehicle’s braking mechanism transforms translational kinetic energy into rotational kinetic energy of massive flywheel. Flywheel has \( I = 11.1 \text{ kg m}^2 \) and maximum angular speed 30,000 rpm. At minimum highway speed of 40 mi/h, air drag and rolling friction dissipate energy at 10.0 kW. If you run out of gas 15 miles from home, with flywheel spinning at maximum speed, can you make it back?

\[
\omega = \left(30,000 \text{ rev/min}\right)\left(\frac{2\pi \text{ rad/rev}}{60 \text{ s/min}}\right) = 3,142 \text{ rad/s} \\
K = \frac{1}{2}I\omega^2 = \frac{1}{2}(11.1 \text{ kg m}^2)(3,142 \text{ rad/s})^2 = 54.9 \text{ MJ} \\
\Delta t = \Delta x / v = (15 \text{ mi}) / (40 \text{ mi/h}) = 0.375 \text{ h} = 1350 \text{ s} \\
\text{Energy needed} = W_{nc} = (10 \text{ kW})(1350 \text{ s}) = 13.5 \text{ MJ} \\
\text{You make it.}
\]

Torque (\( \tau \))

To make an object start rotating, a force is needed; the position and direction of the force matter as well. The perpendicular distance \( r_\perp \) from the axis of rotation to the line along which the force acts is called the lever arm.
Torque
From experience, we know that a force will be more effective at rotating an object such as a nut or door if our hand is not too close to the axis. This is why we have long-handled wrenches, and why doorknobs are not next to hinges.

Torque ($\tau$)
We define a quantity called torque which is a measure of "twisting effort". For force $F$ applied perpendicular distance $r_{\perp}$ from the rotation axis:

\[ \tau = r_{\perp}F \]

SI unit: N·m

The torque increases as the force increases and also as the perpendicular distance $r_{\perp}$ increases.

Other Torque Units
- Common US torque unit: foot-pound (ft-lb)
- Seen in wheel lug nut torques for vehicles:
  - Ford Aspire 1994-97 85 ft-lbs
  - Ford Contour 1999-00 95 ft-lbs
  - Ford Crown Victoria 2000-05 100 ft-lbs

End of Lecture 21
- For Wednesday Oct. 28, read Walker 11.1-3
- Homework assignment PS10b due on WebAssign by 11:00 p.m. on Tues. , Oct. 27.