Unit Vectors

A unit vector is a symbol for a certain defined direction, such as “North”, or “along the x axis”. A unit vector has magnitude equal to 1 and no units, but has a defined direction.

Example: the unit vector in the x direction is usually written: 

\[ \hat{x} \]

Now, I can write a vector as a sum of scalar components multiplied by the corresponding unit vector:

\[ \vec{A} = A_x \hat{x} + A_y \hat{y} \]

where \( A_x \) and \( A_y \) are scalar components (with units).
Unit Vectors

The unit vectors have length 1, no units, and point in the +x-direction and +y-direction.

Example of vector written in terms of scalar components and unit vectors:

$$\mathbf{r} = (2m)\hat{x} + (3m)\hat{y}$$

Motion in Two Dimensions

- Using + or - signs for direction is usually not sufficient to fully describe motion in more than one dimension
- Vectors are used to fully describe motion
- Interested in displacement, velocity, and acceleration vectors
Displacement

- The position of an object is described by its position vector, \( \mathbf{r} \).

- The displacement of the object is defined as the change in its position:
  \[ \Delta \mathbf{r} = \mathbf{r}_f - \mathbf{r}_i \]

The Displacement Vector

\[ \mathbf{r} = x \hat{x} + y \hat{y} \]

\[ \Delta \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 \]

\[ \Delta \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 \]

\[ = (x_2 \hat{x} + y_2 \hat{y}) - (x_1 \hat{x} + y_1 \hat{y}) \]

\[ = \Delta x \hat{x} + \Delta y \hat{y} \]
Velocity

- The average velocity is the ratio of the displacement to the time interval for the displacement

- The direction of $\vec{v}_{av}$ is the direction of $\Delta \vec{r}$
- The instantaneous velocity is the limit of the average velocity as $\Delta t$ approaches zero
- The direction of the instantaneous velocity is along a line tangent to the path of the particle and in the direction of motion

The Average Velocity Vector

Average velocity vector:

$$\vec{v}_{av} = \frac{\Delta \vec{r}}{\Delta t}$$

(3-3)

So $\vec{v}_{av}$ is in the same direction as $\Delta \vec{r}$. 
Example: A Dragonfly

A dragonfly is observed initially at position:
\[ \vec{r}_1 = (2.00 \text{ m})\hat{x} + (3.50 \text{ m})\hat{y} \]
Three seconds later, it is observed at position:
\[ \vec{r}_2 = (-3.00 \text{ m})\hat{x} + (5.50 \text{ m})\hat{y} \]
What was the dragonfly’s average velocity during this time?

\[ \vec{v}_{av} = \frac{\Delta \vec{r}}{\Delta t} = \frac{\vec{r}_2 - \vec{r}_1}{\Delta t} \]
\[ = \left[ (-3.00 \text{ m}) - (2.00 \text{ m}) \right] \hat{x} + \left[ (5.50 \text{ m}) - (3.50 \text{ m}) \right] \hat{y} \]
\[ = (3.00 \text{ s}) \hat{x} + (0.667 \text{ m/s}) \hat{y} \]

Example: Velocity of a Sailboat

Sailboat has coordinates (130 m, 205 m) at \( t_1 = 0.0 \text{ s} \). Two minutes later its position is (110 m, 218 m).
(a) Find \( \vec{v}_{avx} \); (b) Find \( \vec{v}_{avy} \)

\[ \vec{v}_{av} = \vec{v}_{avx} \hat{x} + \vec{v}_{avy} \hat{y} \]

\[ v_{avx} = \frac{\Delta x}{\Delta t} = \frac{110 \text{ m} - 130 \text{ m}}{120 \text{ s}} = -0.167 \text{ m/s} \]

\[ v_{avy} = \frac{\Delta y}{\Delta t} = \frac{218 \text{ m} - 205 \text{ m}}{120 \text{ s}} = 0.108 \text{ m/s} \]

\[ \vec{v}_{av} = (-0.167 \text{ m/s})\hat{x} + (0.108 \text{ m/s})\hat{y} \]

\[ v_{av} = \sqrt{(-0.167 \text{ m/s})^2 + (0.108 \text{ m/s})^2} = 0.199 \text{ m/s} \]

\[ \theta = \arctan \frac{0.108 \text{ m/s}}{-0.167 \text{ m/s}} = 147^\circ \]
Acceleration

► The average acceleration is defined as the rate at which the velocity changes

► The instantaneous acceleration is the limit of the average acceleration as $\Delta t$ approaches zero

Ways an Object Might Accelerate

► The magnitude of the velocity (the speed) can change

► The direction of the velocity can change
  ▪ Then there is acceleration, even though the magnitude is constant -- example: motion in a circle at constant speed

► Both the magnitude and the direction can change
Position, Displacement, Velocity, & Acceleration Vectors

Velocity vector \( \vec{v} \) always points in the direction of motion. The acceleration vector \( \vec{a} \) can point anywhere.

Acceleration on a Curve

Typically, for a vehicle moving on a curve, the magnitude of the velocity stays the same but the direction changes.

For example, consider a car that has an initial velocity of 12 m/s east, and 10 seconds later its velocity is 12 m/s south.
**Vector Motion with Constant Acceleration**

*Average velocity:*
\[ \vec{v}_{av} = \frac{1}{2} (\vec{v}_0 + \vec{v}) \]

*Velocity as a function of time:*
\[ \vec{v}(t) = \vec{v}_0 + \vec{a}t \]

*Position as a function of time:*
\[ \vec{r}(t) = \vec{r}_0 + \vec{v}_{av}t = \vec{r}_0 + \frac{1}{2} (\vec{v}_0 + \vec{v})t \]
\[ \vec{r}(t) = \vec{r}_0 + \vec{v}_0t + \frac{1}{2} \vec{a}t^2 \]

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**Relative Motion**

Suppose you swim downstream at 3 mi/hr and the current is moving at 2 mi/hr. What is your speed relative to the bank of the river?

Velocities must be added as vectors.

\[ \vec{v}_{sw} = 3 \text{ mi/hr} \quad \vec{v}_{wb} = 2 \text{ mi/hr} \quad \vec{v}_{sb} = 5 \text{ mi/hr} \]

velocity of swimmer relative to water + velocity of water relative to bank = velocity of swimmer relative to bank
Relative Motion

In general

\[ \vec{v}_{13} = \vec{v}_{12} + \vec{v}_{23} \]

velocity of 1 relative to 3
velocity of 1 relative to 2
velocity of 2 relative to 3

Note that \[ \vec{v}_{12} = -\vec{v}_{21} \]

Example

A man is walking toward the back of a train at 2 m/s relative to the train, which is heading North. His velocity relative to the ground is 1.1 m/s North. What is the velocity of the train relative to the ground?
Relative Motion

The speed of the passenger with respect to the ground depends on the relative directions of the passenger’s and train’s speeds:

\[ \vec{v}_{pg} = \vec{v}_{pt} + \vec{v}_{tg} = 16.2 \text{ m/s} \]

This also works in two dimensions:

\[ \vec{v}_{pg} = \vec{v}_{pt} + \vec{v}_{tg} = 13.8 \text{ m/s} \]

Relative Motion

\[ \vec{v}_{pg} = \vec{v}_{pt} + \vec{v}_{tg} \]

This also works in two dimensions:
**Example: Flying a Plane**

A pilot wishes to fly a plane due north relative to the ground. The airspeed of the plane is 200 km/h, and the wind is blowing from west to east at 90 km/h.

(a) In which direction should the plane head?
(b) What will be the ground speed of the plane?

\[ \vec{v}_{pG} = \vec{v}_{pA} + \vec{v}_{AG} \]

\[ \theta = \arcsin \frac{v_{AG}}{v_{pA}} = \arcsin \left( \frac{90 \text{ km/h}}{200 \text{ km/h}} \right) = 26.7^\circ \text{ west of north} \]

\[ v_{pG} = \sqrt{v_{pA}^2 - v_{AG}^2} = \sqrt{(200 \text{ km/h})^2 - (90 \text{ km/h})^2} = 179 \text{ km/h} \]

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**Example: Crossing a River**

You are in a boat with a speed relative to the water of \( v_{bw} = 6.1 \text{ m/s} \). Boat points at angle \( \theta = 25^\circ \) upstream on a river flowing at \( v_{wg} = 1.4 \text{ m/s} \).

(a) What is your speed \( v_{bg} \) and angle \( \theta_{bg} \) relative to the ground?

\[ \vec{v}_{bg} = \vec{v}_{bw} + \vec{v}_{wg} \]

\[ \vec{v}_{wg} = (-1.4 \text{ m/s}) \hat{y} \]

\[ \vec{v}_{bw} = (6.1 \text{ m/s}) \cos 25^\circ \hat{x} + (6.1 \text{ m/s}) \sin 25^\circ \hat{y} \]

\[ = (5.5 \text{ m/s}) \hat{x} + (2.6 \text{ m/s}) \hat{y} \]

\[ v_{bg} = \sqrt{(5.5 \text{ m/s})^2 + (1.2 \text{ m/s})^2} = 5.6 \text{ m/s} \]

\[ \vec{v}_{bg} = (5.5 \text{ m/s}) \hat{x} + (2.6 \text{ m/s} - 1.4 \text{ m/s}) \hat{y} \]

\[ = (5.5 \text{ m/s}) \hat{x} + (1.2 \text{ m/s}) \hat{y} \]

\[ \theta_{bg} = \tan^{-1} \left[ \frac{1.2 \text{ m/s}}{5.5 \text{ m/s}} \right] = 12^\circ \]
Key Points of Lecture 6

Unit Vectors $\hat{x}, \hat{y}$

Vector equations for Displacement, Velocity, and Acceleration

Relative Motion $\vec{v}_{13} - \vec{v}_{12} + \vec{v}_{23}$

- Before the next lecture, read *Walker* 4.1-3
- Homework Assignments #3b on WebAssign is due by 11:00 p.m. on Wed. Feb. 11.