Physics 460 Notes Fall 2006 Susan M. Lea

1 Dipole radiation

Electromagnetic waves are produced by accelerating charges, as we have seen. But when lots of charges are involved it is sometimes easier to work with the charge and current distributions. There is no radiation unless these distributions change in time. We'll start with a few simple cases. The first is an ideal dipole that oscillates in time:

$$\vec{p}(t) = \vec{p}_0 \cos \omega t$$

We can create such a source by having a point charge undergo simple harmonic motion, for example. Following Griffiths, we look at the simpler case of two equal and opposite charges fixed in position at $z = \pm d/2$, but with a time-dependent charge

$$q(t) = q_0 \cos \omega t$$

Then the dipole moment is

$$\vec{p} = q_0 d\hat{z} \cos \omega t = \vec{p}_0 \cos \omega t$$

Now we calculate the potentials due to this source using equations (9) and (10) from Notes 4. We will make several simplifying assumptions.

1. $d \ll r$. The source dimension is much less than the distance to the source.
2. $\omega d/c \ll 1$ The dipole oscillates slowly, or equivalently, its length is much less than a wavelength.
3. $\omega r/c \gg 1$. The distance from the source to the observer is much greater than a wavelength.

Then the potential is

$$\begin{aligned} V(\vec{r}, t) &= \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}', t - R/c)}{R} d\tau' \\ &= \frac{q_0}{4\pi\varepsilon_0} \left(\frac{\cos \omega (t - R_+/c)}{R_+} - \frac{\cos \omega (t - R_-/c)}{R_-} \right) \end{aligned}$$

where, using approximation 1, $d \ll r$, we have

$$R_\pm = \sqrt{r^2 + d^2/4 \mp rd \cos \theta} \simeq r \left(1 \mp \frac{d}{2r} \cos \theta \right)$$

$$V(\vec{r}, t) = \frac{q_0}{4\pi\varepsilon_0 r} \left(\frac{\cos \left[\omega t - \omega \frac{r}{c} \left(1 - \frac{d}{2r} \cos \theta \right) \right]}{r \left(1 - \frac{d}{2r} \cos \theta \right)} - \frac{\cos \left[\omega t - \omega \frac{r}{c} \left(1 + \frac{d}{2r} \cos \theta \right) \right]}{r \left(1 + \frac{d}{2r} \cos \theta \right)} \right)$$

Then we apply the second approximation $\omega d/c \ll 1$ and expand the cosine to first order in $\omega d/c$

$$\begin{aligned}
\cos\omega\left(t - R_+/c\right) &= \cos\omega\left(t - \frac{r}{c}\right)\cos\left(\frac{\omega d}{2c}\cos\theta\right) - \sin\omega\left(t - \frac{r}{c}\right)\sin\left(\frac{\omega d}{2c}\cos\theta\right) \\
&\simeq \cos\omega\left(t - \frac{r}{c}\right) - \frac{\omega d}{2c}\cos\theta\sin\omega\left(t - \frac{r}{c}\right)
\end{aligned}$$

Then

$$\begin{aligned}
V &= \frac{q_0}{4\pi\varepsilon_0 r}\left(\frac{\cos\omega\left(t - \frac{r}{c}\right) - \frac{\omega d}{2c}\cos\theta\sin\omega\left(t - \frac{r}{c}\right)}{1 - \frac{d}{2r}\cos\theta} - \frac{\cos\omega\left(t - \frac{r}{c}\right) + \frac{\omega d}{2c}\cos\theta\sin\omega\left(t - \frac{r}{c}\right)}{1 + \frac{d}{2r}\cos\theta}\right) \\
&= \frac{q_0}{4\pi\varepsilon_0 r}\left(\frac{\frac{d}{r}\cos\theta\cos\omega\left(t - \frac{r}{c}\right) - \frac{\omega d}{c}\cos\theta\sin\omega\left(t - \frac{r}{c}\right)}{1 - \left(\frac{d}{2r}\cos\theta\right)^2}\right) \\
&= \frac{p_0}{4\pi\varepsilon_0 r}\left(\frac{1}{r}\cos\theta\cos\omega\left(t - \frac{r}{c}\right) - \frac{\omega}{c}\cos\theta\sin\omega\left(t - \frac{r}{c}\right)\right)
\end{aligned}$$

In the limit $\omega \to 0$ (static dipole) the potential is

$$V \to \frac{p_0\cos\theta}{4\pi\varepsilon_0 r^2}$$

as expected. However, using approximation 3, we find that the second term dominates, and

$$V \simeq -\frac{p_0}{4\pi\varepsilon_0 r}\frac{\omega}{c}\cos\theta\sin\omega\left(t - \frac{r}{c}\right) \tag{1}$$

The vector potential is due to the current, which exists everywhere along the line between the two charges.

$$I(t) = \frac{dq}{dt} = -\omega q_0\sin\omega t$$

We'll need the distance

$$R(z') = \sqrt{r^2 + (z')^2 - 2rz'\cos\theta} \simeq r\left(1 - \frac{z'}{r}\cos\theta\right)$$

Then

$$
\begin{aligned}
\vec{A} &= \frac{\mu_0}{4\pi}\hat{z}\int_{-d/2}^{+d/2}\frac{I\left(t_{\mathrm{ret}}\right)}{R}dz' \\
&= -\frac{\mu_0 q_0}{4\pi}\hat{z}\int_{-d/2}^{+d/2}\frac{\omega\sin\omega\left(t-R/c\right)}{R}dz' \\
&= -\frac{\mu_0 q_0}{4\pi}\frac{\omega}{r}\hat{z}\int_{-d/2}^{+d/2}\frac{\sin\omega\left(t-\frac{r}{c}+\frac{z'}{c}\cos\theta\right)}{1-\frac{z'}{r}\cos\theta}\,dz' \\
&= -\frac{\mu_0 q_0}{4\pi}\frac{\omega}{r}\hat{z}\int_{-d/2}^{+d/2}\left[\sin\omega\left(t-\frac{r}{c}\right)+\omega\frac{z'}{c}\cos\theta\cos\omega\left(t-\frac{r}{c}\right)\right]\left(1+\frac{z'}{r}\cos\theta\right)dz' \\
&= -\frac{\mu_0 q_0}{4\pi}\frac{\omega}{r}\hat{z}\int_{-d/2}^{+d/2}\left\{\sin\omega\left(t-\frac{r}{c}\right)+\frac{z'}{r}\cos\theta\left[\frac{\omega r}{c}\cos\omega\left(t-\frac{r}{c}\right)+1\right]\right\}dz' \\
&\simeq -\frac{\mu_0 q_0}{4\pi}\frac{\omega d}{r}\hat{z}\sin\omega\left(t-\frac{r}{c}\right) = -\frac{\mu_0 p_0\omega}{4\pi r}\hat{z}\sin\omega\left(t-\frac{r}{c}\right) \qquad (2)
\end{aligned}
$$

The second term integrates to zero (odd integrand over even interval) so the next non-zero term would be of order $(\omega d/c)(d/r)$ and we neglect it.

Now that we have the potentials (1) and (2), we calculate the fields.

$$
\begin{aligned}
\vec{E} &= -\vec{\nabla}V-\frac{\partial\vec{A}}{\partial t} \\
&= \vec{\nabla}\frac{p_0}{4\pi\varepsilon_0 r}\frac{\omega}{c}\cos\theta\sin\omega\left(t-\frac{r}{c}\right)+\frac{\partial}{\partial t}\frac{\mu_0 p_0\omega}{4\pi r}\hat{z}\sin\omega\left(t-\frac{r}{c}\right)
\end{aligned}
$$

$$
\begin{aligned}
\vec{\nabla}V &= \frac{\partial V}{\partial r}\hat{r}+\frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\theta} \\
&= \frac{p_0}{4\pi\varepsilon_0}\frac{\omega}{c}\left\{\left[\frac{\cos\theta}{r^2}\sin\omega\left(t-\frac{r}{c}\right)+\frac{\omega}{c}\frac{\cos\theta}{r}\cos\omega\left(t-\frac{r}{c}\right)\right]\hat{r}+\hat{\theta}\frac{\sin\theta}{r^2}\sin\omega\left(t-\frac{r}{c}\right)\right\} \\
&\simeq \frac{p_0}{4\pi\varepsilon_0}\frac{\omega^2}{c^2}\frac{\cos\theta}{r}\cos\omega\left(t-\frac{r}{c}\right)\ \hat{r}
\end{aligned}
$$

where the terms we dropped are of order $c/\omega r$ with respect to the one we kept.

$$
\begin{aligned}
\frac{\partial\vec{A}}{\partial t} &= -\frac{\mu_0 p_0\omega^2}{4\pi r}\hat{z}\cos\omega\left(t-\frac{r}{c}\right) \\
&= -\frac{p_0\omega^2}{4\pi\varepsilon_0 r c^2}\hat{z}\cos\omega\left(t-\frac{r}{c}\right)
\end{aligned}
$$

Thus the total field is

$$
\vec{E} = -\frac{p_0}{4\pi\varepsilon_0 r}\frac{\omega^2}{c^2}\cos\omega\left(t-\frac{r}{c}\right)\left(\,\hat{r}\cos\theta-\hat{z}\right)
$$

But

$$
\hat{z}=\hat{r}\cos\theta-\hat{\theta}\sin\theta
$$

so

$$\vec{E} = -\hat{\theta}\frac{p_0}{4\pi\varepsilon_0 r}\frac{\omega^2}{c^2}\sin\theta\cos\omega\left(t-\frac{r}{c}\right) \tag{3}$$

The magnetic field is

$$\vec{B} = \vec{\nabla}\times\vec{A} = -\vec{\nabla}\times\frac{\mu_0 p_0\omega}{4\pi r}\hat{z}\sin\omega\left(t-\frac{r}{c}\right)$$

$\vec{A}$ has only r and θ components and each depends only on r and θ, so $\vec{B}$ has only a ϕ component:

$$\begin{aligned}
\vec{B} &= \frac{\hat{\phi}}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right] \\
&= \frac{\mu_0 p_0\omega}{4\pi r}\hat{\phi}\left[\frac{\partial}{\partial r}\left(\sin\theta\sin\omega\left(t-\frac{r}{c}\right)\right) + \frac{1}{r}\frac{\partial\cos\theta\sin\omega\left(t-\frac{r}{c}\right)}{\partial\theta}\right] \\
&= -\frac{\mu_0 p_0\omega}{4\pi r}\hat{\phi}\left[\frac{\omega}{c}\sin\theta\cos\omega\left(t-\frac{r}{c}\right) + \frac{1}{r}\sin\theta\sin\omega\left(t-\frac{r}{c}\right)\right]
\end{aligned}$$

Again the first term is much larger than the second, so we have

$$\vec{B} = -\frac{\mu_0 p_0\omega^2}{4\pi rc}\hat{\phi}\sin\theta\cos\omega\left(t-\frac{r}{c}\right) \tag{4}$$

Comparing the two expressions (3) and (4) we see that

$$\vec{B} = \frac{1}{c}\hat{r}\times\vec{E}$$

as we expect for an EM wave. The Poynting flux is

$$\begin{aligned}
\vec{S} &= \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0}\vec{E}\times\left(\frac{1}{c}\hat{r}\times\vec{E}\right) \\
&= \frac{1}{\mu_0 c}\hat{r}E^2 = \frac{1}{\mu_0 c}\hat{r}\left(\frac{p_0}{4\pi\varepsilon_0 r}\right)^2\frac{\omega^4}{c^4}\sin^2\theta\cos^2\omega\left(t-\frac{r}{c}\right)
\end{aligned}$$

The $1/r^2$ dependence on distance is also what we expect. Now we time average to get

$$<\vec{S}> = \hat{r}\frac{p_0^2}{32\pi^2\varepsilon_0 r^2}\frac{\omega^4}{c^3}\sin^2\theta$$

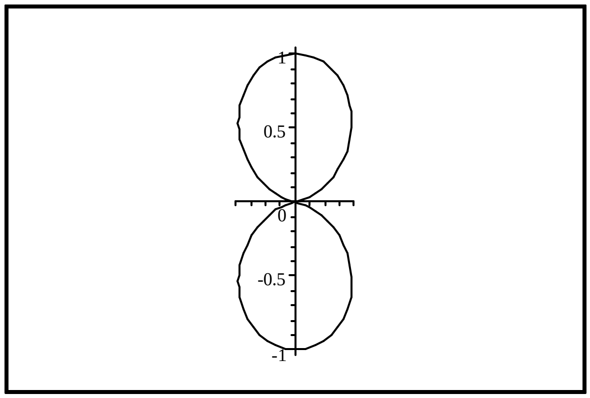
Then

$$\frac{dP}{d} = r^2 < \left|\vec{S}\right| > = \frac{p_0^2}{32\pi^2\varepsilon_0}\frac{\omega^4}{c^3}\sin^2\theta \tag{5}$$

and, with $\mu = \cos\theta$,

$$\begin{aligned}
P &= \int\frac{dP}{d}d = \frac{p_0^2}{16\pi\varepsilon_0}\frac{\omega^4}{c^3}\int_{-1}^{+1}\left(1-\mu^2\right)d\mu \\
&= \frac{p_0^2}{16\pi\varepsilon_0}\frac{\omega^4}{c^3}\frac{4}{3} = \frac{p_0^2}{12\pi\varepsilon_0}\frac{\omega^4}{c^3}
\end{aligned} \tag{6}$$

The total power radiated goes like the frequency to the 4th power and the square of the dipole moment. The angular distribution of power (equation 5) goes as $\sin^2\theta$. With the polar axis plotted to the right, it looks like this:



This is a special case of our point charge radiation formula, because we can model the dipole with an oscillating charge, and so it is no surprise that we get the same angular dependence.

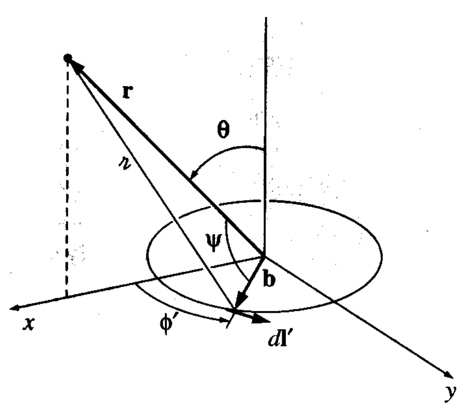
2 Magnetic dipole radiation

An oscillating current in a circular loop of radius b is a good model for an oscillating magnetic dipole. With $I(t) = I_0 \cos\omega t$,

$$\vec{m}(t) = \pi b^2 I(t)\,\hat{n} = \pi b^2 I_0 \cos\omega t\ \hat{n} = \vec{m}_0 \cos\omega t$$

The loop is uncharged, and so we have only a vector potential:

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{I_0\cos\left[\omega\left(t - R/c\right)\right]}{R}\hat{\phi}\,b\,d\phi'$$



We have to take care with the unit vector, since it is not a constant. So we convert to Cartesian unit vectors:

$$\hat{\phi} = -\hat{x}\sin\phi' + \hat{y}\cos\phi'$$

We also need an expression for R :

$$R^2 = r^2 + b^2 - 2\vec{r}\cdot\vec{b}$$

We may place the $x-$axis so that the observation point is in the $x-z-$plane„ as shown in the diagram Then

$$\begin{aligned} \vec{r}\cdot\vec{b} &= r\left(\hat{z}\cos\theta + \hat{x}\sin\theta\right)\cdot b\left(\hat{x}\cos\phi' + \hat{y}\sin\phi'\right) \\ &= rb\sin\theta\cos\phi' \end{aligned}$$

Thus

$$R = r\sqrt{1 + \frac{b^2}{r^2} - 2\frac{b}{r}\sin\theta\cos\phi'} \simeq r\left(1 - \frac{b}{r}\sin\theta\cos\phi'\right)$$

where the analogue of our previous approximation 1 is $b \ll r$. Thus

$$\begin{aligned} \vec{A} &= \frac{\mu_0}{4\pi}\int_0^{2\pi}\frac{I_0\cos\left[\omega\left(t - r/c + \frac{b}{c}\sin\theta\cos\phi'\right)\right]}{r\left(1 - \frac{b}{r}\sin\theta\cos\phi'\right)}\left[-\hat{x}\sin\phi' + \hat{y}\cos\phi'\right]bd\phi' \\ &= \frac{\mu_0 bI_0}{4\pi r}\int_0^{2\pi}\left[\cos\omega\left(t - r/c\right) - \sin\omega\left(t - r/c\right)\frac{\omega b}{c}\sin\theta\cos\phi'\right] \\ &\qquad \times\left(1 + \frac{b}{r}\sin\theta\cos\phi'\right)\left[-\hat{x}\sin\phi' + \hat{y}\cos\phi'\right]\ d\phi' \\ &= \frac{\mu_0 b^2 I_0}{4\pi r}\int_0^{2\pi}\left[\cos\omega\left(t - r/c\right)\frac{\sin\theta}{r}\cos\phi' - \sin\omega\left(t - r/c\right)\frac{\omega}{c}\sin\theta\cos\phi'\right] \\ &\qquad \times\left[-\hat{x}\sin\phi' + \hat{y}\cos\phi'\right]\ d\phi' \end{aligned}$$

where the first term has integrated to zero. The $x-$component also integrates to zero, since we have $\int_0^{2\pi}\sin\phi'\cos\phi' d\phi' = \int_0^{2\pi}\sin 2\phi' d\phi/ = 0$. Thus $\vec{A}$ has only a $y-$component:

$$\begin{aligned} \vec{A} &= \hat{y}\frac{\mu_0\pi b^2 I_0}{4\pi r}\frac{\omega}{c}\sin\theta\left[\cos\omega\left(t - r/c\right)\frac{c}{\omega r} - \sin\omega\left(t - r/c\right)\right] \\ &\simeq -\hat{y}\frac{\mu_0 m_0}{4\pi r}\frac{\omega}{c}\sin\theta\sin\omega\left(t - r/c\right) \qquad (7) \end{aligned}$$

and finally we get the fields:

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t} = \hat{y}\frac{\mu_0 m_0}{4\pi r}\frac{\omega^2}{c}\sin\theta\cos\omega\left(t - r/c\right)$$

We chose a special location for the $x-$axis, but we can see from the diagram, that for this point

$$\hat{y} = \hat{\phi}$$

So, more generally,

$$\vec{E} = \hat{\phi}\frac{\mu_0 m_0}{4\pi r}\frac{\omega^2}{c}\sin\theta\cos\omega\left(t - r/c\right)$$

$$\begin{aligned}
\vec{B} &= \vec{\nabla}\times\vec{A} = -\vec{\nabla}\times\hat{\phi}\frac{\mu_0 m_0}{4\pi r}\frac{\omega}{c}\sin\theta\sin\omega\left(t - r/c\right) \\
&= -\frac{\mu_0 m_0}{4\pi}\frac{\omega}{c}\left[\frac{\hat{r}}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin^2\theta}{r}\sin\omega\left(t - \frac{r}{c}\right)\right) - \frac{\hat{\theta}}{r}\frac{\partial}{\partial r}\left(\sin\theta\sin\omega\left(t - r/c\right)\right)\right] \\
&\simeq -\frac{\mu_0 m_0}{4\pi r}\frac{\omega^2}{c^2}\hat{\theta}\sin\theta\cos\omega\left(t - r/c\right) = \frac{\hat{r}\times\vec{E}}{c}
\end{aligned}$$

This time we get

$$\frac{dP}{d} = \frac{\mu_0}{2}\left(\frac{m_0}{4\pi}\right)^2\frac{\omega^4}{c^2}\sin^2\theta$$

and

$$P = \frac{\mu_0 m_0^2}{12\pi}\frac{\omega^4}{c^2} = \frac{m_0^2}{12\pi\varepsilon_0}\frac{\omega^4}{c^4} \tag{8}$$

The expression (8) looks a lot like (6), but actually the magnetic dipole radiation is much weaker. We can see this by going back to the expressions for $p_0 = q_0 d$ and $m_0 = \pi b^2 I_0$, and remembering that the current in the electric dipole case had amplitude $I_0 = \omega q_0$. Thus, if $d \sim b$,

$$\frac{p_0}{m_0} = \frac{q_0 d}{\pi b^2 I_0} \sim \frac{1}{\omega b}$$

and so

$$\frac{P_{\text{electric dipole}}}{P_{\text{magnetic dipole}}} = \left(\frac{p_0 c}{m_0}\right)^2 = \left(\frac{c}{\omega b}\right)^2 \gg 1$$