Physics 460 Fall 2006 Susan M. Lea

1 Potentials for Dynamic fields

We have spent the last few weeks discussing the propagation of electromagnetic waves, so now it is time to think about how the waves are produced. We have already shown that we can express the fields in terms of the scalar and vector potentials:

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{1}$$

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \tag{2}$$

By inserting these expressions into Gauss' Law and Ampere's law, we can show how the potentials are related to the sources:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} = -\nabla^2 V - \vec{\nabla} \cdot \frac{\partial \vec{A}}{\partial t} \tag{3}$$

and

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A} \right) = \mu_0 \vec{j} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \left(-\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \right)$$

Expand the term on the left, to get:

$$\vec{\nabla} \left(\vec{\nabla} \cdot \vec{A} \right) - \nabla^2 \vec{A} = \mu_0 \vec{j} - \mu_0 \varepsilon_0 \left(\frac{\partial}{\partial t} \vec{\nabla} V + \frac{\partial^2 \vec{A}}{\partial t^2} \right)$$

We can rearrange a bit, to get

$$\frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} = \mu_0 \vec{j} - \vec{\nabla} \left[\frac{1}{c^2} \frac{\partial V}{\partial t} + \vec{\nabla} \cdot \vec{A} \right]$$

The left hand side of this equation is nice: it has the wave operator operating on $\vec{A}$, and the right hand side has the source $\vec{j}$ of $\vec{A}$, but there is an additional term that spoils it. But now we recall that V and $\vec{A}$ are not unique. Since the curl of a gradient is zero, we can add the gradient of a scalar to $\vec{A}$ without changing $\vec{B}$

$$\vec{A}' = \vec{A} + \vec{\nabla} \chi \tag{4}$$
$$\vec{B}' = \vec{\nabla} \times \vec{A}' = \vec{\nabla} \times \vec{A} = \vec{B}$$

Then for the new $\vec{E}$ field we have:

$$\vec{E}' = -\vec{\nabla} V' - \frac{\partial \vec{A}'}{\partial t} = -\vec{\nabla} V' - \frac{\partial \vec{A}}{\partial t} - \frac{\partial}{\partial t} \vec{\nabla} \chi = \vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$

So to get the same $\vec{E}$ we need a new V :

$$V' = V - \frac{\partial \chi}{\partial t} \tag{5}$$

Equations (4) and (5) guarantee that the fields are the same whether we use the unprime or prime potentials. They describe the *Gauge transformations* for the fields.

Knowing that we have this flexibility, we can choose the potentials so that

$$\frac{1}{c^2}\frac{\partial V}{\partial t} + \vec{\nabla} \cdot \vec{A} = 0 \tag{6}$$

For suppose this is not true. Then we apply a Gauge tranformation, and require that the new potentials satisfy equation (6). Then

$$\frac{1}{c^2}\frac{\partial}{\partial t}\left(V - \frac{\partial \chi}{\partial t}\right) + \vec{\nabla} \cdot \left(\vec{A} + \vec{\nabla}\chi\right) = 0$$

or

$$\nabla^2 \chi - \frac{1}{c^2}\frac{\partial^2 \chi}{\partial t^2} = -\left(\frac{1}{c^2}\frac{\partial V}{\partial t} + \vec{\nabla} \cdot \vec{A}\right)$$

So we have a prescription for finding the necessary χ. In fact any two different values of χ that differ by a solution of the homogeneous wave equation are equally valid. Equation (6) is the Lorentz Gauge condition. Choosing this gauge, the equation for $\vec{A}$ is a wave equation with source $\vec{j}$:

$$\frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} = \mu_0 \vec{j} \tag{7}$$

Then putting the gauge condition into the equation (3) for V, we have:

$$\begin{aligned} \frac{\rho}{\varepsilon_0} &= -\nabla^2 V - \frac{\partial}{\partial t}\vec{\nabla} \cdot \vec{A} \\ &= -\nabla^2 V - \frac{\partial}{\partial t}\left(-\frac{1}{c^2}\frac{\partial V}{\partial t}\right) \end{aligned}$$

or

$$\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} - \nabla^2 V = \frac{\rho}{\varepsilon_0} \tag{8}$$

Thus each Cartesian component of $\vec{A}$ and the scalar potential V satisfy similar equations: a wave equation with source equal to a component of $\vec{j}$ (for $\vec{A}$) or the charge density ρ (for V).

Because of this nice symmetry (and some other reasons we'll discuss when we study relativity) the Lorentz gauge is particularly useful for calculating wave fields. But it is not the only choice. We could use the Coulomb gauge

$$\vec{\nabla} \cdot \vec{A} = 0$$

which gives a very simple equation for V. In fact it is the same equation as when the sources are time independent:

$$\nabla^2 V(\vec{r},t) = -\frac{\rho(\vec{r},t)}{\varepsilon_0}$$

This is rather odd, however, because it means that as we change ρ, the potential changes instantly everywhere in the universe! This defies all of our expectations about causality. The equation for $\vec{A}$ is also rather ugly:

$$\frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} = \mu_0 \vec{j} - \vec{\nabla}\frac{1}{c^2}\frac{\partial V}{\partial t}$$

There are ways to simplify this (see Physics 704) but we still have to deal with the acausal nature of V. In classical physics the potentials are merely useful tools for calculating $\vec{E}$ and $\vec{B}$, and it turns out tha the physical fields $\vec{E}$ and $\vec{B}$ don't change everywhere at once, but obey the expected rules that signals travel at c. For now, however, let's stick with Lorentz gauge.

The wave equations we have found show that electromagnetic signals travel at the speed of light. If a source 1 km from me changes its magnitude at time t_0, I won't know about it until time $t = t_0 + 1000\text{ m}/(3\times 10^8\text{ m/s}) = t_0 + 30$ μs. Another way of saying this is that the charge density that determines V at my position is the charge density 30 μs ago. We can formalize this idea by noting that we must use the charge density at the *retarded time*:

$$t_{\text{ret}} = t - R/c$$

to get the potential at time t. Using this idea, we guess that the correct expression for potential is

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,',t-R/c)}{R}d\tau' \tag{9}$$

where, as usual,

$$R = |\vec{r} - \vec{r}\,'|$$

Equation (9) is the retarded potential. Let's check that it works by stuffing into the wave equation. We have to be careful in taking the space derivatives because $\vec{r}$ appears in R and R appears in ρ as well as in the denominator. We use the chain rule to take the derivatives of ρ.

$$\begin{aligned}\frac{\partial \rho}{\partial x} &= \frac{\partial \rho}{\partial t_{\text{ret}}}\frac{\partial t_{\text{ret}}}{\partial x} = \frac{\partial \rho}{\partial t_{\text{ret}}}\left(-\frac{1}{c}\frac{\partial R}{\partial x}\right)\\ \frac{\partial \rho}{\partial t} &= \frac{\partial \rho}{\partial t_{\text{ret}}}\frac{\partial t_{\text{ret}}}{\partial t} = \frac{\partial \rho}{\partial t_{\text{ret}}}\end{aligned}$$

Thus

$$\begin{aligned}
\nabla^2 V &= \vec{\nabla}\cdot\left(\vec{\nabla}V\right) = \vec{\nabla}\cdot\frac{\vec{\nabla}}{4\pi\varepsilon_0}\int\frac{\rho\left(\vec{r}',t-R/c\right)}{R}d\tau' \\
&= \vec{\nabla}\cdot\frac{1}{4\pi\varepsilon_0}\int\left[\rho\left(\vec{r}',t_{\mathrm{ret}}\right)\vec{\nabla}\frac{1}{R}+\frac{1}{R}\vec{\nabla}\rho\left(\vec{r}',t_{\mathrm{ret}}\right)\right]d\tau' \\
&= \vec{\nabla}\cdot\frac{1}{4\pi\varepsilon_0}\int\left[\rho\left(\vec{r}',t_{\mathrm{ret}}\right)\vec{\nabla}\frac{1}{R}-\frac{1}{Rc}\frac{\partial\rho\left(\vec{r}',t_{\mathrm{ret}}\right)}{\partial t_{\mathrm{ret}}}\vec{\nabla}R\right]d\tau' \\
&= \frac{1}{4\pi\varepsilon_0}\int\left(T_1+T_2\right)\ d\tau'
\end{aligned}$$

Now we take the next derivative. The first term in the integrand is

$$T_1=\rho\left(\vec{r}',t_{\mathrm{ret}}\right)\nabla^2\frac{1}{R}-\vec{\nabla}\frac{1}{R}\cdot\frac{1}{Rc}\frac{\partial\rho\left(\vec{r}',t_{\mathrm{ret}}\right)}{\partial t_{\mathrm{ret}}}\vec{\nabla}R$$

where

$$\nabla^2\frac{1}{R}=-4\pi\delta\left(\vec{r}-\vec{r}'\right)$$

$$\vec{\nabla}\frac{1}{R}=-\frac{\vec{R}}{R^2},\qquad (R\neq 0)$$

and

$$\vec{\nabla}R=\frac{\vec{R}}{R}=\hat{R}$$

so

$$\begin{aligned}
T_1 &= -4\pi\rho\left(\vec{r}',t_{\mathrm{ret}}\right)\delta\left(\vec{r}-\vec{r}'\right)+\frac{\vec{R}}{R^2}\cdot\frac{1}{Rc}\frac{\partial\rho\left(\vec{r}',t_{\mathrm{ret}}\right)}{\partial t_{\mathrm{ret}}}\hat{R} \\
&= -4\pi\rho\left(\vec{r}',t_{\mathrm{ret}}\right)\delta\left(\vec{r}-\vec{r}'\right)+\frac{1}{cR^2}\frac{\partial\rho\left(\vec{r}',t_{\mathrm{ret}}\right)}{\partial t_{\mathrm{ret}}}
\end{aligned}$$

The second term is

$$\begin{aligned}
T_2 &= -\frac{1}{c}\left\{\vec{\nabla}\left[\frac{1}{R}\frac{\partial\rho\left(\vec{r}',t_{\mathrm{ret}}\right)}{\partial t_{\mathrm{ret}}}\right]\cdot\hat{R}+\frac{1}{R}\frac{\partial\rho\left(\vec{r}',t_{\mathrm{ret}}\right)}{\partial t_{\mathrm{ret}}}\vec{\nabla}\cdot\frac{\vec{R}}{R}\right\} \\
&= -\frac{1}{c}\left\{\frac{\partial\rho\left(\vec{r}',t_{\mathrm{ret}}\right)}{\partial t_{\mathrm{ret}}}\vec{\nabla}\frac{1}{R}\cdot\hat{R}+\frac{\partial}{\partial t_{\mathrm{ret}}}\left(-\frac{1}{c}\frac{\partial\rho}{\partial t_{\mathrm{ret}}}\right)\frac{\vec{\nabla}R}{R}\cdot\hat{R}+\frac{1}{R}\frac{\partial\rho\left(\vec{r}',t_{\mathrm{ret}}\right)}{\partial t_{\mathrm{ret}}}\left(-\frac{\hat{R}\cdot\vec{R}}{R^2}+\frac{3}{R}\right)\right\} \\
&= -\frac{1}{c}\left\{\frac{\partial\rho\left(\vec{r}',t_{\mathrm{ret}}\right)}{\partial t_{\mathrm{ret}}}\left(-\frac{\vec{R}}{R^2}\cdot\hat{R}\right)-\frac{1}{c}\frac{\partial^2\rho}{\partial t_{\mathrm{ret}}^2}\frac{1}{R}+\frac{2}{R^2}\frac{\partial\rho\left(\vec{r}',t_{\mathrm{ret}}\right)}{\partial t_{\mathrm{ret}}}\right\} \\
&= \frac{1}{Rc^2}\frac{\partial^2\rho}{\partial t_{\mathrm{ret}}^2}-\frac{1}{R^2c}\frac{\partial\rho\left(\vec{r}',t_{\mathrm{ret}}\right)}{\partial t_{\mathrm{ret}}}
\end{aligned}$$

Putting it all together, the first order time derivatives cancel:

$$\begin{aligned}
\nabla^2 V &= \frac{1}{4\pi\varepsilon_0}\int\left[-4\pi\rho\left(\vec{r}',t_{\mathrm{ret}}\right)\delta\left(\vec{r}-\vec{r}'\right)+\frac{1}{cR^2}\frac{\partial\rho\left(\vec{r}',t_{\mathrm{ret}}\right)}{\partial t_{\mathrm{ret}}}+\frac{1}{Rc^2}\frac{\partial^2\rho}{\partial t_{\mathrm{ret}}^2}-\frac{1}{R^2c}\frac{\partial\rho\left(\vec{r}',t_{\mathrm{ret}}\right)}{\partial t_{\mathrm{ret}}}\right]d\tau' \\
&= \frac{\rho\left(\vec{r},t\right)}{\varepsilon_0}+\frac{1}{c^2}\frac{\partial^2}{\partial t^2}V
\end{aligned}$$

Thus we have retrieved equation (8).

Similarly, the solution for $\vec{A}$ is

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{j}(\vec{r}', t_{\text{ret}})}{R} d\tau' \tag{10}$$

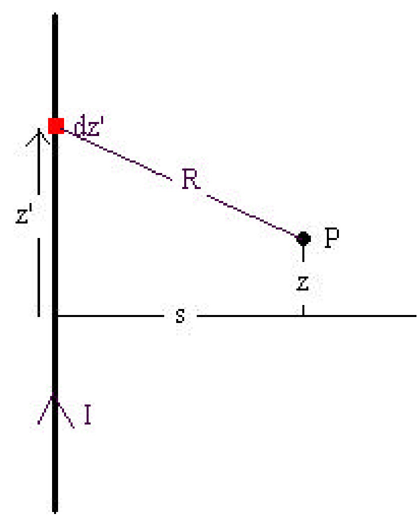
It remains to show that these potentials also satisfy the Lorentz gauge condition. See problem 10.8

The retarded potentials (9) and (10) are often easier to write down than they are to use. But we can calculate the potentials in some simple cases. Imagine that we could set the current up everywhere in an infinitely long wire at time $t = 0$. (I don't know how we'd do this, but let's pretend.)

$$\begin{aligned} I &= 0 \quad t < 0 \\ &= I_0 \quad t > 0 \end{aligned}$$

Then $V \equiv 0$ since $\rho \equiv 0$ everywhere, (notice that $\vec{\nabla} \cdot \vec{j} = 0$, so if ρ is zero at any time it stays zero), and, with $z-$axis along the wire:

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\hat{z}\int_{-\infty}^{+\infty} \frac{I(t_{\text{ret}})}{R} dz'$$



The current function is zero unless $t_{\text{ret}} = t - R/c > 0$, so we need

$$R = \sqrt{s^2 + (z - z')^2} < ct$$

This condition restricts the range of integration:

$$\begin{aligned} (z - z')^2 &< c^2t^2 - s^2 \\ |z - z'| &< \sqrt{c^2t^2 - s^2} \end{aligned}$$

or

$$z-\sqrt{c^2t^2-s^2}<z'<z+\sqrt{c^2t^2-s^2}$$

Note that $s<ct$ is also required, so that the square root is real.

$$\vec{A}(\vec{r},t)=\frac{\mu_0}{4\pi}\hat{z}\int_{z-\sqrt{c^2t^2-s^2}}^{z+\sqrt{c^2t^2-s^2}}\frac{I_0}{\sqrt{s^2+(z-z')^2}}dz'$$

Let $z'-z=s\tan\theta$. Then

$$\begin{aligned}
\vec{A}(\vec{r},t) &= \frac{\mu_0 I_0}{4\pi}\hat{z}\int_{\theta_1}^{\theta_2}\frac{s\sec^2\theta}{s\sec\theta}d\theta \\
&= \frac{\mu_0 I_0}{4\pi}\hat{z}\ln\left(\sec\theta+\tan\theta\right)\Big|_{\theta_1}^{\theta_2} \\
&= \frac{\mu_0 I_0}{4\pi}\hat{z}\ln\left(\sqrt{1+\left(\frac{z-z'}{s}\right)^2}+\frac{z-z'}{s}\right)\Bigg|_{z-\sqrt{c^2t^2-s^2}}^{z+\sqrt{c^2t^2-s^2}} \\
&= \frac{\mu_0 I_0}{4\pi}\hat{z}\ln\left(\sqrt{1+\frac{c^2t^2-s^2}{s^2}}+\frac{\sqrt{c^2t^2-s^2}}{s}\right)-\ln\left(\sqrt{1+\frac{c^2t^2-s^2}{s^2}}-\frac{\sqrt{c^2t^2-s^2}}{s}\right) \\
&= \frac{\mu_0 I_0}{4\pi}\hat{z}\ln\left(\frac{ct+\sqrt{c^2t^2-s^2}}{ct-\sqrt{c^2t^2-s^2}}\right)\quad \text{for } s<ct \\
&= 0 \text{ otherwise}
\end{aligned}$$

Note that the result is independent of z, as we would expect[1].

Now we can find the fields:

$$\begin{aligned}
\vec{E} &= -\frac{\partial\vec{A}}{\partial t} \\
&= \frac{\mu_0 I_0}{4\pi}\hat{z}\left\{\left(\frac{c+c^2t/\sqrt{c^2t^2-s^2}}{ct+\sqrt{c^2t^2-s^2}}\right)-\left(\frac{c-c^2t/\sqrt{c^2t^2-s^2}}{ct-\sqrt{c^2t^2-s^2}}\right)\right\}S(ct-s) \\
&\quad -\frac{\mu_0 I_0}{4\pi}\hat{z}\ln\left(\frac{ct+\sqrt{c^2t^2-s^2}}{ct-\sqrt{c^2t^2-s^2}}\right)\delta\left(t-\frac{s}{c}\right) \\
&= \frac{\mu_0 I_0 c}{4\pi}\hat{z}\left(\frac{\left(\sqrt{c^2t^2-s^2}+ct\right)\left(ct-\sqrt{c^2t^2-s^2}\right)-\left(\sqrt{c^2t^2-s^2}-ct\right)\left(ct+\sqrt{c^2t^2-s^2}\right)}{c^2t^2-(c^2t^2-s^2)}\right)\frac{S(ct-s)}{\sqrt{c^2t^2-s^2}} \\
&= \frac{\mu_0 I_0 c}{4\pi}\hat{z}\frac{2}{\sqrt{c^2t^2-s^2}}S(ct-s) \\
&= \frac{I_0}{2\pi\varepsilon_0 c}\frac{\hat{z}}{\sqrt{c^2t^2-s^2}}S(ct-s)
\end{aligned}$$

[1] This is not quite the same as Griffiths' result. The two differ by a gauge transformation. Try to fix up Griffith's sloppy derivation by putting in the necessary sentences!

In the second form it is easier to check that the result is dimensionally correct. The term with the $\delta-$function is zero because when $s = ct$ the log is $\ln(1) = 0$.

Finally we get $\vec{B}$:

$$\begin{aligned}
\vec{B} &= \vec{\nabla} \times A\hat{z} = \frac{\partial A_z}{\partial \phi}\hat{s} - \frac{\partial A_z}{\partial s}\hat{\phi} \\
&= -\frac{\mu_0 I_0}{4\pi}\hat{\phi}\frac{\partial}{\partial s}\ln\left(\frac{ct + \sqrt{c^2t^2 - s^2}}{ct - \sqrt{c^2t^2 - s^2}}\right) S(ct - s) \\
&= -\frac{\mu_0 I_0}{4\pi}\hat{\phi}\left\{\left(\frac{-s/\sqrt{c^2t^2 - s^2}}{ct + \sqrt{c^2t^2 - s^2}} - \frac{s/\sqrt{c^2t^2 - s^2}}{ct - \sqrt{c^2t^2 - s^2}}\right) S(ct - s) - \ln\left(\frac{ct + \sqrt{c^2t^2 - s^2}}{ct - \sqrt{c^2t^2 - s^2}}\right)\delta(s - ct)\right\} \\
&= \frac{\mu_0 I_0}{4\pi}\hat{\phi}\frac{s}{\sqrt{c^2t^2 - s^2}}\left(\frac{2ct}{(ct)^2 - c^2t^2 + s^2}\right) S(ct - s) \\
&= \frac{\mu_0 I_0}{2\pi}\hat{\phi}\frac{ct}{s\sqrt{c^2t^2 - s^2}} S(ct - s)
\end{aligned}$$

As $t \to \infty$ we get back the static limit:

$$\begin{aligned}
\vec{E} &\to 0 \\
\vec{B} &\to \frac{\mu_0 I_0}{2\pi s}\hat{\phi}
\end{aligned}$$

2 Fields due to a moving point charge

2.1 Potentials

One of the important applications of the retarded potentials is to calculate the fields due to a moving point charge. The charge is located at a point with position vector $r_0(t)$, where its velocity is

$$\vec{v} = \frac{d\vec{r}_0}{dt}$$

and its acceleration is

$$\vec{a} = \frac{d\vec{v}}{dt}$$

The charge density is then

$$\rho(\vec{r}, t) = q\delta(\vec{r} - \vec{r}_0(t))$$

and the current density is

$$\vec{j} = \rho\vec{v} = q\vec{v}\delta(\vec{r} - \vec{r}_0(t))$$

The potential at a point P with position $\vec{r}_P$ at time t is due to the charge at position $\vec{r}_0(t_{\text{ret}})$. Thus t_{ret} is determined by the equation

$$\begin{aligned}
R(t_{\text{ret}}) &= c(t - t_{\text{ret}}) \\
|\vec{r}_P - \vec{r}_0(t_{\text{ret}})| &= c(t - t_{\text{ret}})
\end{aligned} \tag{11}$$

To get the potential we back up one step and write:

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}',t')\,\delta(t'-t_{ret})}{R}d\tau' dt'$$

The potential is then

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{q\delta(\vec{r}-\vec{r}_0(t))\,\delta(t'-t_{ret})}{R}d\tau' dt'$$

We do the integral over the volume (in primed variables) *first,* to get

$$V(\vec{r},t) = \frac{q}{4\pi\varepsilon_0}\int \frac{\delta(t'-t_{\text{ret}}(\vec{r}_0))}{R}dt'$$

We can do the integral if we can get the delta function into the form
$\delta(t' - \text{something independent of } t')$. It is not in that form yet, because t_{ret} contains $\vec{r}_0(t')$. We have $\delta[f(t')]$ where

$$f(t') = t' - \left(t - \frac{|\vec{r}-\vec{r}_0(t')|}{c}\right)$$

Those of you taking Physics 485/785 know that

$$\delta[f(x)] = \frac{\delta(x-x_0)}{|f'(x_0)|}$$

where $f(x_0) = 0$. So here the root is t_{ret} and

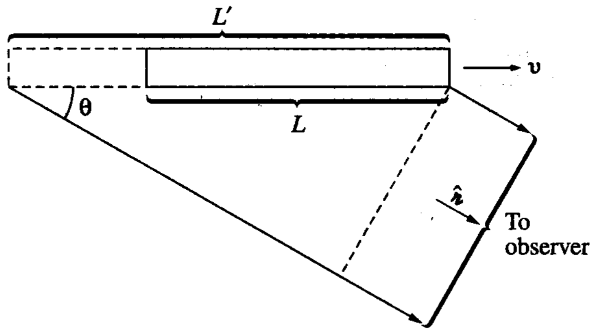
$$\begin{aligned} f'(t') &= 1 - \frac{1}{c}\frac{\partial}{\partial t'}|\vec{r}-\vec{r}_0(t')| = 1 - \vec{v}\cdot\frac{1}{2}2\frac{(\vec{r}-\vec{r}_0)}{c\,|(\vec{r}-\vec{r}_0)|} \\ &= 1 - \frac{\vec{v}\cdot\hat{R}}{c} \end{aligned}$$

Thus

$$V(\vec{r},t) = \frac{q}{4\pi\varepsilon_0\,|\vec{r}-\vec{r}_0(t_{\text{ret}})|\left(1-\vec{v}\cdot\hat{R}/c\right)} \tag{12}$$

Thus the potential looks exactly like the potential due to a static point charge except that $\vec{r}_0$ now changes with time, and must be evaluated at t_{ret}, and there is an extra geometric factor $1 - \vec{v}\cdot\hat{R}/c$ in the denominator. When the speed of the charge approches the speed of light, this factor makes the potential very large at points where $\vec{v}\cdot\hat{R}/c \sim 1$, that is, where $\hat{R}$ is parallel to $\vec{v}$.

To understand why this factor appears, consider the observed volume of a train travelling at speed v, as shown in the diagram.



Light from the front of the train leaves the train at time t_0 and reaches the observer at time T where $T - t_0 = R/c$. If light from the rear is to reach the observed at the same time, it has to start earlier, at time t_1, but then the rear of the train was father back along the tracks, where $L' - L = v\,(t_0 - t_1)$ and

$$T - t_1 = \frac{R + L' \cos\theta}{c}$$

or

$$\frac{R}{c} + t_0 - t_1 = \frac{R + L' \cos\theta}{c}$$

So

$$\frac{L' - L}{v} = \frac{L'}{c} \cos\theta$$

or

$$L' \left(1 - \frac{v}{c} \cos\theta\right) = L$$

Then the observed volume of the train is

$$V' = L'A = \frac{LA}{\left(1 - \frac{v}{c}\cos\theta\right)} = \frac{V}{\left(1 - \frac{v}{c}\cos\theta\right)}$$

The geometrical factor in the denominator here is exactly the same as the factor in the potenial (12).

The integral we need to do to get $\vec{A}$ is the same as for V, so together we have the Lienard Wiechert potentials:

$$\begin{aligned} V &= \left.\frac{q}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r}_0\right| \left(1 - \vec{v}\cdot\hat{R}/c\right)}\right|_{t_{\rm ret}} \\ \vec{A} &= \left.\frac{q\vec{v}}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r}_0\right| \left(1 - \vec{v}\cdot\hat{R}/c\right)}\right|_{t_{\rm ret}} \end{aligned} \tag{13}$$

where the whole expression is evaluated at the retarded time. As $v \to 0$ we get back the usual result for a stationary charge.

2.2 Fields due to a moving charge.

To get the fields from the potentials we use equations (1) and (2), This is not an easy task because t_{ret} depends on the spatial coordinates.

2.2.1 Non-relativistic limit:

In Lorentz Gauge, the fields are found using

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}$$

and

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

But our expressions for the potentials are in terms of $\vec{x}$ and t_{ret}, not $\vec{x}$ and t, so we have to be very careful in taking the partial derivatives. We can put the origin at the instantaneous position of the charge to simplify things. Then $R = r$. Our potential may be written:

$$\Phi\left(\vec{x}, t\right) \equiv \Psi\left(\vec{x}, t_{\text{ret}}\right)$$

Then

$$d\Phi = \left.\vec{\nabla}\Phi\right|_{\text{const t}} \cdot d\vec{x} + \frac{\partial \Phi}{\partial t} dt \equiv \left.\vec{\nabla}\Psi\right|_{\text{const t}_{ret}} \cdot d\vec{x} + \frac{\partial \Psi}{\partial t_{\text{ret}}} dt_{\text{ret}}$$

But $dt_{\text{ret}} = dt - dr/c$, so

$$\left.\vec{\nabla}\Phi\right|_{\text{const t}} \cdot d\vec{x} + \frac{\partial \Phi}{\partial t} dt \equiv \left.\vec{\nabla}\Psi\right|_{\text{const t}_{ret}} \cdot d\vec{x} - \frac{\partial \Psi}{\partial t_{\text{ret}}} \frac{dr}{c} + \frac{\partial \Psi}{\partial t_{\text{ret}}} dt$$

Thus the $r-$component of $\vec{\nabla}\Phi$ must be modified:

$$\left.\frac{\partial \Phi}{\partial r}\right|_{\text{const t}} = \left.\frac{\partial \Psi}{\partial r}\right|_{\text{const t}_{ret}} - \frac{1}{c} \frac{\partial \Psi}{\partial t_{\text{ret}}} \tag{14}$$

Now we can calculate the fields:

$$\vec{\nabla}\Phi = \frac{1}{4\pi\varepsilon_0} \vec{\nabla} \frac{q}{r_v} = -\frac{1}{4\pi\varepsilon_0} \frac{q}{r_v^2} \vec{\nabla} r_v$$

and

$$\vec{\nabla} r_v = \frac{\partial}{\partial r}\left(r - \frac{\vec{r}\cdot\vec{v}}{c}\right)\hat{r} + \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial}{\partial \theta}\left(r - \frac{\vec{r}\cdot\vec{v}}{c}\right) + \frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\frac{\partial}{\partial \phi}\left(r - \frac{\vec{r}\cdot\vec{v}}{c}\right)$$

We can choose our axes with polar axis along the instantaneous direction of $\vec{v}$. Then $\vec{r}\cdot\vec{v} = rv\cos\theta$, and

$$\vec{\nabla} r_v = \left(1 - \frac{v}{c}\cos\theta\right)\hat{r} + \frac{\hat{\boldsymbol{\theta}}}{r}\left(r\frac{v}{c}\sin\theta\right)$$

In the non-relativistic limit, $v/c \ll 1$, to zeroth order in v/c, this is

$$\vec{\nabla} r_v = \hat{r}$$

We are also going to need

$$\frac{\partial r_v}{\partial t} = -\frac{\vec{r}\cdot\vec{a}}{c}$$

Then we have

$$\begin{aligned}
\vec{E} &= -\frac{1}{4\pi\varepsilon_0}\left.\vec{\nabla}\Phi\right|_{\text{const t}_{ret}} + \frac{1}{4\pi\varepsilon_0 c}\frac{\partial\Phi}{\partial t}\hat{r} - \frac{\partial\vec{A}}{\partial t} \\
&= \frac{1}{4\pi\varepsilon_0}\frac{q}{r_v^2}\hat{\mathbf{r}} - \frac{1}{4\pi\varepsilon_0 c}\frac{q}{r_v^2}\left(-\frac{\vec{r}\cdot\vec{a}}{c}\right)\hat{\mathbf{r}} - \frac{\mu_0}{4\pi}q\frac{\partial}{\partial t}\frac{\vec{v}}{r_v} \\
&= \frac{1}{4\pi\varepsilon_0}\frac{q}{r_v^2}\hat{\mathbf{r}}\left(1+\frac{\vec{r}\cdot\vec{a}}{c^2}\right) - \frac{\mu_0}{4\pi}q\frac{\vec{a}}{r_v} + \frac{\mu_0}{4\pi}\frac{q\vec{v}}{r_v^2}\left(r - \frac{\vec{r}\cdot\vec{a}}{c}\right)
\end{aligned}$$

and again taking the non-relativistic limit, this becomes:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{r^2}\hat{\mathbf{r}}\left(1+\frac{\vec{r}\cdot\vec{a}}{c^2}\right) - \frac{q}{c^2}\frac{\vec{a}}{r}\right]$$

The first term is the usual Coulomb field. The other two terms depend on $\vec{a}$: these are the radiation field.

$$\begin{aligned}
\vec{E}_{\text{rad}} &= \frac{1}{4\pi\varepsilon_0 c^2}\frac{q}{r}\left[(\hat{r}\cdot\vec{a})\,\hat{r} - \vec{a}\right] \\
&= \frac{\mu_0}{4\pi}\frac{q}{r}\left[\hat{r}\times(\hat{r}\times\vec{a})\right] \qquad (15)
\end{aligned}$$

Next let's calculate the magnetic field:

$$\begin{aligned}
\vec{B} &= \left.\vec{\nabla}\times\vec{A}\right|_{\text{const } t} \\
&= \left(\left.\vec{\nabla}\right|_{\text{const } t_{\text{ret}}} - \frac{1}{c}\hat{r}\frac{\partial}{\partial t}\right)\times\frac{\mu_0}{4\pi}\frac{q\vec{v}}{r_v} \\
&= \frac{\mu_0}{4\pi}\left[q\vec{\nabla}\left(\frac{1}{r_v}\right)\times\vec{v} - q\hat{r}\times\frac{1}{c}\left(\frac{\vec{a}}{r_v} - \frac{\vec{v}}{r_v^2}\frac{\partial r_v}{\partial t}\right)\right] \\
&= -\frac{\mu_0}{4\pi}\left[\frac{q}{r^2}\hat{r}\times\vec{v} + \frac{q}{c}\hat{r}\times\left(\frac{\vec{a}}{r_v} + \frac{\vec{v}}{r_v^2}\frac{\vec{r}\cdot\vec{a}}{c}\right)\right] \\
&\simeq \frac{\mu_0}{4\pi}\left[\frac{q}{r^2}\vec{v}\times\hat{r} - \frac{q}{r}\frac{\hat{r}\times\vec{a}}{c}\right]
\end{aligned}$$

where again we have negelected terms of order v/c compared with those retained. The first term is the usual Biot-Savart law result. The second term is the radiation field:

$$\vec{B}_{\text{rad}} = \frac{\mu_0}{4\pi}\frac{q}{rc}\vec{a}\times\hat{r} \qquad (16)$$

Notice that

$$\vec{E}_{\text{rad}} = c\vec{B}_{\text{rad}} \times \hat{r}$$

as expected for a plane wave.

The Poynting flux is:

$$\begin{aligned}\vec{S} &= \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0}\vec{E}\times\left(\frac{\hat{r}\times\vec{E}}{c}\right) \\ &= \frac{1}{\mu_0 c}E^2\ \hat{r}\end{aligned}$$

where from equation 25:

$$E = \frac{\mu_0}{4\pi}\frac{q}{r}a\sin\theta = \frac{1}{4\pi\varepsilon_0}\frac{q}{rc^2}a\sin\theta$$

and θ is the angle between $\vec{a}$ and $\hat{r}$. Thus

$$S = \frac{1}{\mu_0 c}\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{rc^2}a\sin\theta\right)^2 = \frac{q^2}{(4\pi)^2\varepsilon_0 c^3}\frac{a^2}{r^2}\sin^2\theta$$

and the power radiated per unit solid angle is:

$$\frac{dP}{d} = r^2 S = \frac{q^2a^2}{(4\pi)^2\varepsilon_0 c^3}\sin^2\theta \tag{17}$$

Finally the total power radiated is

$$P = \int\frac{dP}{d}d\quad = \frac{q^2a^2}{(4\pi)^2\varepsilon_0 c^3}\int_0^{2\pi}\int_{-1}^{+1}\left(1-\mu^2\right)d\phi d\mu$$

where as usual $\mu = \cos\theta$. Thus

$$\begin{aligned}P &= \frac{q^2a^2}{(4\pi\varepsilon_0)\,2c^3}\left(\mu - \frac{\mu^3}{3}\right)\Bigg|_{-1}^{+1} \\ &= \frac{2}{3}\frac{q^2a^2}{4\pi\varepsilon_0 c^3}\end{aligned} \tag{18}$$

This result is called the Larmor formula.

2.2.2 Relativistic "brute force" caculation, as per Griffiths

Ths is tough. However, we did some of the work already in finding the potentials. First let's look at the gradient:

$$\begin{aligned}\frac{4\pi\varepsilon_0}{q}\vec{\nabla}V &= \vec{\nabla}\frac{1}{R\left(1-\vec{v}\cdot\hat{R}/c\right)} = \vec{\nabla}\frac{1}{\left(R-\vec{v}\cdot\vec{R}/c\right)} \\ &= \frac{-\vec{\nabla}\left(R-\vec{v}\cdot\vec{R}/c\right)}{\left(R-\vec{v}\cdot\vec{R}/c\right)^2}\end{aligned}$$

where

$$\vec{R} = \vec{r} - \vec{r}_0\left(t_{\mathrm{ret}}\right)$$

and

$$t_{\mathrm{ret}} = t - R/c$$

Thus

$$\vec{\nabla} R = \vec{\nabla} c\left(t - t_{\mathrm{ret}}\right) = -c\vec{\nabla} t_{\mathrm{ret}}$$

while

$$\vec{\nabla}\left[\vec{R}\cdot\vec{v}\left(t_{\mathrm{ret}}\right)\right] = \left(\vec{R}\cdot\vec{\nabla}\right)\vec{v} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{R} + \vec{R}\times\left(\vec{\nabla}\times\vec{v}\right) + \vec{v}\times\left(\vec{\nabla}\times\vec{R}\right)$$

To evaluate these derivatives, again we use the chain rule. So, for example,

$$\frac{\partial\vec{v}}{\partial x} = \frac{d\vec{v}}{dt_{\mathrm{ret}}}\frac{\partial t_{\mathrm{ret}}}{\partial x} = \vec{a}\frac{\partial t_{\mathrm{ret}}}{\partial x}$$

Thus

$$\vec{\nabla}\left[\vec{R}\cdot\vec{v}\left(t_{\mathrm{ret}}\right)\right] = \vec{a}\left(\vec{R}\cdot\vec{\nabla}\right)t_{\mathrm{ret}} + \left(\vec{v}\cdot\vec{\nabla}\right)\left(\vec{r}-\vec{r}_0\right) - \vec{R}\times\left(\vec{a}\times\vec{\nabla}t_{\mathrm{ret}}\right) + \vec{v}\times\left[\vec{\nabla}\times\left(\vec{r}-\vec{r}_0\right)\right]$$

But $\vec{\nabla}\times\vec{r} = 0$, $\left(\vec{v}\cdot\vec{\nabla}\right)\vec{r} = \vec{v}$, and

$$\frac{\partial\vec{r}_0}{\partial x} = \frac{d\vec{r}_0}{dt_{\mathrm{ret}}}\frac{\partial t_{\mathrm{ret}}}{\partial x} = \vec{v}\frac{\partial t_{\mathrm{ret}}}{\partial x} \tag{19}$$

So, using (19), we get

$$\begin{aligned}
\vec{\nabla}\left[\vec{R}\cdot\vec{v}\left(t_{\mathrm{ret}}\right)\right] &= \vec{a}\left(\vec{R}\cdot\vec{\nabla}\right)t_{\mathrm{ret}} + \vec{v} - \vec{v}\left(\vec{v}\cdot\vec{\nabla}t_{\mathrm{ret}}\right) - \vec{a}\left(\vec{R}\cdot\vec{\nabla}\right)t_{\mathrm{ret}} \\
&\quad + \left(\vec{R}\cdot\vec{a}\right)\vec{\nabla}t_{\mathrm{ret}} + \vec{v}\times\left[\vec{v}\times\vec{\nabla}t_{\mathrm{ret}}\right] \\
&= \vec{v} + \left(\vec{R}\cdot\vec{a}\right)\vec{\nabla}t_{\mathrm{ret}} - \vec{v}\left(\vec{v}\cdot\vec{\nabla}t_{\mathrm{ret}}\right) + \vec{v}\left(\vec{v}\cdot\vec{\nabla}t_{\mathrm{ret}}\right) - v^2\vec{\nabla}t_{\mathrm{ret}} \\
&= \vec{v} + \left[\vec{R}\cdot\vec{a} - v^2\right]\vec{\nabla}t_{\mathrm{ret}}
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{4\pi\varepsilon_0}{q}\vec{\nabla}V &= \frac{-1}{\left(R - \vec{v}\cdot\vec{R}/c\right)^2}\left\{-c\vec{\nabla}t_{\mathrm{ret}} - \frac{\vec{v}}{c} - \frac{1}{c}\left[\left(\vec{R}\cdot\vec{a}\right) - v^2\right]\vec{\nabla}t_{\mathrm{ret}}\right\} \\
&= \frac{1}{\left(R - \vec{v}\cdot\vec{R}/c\right)^2}\left[\frac{\vec{v}}{c} + \frac{1}{c}\left[\left(\vec{R}\cdot\vec{a}\right) + c^2 - v^2\right]\vec{\nabla}t_{\mathrm{ret}}\right]
\end{aligned} \tag{20}$$

Now what about $\vec{\nabla}t_{\mathrm{ret}}$:

$$\begin{aligned}
\vec{\nabla}R &= -c\vec{\nabla}t_{\mathrm{ret}} = \vec{\nabla}\sqrt{\vec{R}\cdot\vec{R}} = \frac{1}{2}\frac{\vec{\nabla}\left(\vec{R}\cdot\vec{R}\right)}{R} \\
&= \frac{1}{R}\left[\left(\vec{R}\cdot\vec{\nabla}\right)\vec{R} + \vec{R}\times\left(\vec{\nabla}\times\vec{R}\right)\right]
\end{aligned}$$

where we used product rule (4) from the front cover of G with $\vec{A} = \vec{B} = \vec{R}$. Thus, again using (19), we have

$$\begin{aligned} -c\vec{\nabla}t_{\mathrm{ret}} &= \frac{1}{R}\left[\vec{R} - \vec{v}\left(\vec{R}\cdot\vec{\nabla}\right)t_{\mathrm{ret}} + \vec{R}\times\left(\vec{v}\times\vec{\nabla}t_{\mathrm{ret}}\right)\right] \\ &= \frac{1}{R}\left[\vec{R} - \vec{v}\left(\vec{R}\cdot\vec{\nabla}\right)t_{\mathrm{ret}} + \vec{v}\left(\vec{R}\cdot\vec{\nabla}\right)t_{\mathrm{ret}} - \left(\vec{R}\cdot\vec{v}\right)\vec{\nabla}t_{\mathrm{ret}}\right] \\ &= \hat{R} - \left(\hat{R}\cdot\vec{v}\right)\vec{\nabla}t_{\mathrm{ret}} \end{aligned}$$

Rearranging, we get

$$\vec{\nabla}t_{\mathrm{ret}} = \frac{\hat{R}}{\hat{R}\cdot\vec{v} - c} = -\frac{\hat{R}/c}{1 - \hat{R}\cdot\vec{v}/c} \tag{21}$$

Then putting this result into (20), we get

$$\vec{\nabla}V = \frac{q}{4\pi\varepsilon_0}\frac{1}{R^2\left(1 - \vec{v}\cdot\hat{R}/c\right)^2}\left\{\frac{\vec{v}}{c} - \frac{1}{c^2}\left[\left(\vec{R}\cdot\vec{a}\right) + c^2 - v^2\right]\frac{\hat{R}}{1 - \hat{R}\cdot\vec{v}/c}\right\}$$

Next

$$\frac{4\pi\varepsilon_0}{q}\frac{\partial\vec{A}}{\partial t} = \frac{\partial}{\partial t}\frac{\vec{v}}{c^2R\left(1 - \vec{v}\cdot\hat{R}/c\right)}$$

So here we need

$$\frac{\partial R}{\partial t} = \frac{\partial R}{\partial t_{\mathrm{ret}}}\frac{\partial t_{\mathrm{ret}}}{\partial t}$$

$$\frac{\partial t_{\mathrm{ret}}}{\partial t} = \frac{\partial}{\partial t}\left(t - \frac{R}{c}\right) = 1 - \frac{1}{c}\frac{\partial R}{\partial t}$$

$$\begin{aligned} \frac{\partial R}{\partial t_{\mathrm{ret}}} &= \frac{\partial}{\partial t_{\mathrm{ret}}}\sqrt{\vec{R}\cdot\vec{R}} = \frac{\vec{R}\cdot}{R}\frac{\partial}{\partial t_{\mathrm{ret}}}\left(\vec{r} - \vec{r}_0\left(t_{\mathrm{ret}}\right)\right) \\ &= -\frac{\vec{v}\cdot\vec{R}}{R} \end{aligned}$$

Thus

$$\begin{aligned} \frac{\partial R}{\partial t} &= -\hat{R}\cdot\vec{v}\left(1 - \frac{1}{c}\frac{\partial R}{\partial t}\right) \\ \frac{\partial R}{\partial t}\left(1 - \frac{\hat{R}\cdot\vec{v}}{c}\right) &= -\hat{R}\cdot\vec{v} \\ \frac{\partial R}{\partial t} &= \frac{-\hat{R}\cdot\vec{v}}{\left(1 - \hat{R}\cdot\vec{v}/c\right)} \end{aligned} \tag{22}$$

and

$$\frac{\partial t_{\rm ret}}{\partial t} = 1 - \frac{-\hat{R}\cdot\vec{v}/c}{\left(1-\hat{R}\cdot\vec{v}/c\right)} = \frac{1}{\left(1-\hat{R}\cdot\vec{v}/c\right)} \tag{23}$$

Thus

$$\frac{\partial\vec{v}\,(t_{\rm ret})}{\partial t} = \frac{\vec{a}}{\left(1-\hat{R}\cdot\vec{v}/c\right)}$$

and

$$\begin{aligned}
\frac{4\pi\varepsilon_0}{q}\frac{\partial\vec{A}}{\partial t} &= \frac{\partial}{\partial t}\frac{\vec{v}}{c^2R\left(1-\vec{v}\cdot\hat{R}/c\right)} \\
&= \frac{1}{c^2}\frac{\partial}{\partial t_{\rm ret}}\left[\frac{\vec{v}}{R\left(1-\vec{v}\cdot\hat{R}/c\right)}\right]\frac{1}{\left(1-\hat{R}\cdot\vec{v}/c\right)} \\
&= \frac{1}{c^2}\frac{1}{\left(1-\hat{R}\cdot\vec{v}/c\right)}\left[\frac{\vec{a}}{R-\vec{v}\cdot\vec{R}/c} - \vec{v}\left(\frac{-\hat{R}\cdot\vec{v}-\vec{a}\cdot\vec{R}/c+\vec{v}\cdot\vec{v}/c}{\left(R-\vec{v}\cdot\vec{R}/c\right)^2}\right)\right] \\
&= \frac{1}{c^2}\frac{1}{\left(1-\hat{R}\cdot\vec{v}/c\right)}\left[\frac{\vec{a}}{R-\vec{v}\cdot\vec{R}/c} + \vec{v}\left(\frac{\hat{R}\cdot\vec{v}+\vec{a}\cdot\vec{R}/c-v^2/c}{\left(R-\vec{v}\cdot\vec{R}/c\right)^2}\right)\right]
\end{aligned}$$

So, putting the pieces together, we get

$$
\begin{aligned}
\frac{4\pi\varepsilon_0}{q}\vec{E} &= \frac{-1}{R^2\left(1-\vec{v}\cdot\hat{R}/c\right)^2}\left\{\frac{\vec{v}}{c}-\frac{1}{c^2}\left[\left(\vec{R}\cdot\vec{a}\right)+c^2-v^2\right]\frac{\hat{R}}{\left(1-\hat{R}\cdot\vec{v}/c\right)}\right\} \\
&\quad -\frac{1}{c^2}\frac{1}{\left(1-\hat{R}\cdot\vec{v}/c\right)}\left[\frac{\vec{a}}{R-\vec{v}\cdot\vec{R}/c}+\vec{v}\left(\frac{\hat{R}\cdot\vec{v}+\vec{a}\cdot\vec{R}/c-v^2/c}{\left(R-\vec{v}\cdot\vec{R}/c\right)^2}\right)\right] \\
&= \frac{-1}{R^2\left(1-\vec{v}\cdot\hat{R}/c\right)^2}\left[\frac{\vec{v}}{c}\left(1+\frac{\frac{\hat{R}\cdot\vec{v}}{c}-\frac{v^2}{c^2}}{\left(1-\hat{R}\cdot\vec{v}/c\right)}\right)-\left(1-\frac{v^2}{c^2}\right)\frac{\hat{R}}{\left(1-\hat{R}\cdot\vec{v}/c\right)}\right] \\
&\quad +\frac{1}{c^2R\left(1-\hat{R}\cdot\vec{v}/c\right)^2}\left\{\frac{\left(\hat{R}\cdot\vec{a}\right)\hat{R}}{\left(1-\vec{v}\cdot\hat{R}/c\right)}-\vec{a}-\frac{\vec{v}\left(\vec{a}\cdot\hat{R}\right)/c}{\left(1-\vec{v}\cdot\hat{R}/c\right)}\right\} \\
&= \frac{1-v^2/c^2}{R^2\left(1-\vec{v}\cdot\hat{R}/c\right)^2}\left[\frac{\hat{R}-\vec{v}/c}{\left(1-\hat{R}\cdot\vec{v}/c\right)}\right] \\
&\quad +\frac{1}{c^2R\left(1-\hat{R}\cdot\vec{v}/c\right)^2}\left\{\frac{\left(\hat{R}\cdot\vec{a}\right)\hat{R}-\vec{a}+\vec{a}\left(\vec{v}\cdot\hat{R}\right)/c-\vec{v}\left(\vec{a}\cdot\hat{R}\right)/c}{\left(1-\vec{v}\cdot\hat{R}/c\right)}\right\}
\end{aligned}
$$

Thus the electric field has two terms. The first goes as $1/R^2$, and in the limit $v \ll c$ becomes the usual Coulomb field

$$
\vec{E}_{\rm Coul} = \frac{q}{4\pi\varepsilon_0}\frac{1-v^2/c^2}{R^2\left(1-\vec{v}\cdot\hat{R}/c\right)^2}\left[\frac{\hat{R}-\vec{v}/c}{\left(1-\hat{R}\cdot\vec{v}/c\right)}\right] \to \frac{q}{4\pi\varepsilon_0}\frac{\hat{R}}{R^2} \text{ for } v \ll c \quad (24)
$$

The second term is

- proportional to the acceleration
- proportional to $1/R$ not $1/R^2$

This term is the *radiation field.*

$$
\begin{aligned}
\vec{E}_{\rm rad} &= \frac{q}{4\pi\varepsilon_0 R}\left\{\frac{\left(\hat{R}\cdot\vec{a}\right)\left(\hat{R}-\vec{v}/c\right)-\vec{a}\hat{R}\cdot\left(\hat{R}-\vec{v}/c\right)}{\left(1-\vec{v}\cdot\hat{R}/c\right)^3c^2}\right\} \\
&= \frac{q}{4\pi\varepsilon_0 R}\frac{\hat{R}\times\left[\left(\hat{R}-\vec{v}/c\right)\times\vec{a}\right]}{\left(1-\vec{v}\cdot\hat{R}/c\right)^3c^2} \quad (25)
\end{aligned}
$$

Before we comment further on these terms, let's work on $\vec{B}$.

$$\begin{aligned}\vec{B} &= \vec{\nabla}\times\vec{A} = \vec{\nabla}\times(V\vec{v}) \\ &= V\vec{\nabla}\times\vec{v} + \vec{\nabla}V\times\vec{v}\end{aligned}$$

So, making use of our previous results, we get

$$\begin{aligned}\frac{4\pi\varepsilon_0}{q}\vec{B} &= \frac{1}{R\left(1-\vec{v}\cdot\hat{R}/c\right)}\left(\vec{a}\times\frac{\hat{R}/c}{1-\hat{R}\cdot\vec{v}/c}\right) \\ &\quad + \frac{1}{R^2\left(1-\vec{v}\cdot\hat{R}/c\right)^2}\left\{\frac{\vec{v}}{c}-\frac{1}{c^2}\left[\left(\vec{R}\cdot\vec{a}\right)+c^2-v^2\right]\frac{\hat{R}}{1-\hat{R}\cdot\vec{v}/c}\right\}\times\vec{v} \\ &= \frac{\vec{a}\times\hat{R}/c}{R\left(1-\vec{v}\cdot\hat{R}/c\right)^2} - \frac{1}{R^2\left(1-\vec{v}\cdot\hat{R}/c\right)^2}\frac{1}{c^2}\left[\left(\vec{R}\cdot\vec{a}\right)+c^2-v^2\right]\frac{\hat{R}\times\vec{v}}{1-\hat{R}\cdot\vec{v}/c}\end{aligned}$$

Again we get a term in $1/R^2$. This is the Biot-Savart law field.

$$\vec{B}_{\text{B-S}} = \frac{q\vec{v}\times\vec{R}}{4\pi\varepsilon_0 R^2}\frac{\left(1-v^2/c^2\right)}{\left(1-\vec{v}\cdot\hat{R}/c\right)^3} \tag{26}$$

The radiation field is again proportional to a and to $1/R$:

$$\begin{aligned}\vec{B}_{\text{rad}} &= \frac{q}{4\pi\varepsilon_0 R}\frac{\vec{a}\times\hat{R}/c\left(1-\vec{v}\cdot\hat{R}/c\right)-\left(\vec{R}\cdot\vec{a}\right)\left(\hat{R}\times\vec{v}\right)/c^2}{\left(1-\vec{v}\cdot\hat{R}/c\right)^3} \\ &= \frac{q}{4\pi\varepsilon_0 cR}\frac{-\hat{R}\times\left[\left(\vec{R}\cdot\vec{a}\right)\vec{v}/c+\vec{a}\left(1-\vec{v}\cdot\hat{R}/c\right)\right]}{\left(1-\vec{v}\cdot\hat{R}/c\right)^3} \\ &= \frac{q}{4\pi\varepsilon_0 cR}\frac{-\hat{R}\times\left\{-\left(\vec{R}\cdot\vec{a}\right)\left(\hat{R}-\vec{v}/c\right)+\vec{a}\left[\hat{R}\cdot\left(\hat{R}-\vec{v}/c\right)\right]\right\}}{\left(1-\vec{v}\cdot\hat{R}/c\right)^3}\end{aligned}$$

In the last step we added a term proportional to $\hat{R}\times\hat{R}$, which is identically zero. Thus

$$\begin{aligned}\vec{B}_{\text{rad}} &= \frac{q}{4\pi\varepsilon_0 cR}\frac{-\hat{R}\times\left\{\hat{R}\times\left[\left(\hat{R}-\vec{v}/c\right)\times\vec{a}\right]\right\}}{\left(1-\vec{v}\cdot\hat{R}/c\right)^3} \\ &= \frac{1}{c}\hat{R}\times\vec{E}_{\text{rad}}\end{aligned} \tag{27}$$

Relation (27) is exactly what we expect for an EM wave.

2.2.3 Properties of the Coulomb fields

It's interesting to see what happens to the Coulomb fields as v approaches c. Let's look at $\vec{E}$ (equation 24).

$$\begin{aligned}
\vec{E}_{\text{Coul}} &= \frac{q}{4\pi\varepsilon_0}\frac{1-v^2/c^2}{R^2\left(1-\vec{v}\cdot\hat{R}/c\right)^2}\left[\frac{\hat{R}-\vec{v}/c}{\left(1-\hat{R}\cdot\vec{v}/c\right)}\right] \\
&= \frac{q}{4\pi\varepsilon_0}\frac{(1+v/c)(1-v/c)}{R^2\left(1-\vec{v}\cdot\hat{R}/c\right)^2}\left[\frac{\hat{R}-\vec{v}/c}{\left(1-\hat{R}\cdot\vec{v}/c\right)}\right] \\
&\to \frac{q}{2\pi\varepsilon_0}\frac{(1-v/c)\left(\hat{R}-\vec{v}/c\right)}{R^2\left(1-\vec{v}\cdot\hat{R}/c\right)^3} \quad \text{as } v\to c
\end{aligned}$$

First look in the direction $\hat{R}\|\vec{v}$. We get

$$\vec{E}_{\text{Coul}}\left(\hat{R}=\hat{v}\right)\to\frac{q}{2\pi\varepsilon_0}\frac{\hat{R}}{R^2(1-v/c)}$$

which becomes very large as $v\to c$. In the direction with $\hat{R}\perp\vec{v}$, we get

$$\vec{E}_{\text{Coul}}\left(\hat{R}\perp\hat{v}\right)\to\frac{q}{2\pi\varepsilon_0}\frac{(1-v/c)\left(\hat{R}-\vec{v}/c\right)}{R^2}\to 0$$

But we must be careful in interpreting these results, because

$$\vec{R}=\vec{r}-\vec{r}_0\left(t_{\text{ret}}\right)$$

If the charge has a constant velocity ($\vec{a}=0$), then

$$\vec{r}_0=\vec{v}t_{\text{ret}}$$

and

$$\begin{aligned}
\vec{R}-R\frac{\vec{v}}{c} &= \vec{r}-\vec{v}t_{\text{ret}}-R\frac{\vec{v}}{c}=\vec{r}-\vec{v}t_{\text{ret}}-\frac{\vec{v}}{c}c\left(t-t_{\text{ret}}\right) \\
&= \vec{r}-\vec{v}t\equiv\vec{s}
\end{aligned}$$

$$\begin{aligned}
R^2 &= \left[\vec{r}-\vec{v}\left(t-\frac{R}{c}\right)\right]^2 \\
&= \left(\vec{s}+\vec{v}R/c\right)^2 \\
&= s^2+2\vec{s}\cdot\vec{v}R/c+R^2v^2/c^2
\end{aligned}$$

where $\vec{s}=\vec{r}-\vec{v}t$, and so we can solve for R as a function of $\vec{s}$:

$$R^2\left(1-\frac{v^2}{c^2}\right)-2R\frac{\vec{s}\cdot\vec{v}}{c}-s^2=0$$

$$
\begin{aligned}
R &= \frac{2\frac{\vec{s}\cdot\vec{v}}{c} \pm \sqrt{4s^2\frac{v^2}{c^2}\cos^2\theta + 4s^2\left(1-\frac{v^2}{c^2}\right)}}{2\left(1-\frac{v^2}{c^2}\right)} \\
&= \frac{\vec{s}\cdot\vec{v}/c \pm s\sqrt{1-v^2\sin^2\theta/c^2}}{1-v^2/c^2}
\end{aligned}
$$

Since R is positive, we need the plus sign. (Check the limit $v \to 0$.) Then

$$
\begin{aligned}
cR - \vec{v}\cdot\vec{R} &= cR - \vec{v}\cdot\vec{r} + v^2\left(t - \frac{R}{c}\right) \\
&= cR\left(1-\frac{v^2}{c^2}\right) - \vec{v}\cdot\vec{r} + v^2 t \\
&= \vec{s}\cdot\vec{v} + s\sqrt{c^2 - v^2\sin^2\theta} - \vec{v}\cdot\vec{s} \\
&= s\sqrt{c^2 - v^2\sin^2\theta}
\end{aligned}
$$

Thus

$$
\vec{E}_{\mathrm{Coul}} = \frac{q}{4\pi\varepsilon_0}\frac{1-v^2/c^2}{s^2\left(1-v^2\sin^2\theta/c^2\right)}\left[\frac{\vec{s}}{s\sqrt{1-v^2\sin^2\theta/c^2}}\right]
$$

Thus

$$
\vec{E}_{\mathrm{Coul}} = \frac{q}{4\pi\varepsilon_0}\frac{\left(1-v^2/c^2\right)\hat{s}}{s^2\left(1-v^2\sin^2\theta/c^2\right)^{3/2}}
$$

In the direction $\theta \simeq 0$, $(\vec{s}\|\vec{v})$, we get

$$
\vec{E}_{\mathrm{Coul}} = \frac{q\hat{s}}{4\pi\varepsilon_0 s^2}\hat{s}\left(1-v^2/c^2\right) \to 0 \quad \text{as } v \to c
$$

whereas for $\theta \simeq \pi/2$, $(\vec{s}\perp\vec{v})$, we get

$$
\vec{E}_{\mathrm{coul}} = \frac{q}{4\pi\varepsilon_0 s^2}\frac{\hat{s}}{\sqrt{1-v^2/c^2}} \to \infty \quad \text{as } v \to c
$$

Thus the field is concentrated in the direction $\vec{s}\perp\vec{v}$, and becomes very large as v increases. A stationary observer sees a pulse of field as the charge moves past.

2.2.4 Properties of the radiation fields

The radiation field decreases more slowly with distance than the Coulomb field, so at large distances, the radiation field dominates. The Poynting vector describes the energy radiated:

$$
\begin{aligned}
\vec{S}_{\mathrm{rad}} &= \frac{1}{\mu_0}\vec{E}_{\mathrm{rad}} \times \vec{B}_{\mathrm{rad}} \\
&= \frac{1}{\mu_0}\vec{E}_{\mathrm{rad}} \times \left(\frac{1}{c}\hat{R} \times \vec{E}_{\mathrm{rad}}\right) \\
&= \frac{1}{\mu_0 c}\hat{R}E_{\mathrm{rad}}^2 - \vec{E}_{\mathrm{rad}}\left(\hat{R}\cdot\vec{E}_{\mathrm{rad}}\right)
\end{aligned}
$$

From equation (25), we see that $\hat{R} \cdot \vec{E}_{\rm rad} = 0$, and then

$$\vec{S} = \frac{1}{\mu_0 c}\hat{R}\left(\frac{q}{4\pi\varepsilon_0 R}\right)^2 \frac{\left(\hat{R}\times\left[\left(\hat{R}-\vec{v}/c\right)\times\vec{a}\right]\right)^2}{\left(1-\vec{v}\cdot\hat{R}/c\right)^6 c^4}$$

In the non-relativistic case, $v \ll c$,

$$\begin{aligned} \vec{S} &= \frac{1}{\mu_0\varepsilon_0^2 c}\hat{R}\left(\frac{q}{4\pi R}\right)^2 \frac{\left(\hat{R}\times\left[\hat{R}\times\vec{a}\right]\right)^2}{c^4} \\ &= \frac{1}{\varepsilon_0 c^3}\hat{R}\left(\frac{q}{4\pi R}\right)^2 a^2 \sin^2\theta \end{aligned}$$

where θ is the angle between $\hat{R}$ and $\vec{a}$. Notice that $\vec{S} \propto 1/R^2$. This is the usual inverse square law for light. We can compute the power radiated per unit solid angle:

$$\frac{dP}{d} = R^2 S = \frac{1}{\varepsilon_0 c^3}\left(\frac{q}{4\pi}\right)^2 a^2 \sin^2\theta \tag{28}$$

This is the Larmor formula for power radiated. The power is proportional to the square of the particle's acceleration. Notice that no power is radiated along the line of the acceleration ($\theta = 0$) and power radiated is maximum perpendicular to the acceleration. See LB page 763.

The total power radiated is

$$P = \int \frac{dP}{d} d \quad = \frac{1}{\varepsilon_0 c^3}\left(\frac{q}{4\pi}\right)^2 a^2 \int_0^\pi \sin^2\theta 2\pi \sin\theta d\theta$$

Let $\mu = \cos\theta$. then

$$\begin{aligned} P &= \frac{q^2}{8\pi\varepsilon_0 c^3}a^2 \int_{-1}^{+1}\left(1-\mu^2\right)d\mu = \frac{q^2}{8\pi\varepsilon_0 c^3}a^2\left(\mu - \frac{\mu^3}{3}\right)\Bigg|_{-1}^{+1} \\ &= \frac{q^2}{8\pi\varepsilon_0 c^3}a^2\frac{4}{3} = \frac{q^2}{6\pi\varepsilon_0 c^3}a^2 \end{aligned} \tag{29}$$

There are some interesting and important corrections when the motion is relativistic. Most importantly, the total power radiated is increased significantly, and it is also strongly beamed in the direction of the particle's velocity $\vec{v}$. With the relativistic factor γ defined as

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$$

so that $\gamma \simeq 1$ corresponds to non-relativistic motion, and $\gamma \to \infty$ as $v \to c$, we find that

$$P = \gamma^6 \frac{q^2}{6\pi\varepsilon_0 c^3}\left(a^2 - \frac{|\vec{v}\times\vec{a}|^2}{c^2}\right)$$

and the radiation is emitted within a cone of opening angle of about $1/\gamma$ around the direction of $\vec{v}$. The derivation of these results is not easy, and we shall omit it.