Physics 460   Fall 2006   Susan M. Lea

Let's start off by reviewing what we accomplished in Physics 360. We learned how to investigate the behavior of electromagnetic systems that are constant in time. We derived Maxwell's equations in the static limit. They are:

$$\text{Coulomb's law (Gauss's Law)} \quad : \quad \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \tag{1}$$

$$\text{Gauss's Law for } \vec{B} \quad : \quad \vec{\nabla}\cdot\vec{B} = 0 \tag{2}$$

$$\vec{\nabla}\times\vec{E} \quad = \quad 0 \tag{3}$$

$$\text{Ampere's Law} \quad : \quad \vec{\nabla}\times\vec{B} = \mu_0\vec{j} \tag{4}$$

We used these equations to derive the potentials V and $\vec{A}$ and the equations they satisfy:

$$\begin{aligned} \vec{\nabla}\times\vec{E} &= 0 \Rightarrow \vec{E} = -\vec{\nabla}V \\ \vec{\nabla}\cdot\vec{E} &= \frac{\rho}{\varepsilon_0} \Rightarrow \nabla^2 V = -\frac{\rho}{\varepsilon_0} \end{aligned}$$

$$\begin{aligned} \vec{\nabla}\cdot\vec{B} &= 0 \Rightarrow \vec{B} = \vec{\nabla}\times\vec{A} \\ \vec{\nabla}\times\vec{B} &= \mu_0\vec{j} \Rightarrow \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right) - \nabla^2\vec{A} = -\mu_0\vec{j} \end{aligned}$$

Here we add the Gauge condition

$$\text{Coulomb Gauge:} \quad \vec{\nabla}\cdot\vec{A} = 0$$

to obtain

$$\nabla^2\vec{A} = \mu_0\vec{j}$$

If we use Cartesian components, each component of $\vec{A}$ as well as V satisfies Poisson's equation:

$$\nabla^2(\text{function}) = (\text{source})$$

In a region where the sources are zero, we have Laplace's equation:

$$\nabla^2(\text{function}) = 0$$

and we learned how to solve this equation making use of boundary conditions at the edges of the region.

We also derived expressions for the electric energy density:

$$u_E = \frac{1}{2}\varepsilon_0 E^2$$

and the charge conservation equation

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0 \tag{5}$$

Finally, we showed how to use the auxiliary fields $\vec{D} = \varepsilon\vec{E}$ and $\vec{H} = \vec{B}/\mu$ to simplify discussion of fields in media.

Now it is time to expand our discussion to allow for time variation in the sources and the fields: we are to study **Electrodynamics.**

1 Current and resistance

1.1 Current

We start with some practical examples of systems where charges are moving. Every current consists of moving charges. As we found in Physics 360 (see 360notes12)

$$\vec{j} = ne\vec{v}$$

where n is the number density of charged particles, e the charge per particle, and $\vec{v}$ the velocity of each particle. Conductors are systems than contain charges free to move under the application of applied forces. Such systems usually exhibit *resistance*. Like mechanical systems with friction, a force is needed to start a particle moving, and the moving particles may reach a constant speed with the applied force balancing the frictional force. The electrical systems reach a state with constant current

$$\vec{j} = \sigma \frac{\vec{F}}{q} = \sigma \vec{f}$$

where

$$\vec{f} = \frac{\vec{F}}{q}$$

is the force per unit charge. The constant of proportionality σ is the conductivity

$$\sigma = \frac{1}{\rho}$$

and ρ is the resistivity. The resistivity of a good conductor like copper is around 10^{-8} ·m; for semiconductors like silicon the value is around 10^3, while for insulators like wood the value is around 10^{11}.

While any force can in principle cause a current, we are going to be interested in electromagnetic forces:

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

so that

$$\vec{j} = \sigma\left(\vec{E} + \vec{v} \times \vec{B}\right) \tag{6}$$

This relation is often called Ohm's law, although strictly it is not. What Ohm actually stated is that resistance is independent of current. This result follows from equation (6) if σ is independent of current. This is almost true for materials like copper, as we shall see. In most ordinary conductors we may often ignore the $\vec{v} \times \vec{B}$ term because v is small ($\ll c$), and the effects of $\vec{B}$ are often balanced by additional components of $\vec{E}$ (as in the Hall effect- see below).

1.2 Resistance

Let's find the relation between the resistance of a circuit component and its fundamental properties (shape, size, material). Let the resistor be made of uniform material of resistivity ρ, and have a length ℓ (parallel to $\vec{E}$) and a cross-sectional area A that is constant along its length. Then the current may be expressed in terms of the current density as

$$I = jA$$

(Remember: current = flux of current density)

$$I = jA = \sigma EA$$

Now we may express E (assumed uniform along the length ℓ) in terms of the potential difference:

$$E = \frac{\Delta V}{\ell}$$

and so

$$I = \frac{\sigma A}{\ell} \Delta V$$

or

$$\Delta V = I \left(\frac{\ell \rho}{A} \right) = IR \tag{7}$$

a more familiar version of "Ohm's Law". Thus the resistance is

$$R = \rho \frac{\ell}{A} \tag{8}$$

It is easy to understand this result: like water flowing down a pipe, current flows more easily when the area of the "pipe" (the circuit component) is greater, and less easily when the length is greater.

We made quite a few assumptions here, and we should verify them. First, is the electric field uniform? We would set up the circuit by applying a fixed potential difference ΔV using a battery or generator. Once the system reaches equilibrium, which it will do in a time of order ℓ/c, that leaves us with a static boundary value problem to solve. Let's ground one end of the resistor for simplicity. Then we have

$$\begin{aligned} V &= 0 \text{ at } z = 0 \\ V &= V_0 \text{ at } z = \ell \end{aligned}$$

and

$$\nabla^2 V = -\frac{\rho_q}{\varepsilon_0}$$

where I have used the subscript q on the charge density to distinguish it from resistivity. But what is ρ_q? Well, we found that in a static situation without currents, the charge density is zero inside a conductor. If the currents are steady (we have reached equilibrium) then the charge density, zero or not, is also constant. But then, from charge conservation (eqn 5)

$$\frac{\partial \rho_q}{\partial t} = 0 = -\vec{\nabla} \cdot \vec{j} = -\vec{\nabla} \cdot \left(\sigma \vec{E}\right)$$

and if the material is uniform so that $\sigma =$ constant, then, using equation (1)

$$0 = \vec{\nabla} \cdot \left(\sigma \vec{E}\right) = \sigma \vec{\nabla} \cdot \vec{E} = \sigma \frac{\rho_q}{\varepsilon_0}$$

Then the charge density **is** zero, and so V satisfies Laplace's equation. In addition, current flows along the cylinder, but not across the boundaries into the space outside, so

$$\vec{j} \cdot \hat{n} = \sigma \vec{E} \cdot \hat{n} = -\sigma \hat{n} \cdot \vec{\nabla} V = 0$$

Thus either V or its normal derivative is known everywhere on the surface of the resistor, so the uniqueness theorem tells us that there is one unique solution for the potential. A simple solution that satisfies all the conditions is

$$V = V_0 \frac{z}{\ell}$$

with a uniform $\vec{E}$

$$\vec{E} = -\vec{\nabla} V = -V_0 \frac{\hat{z}}{\ell}$$

as we assumed above.

We have proved a very powerful theorem:

In a material of *uniform* conductivity carrying a steady current, the charge density is zero.

Note that this does not prohibit non-zero *surface* charge density on the boundaries, and in general these surface charges are necessary for a self-consistent solution.

Now let's investigate what happens when the cross-sectional area is *not* a constant. Suppose the circuit component is a cylindrical shell with radii $a < b$, and we apply the potential difference so that the inner surface at $s = a$ has potential V_0 and the outer surface at $s = b$ has potential zero. Then the appropriate solution of Laplace's equation is (360notes8 page 8)

$$V = C_1 \ln s + C_2$$

To get $V = 0$ at $s = b$ and $V = V_0$ at $s = a$, we choose the constants as follows:

$$V = \frac{V_0}{\ln a/b} \ln \frac{\rho}{b}$$

with electric field

$$\vec{E} = -\vec{\nabla} V = -\frac{V_0}{\ln a/b} \frac{\hat{s}}{s} = \frac{V_0}{\ln b/a} \frac{\hat{s}}{s}$$

where the second expression has $\ln b/a > 0$, and thus shows that $\vec{E}$ points outward. The total current through a cylindrical surface of radius s

$$\begin{aligned} I &= \int \vec{j} \cdot \hat{n}\; dA = \sigma \int_0^\ell \int_0^{2\pi} \vec{E} \cdot \hat{s}\; s d\theta dz \\ &= \sigma \frac{V_0}{\ln b/a} \ell \int \frac{s d\theta}{s} = \sigma \frac{V_0 \ell}{\ln b/a} 2\pi \end{aligned}$$

The result is independent of s, showing that the current is constant throughout the resistor, as expected. The resistance is

$$R = \frac{\Delta V}{I} = \frac{V_0}{2\pi\sigma\ell V_0} \ln \frac{b}{a} = \rho \frac{\ln b/a}{2\pi\ell}$$

Griffiths' Example 7.2 introduces the charge per unit length on the cylinder, which is unnecessary and involves the additional assumption that the electric field inside the inner cylinder is zero. If the field for $s < a$ is zero, we could use our solution to find λ from the given value of V_0.

Both Griffiths and LB discuss the "Drude" model for conductivity, and you should definitely look at it, bearing in mind that modern quantum theories are quite a bit different. Graduate students should look up what Feynmann has to say on the subject.

As a result of the many collisions undergone by the moving electrons, the work done by the battery increases the thermal energy of the resistor. The amount of charge passing through potential difference ΔV in time t is

$$Q = It$$

and the work done by the battery is

$$W = Q\Delta V = It\Delta V$$

thus the power delivered to the charges is

$$P = \frac{W}{t} = I\Delta V = I^2 R = \frac{R^2}{\Delta V}$$

This power serves to heat the resistor, thus increasing the temperature and changing the resistance. This effect is used to advantage in electric toasters, hair dryers, and light bulbs, but is a nuisance in TV sets, for example. It is sometmes called Joule heating.

We should remember several important facts from this discussion of steady currents in circuits. First, each circuit has a self-consistent distribution of charge, with resulting fields and potential differences. Charge densities occur only where the electrical properties change– usually at the surface of wires, the ends of resistors, and similar places. As we saw with conductors in Phys 360, each charge moves a tiny distance to establish the equilibrium, and the timescale to establish equilibrium is the time for the fields to adjust, equal to (length scale of system)/(speed of light). The self-consistent fields serve to distribute the effects of batteries, for example, around the circuit.

2 EMF

2.1 Batteries

The influence of the external agent that drives the circuit (battery, photvoltaic cell, whatever) is often described by a quantity called EMF or elctro-motive force. It is not a force at all, but the line integral of the force per unit charge around the circuit:

$$\mathcal{E} = \oint \vec{f}_{\rm ext} \cdot d\vec{\ell}$$

Once the circuit reaches equilibrium (very fast! see above), the self-consistent electrostatic fields don't contribute, because (eqn 3)

$$\vec{\nabla} \times \vec{E} = 0 \Longleftrightarrow \oint \vec{E} \cdot d\vec{\ell} = 0$$

Thus $\mathcal{E}$ reflects the contribution of the external agent.

For an ideal battery with zero internal resistance, the net force on the charges inside the battery is zero. This force has two components, one electrical and chemical, so they have to balance.

$$\int_{\rm terminal\ a}^{\rm terminal\ b} \left(\vec{E} + \vec{f}_{\rm ext}\right) \cdot d\vec{\ell} = 0$$

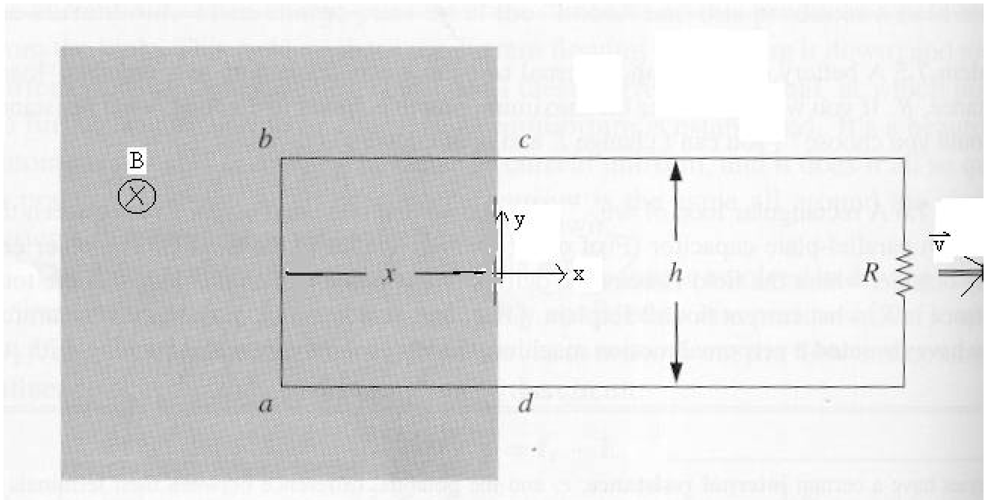
So the potential difference across the battery is

$$\begin{aligned} V_a - V_b &= \int_{\rm terminal\ a}^{\rm terminal\ b} \vec{E} \cdot d\vec{\ell} = -\int_{\rm terminal\ a}^{\rm terminal\ b} \vec{f}_{\rm ext} \cdot d\vec{\ell} \\ &= -\int_{\rm terminal\ a, inside\ battery}^{\rm terminal\ b} \vec{f}_{\rm ext} \cdot d\vec{\ell} - \int_{\rm terminal\ a, outside\ batteryt}^{\rm terminal\ b} \vec{f}_{\rm ext} \cdot d\vec{\ell} \\ &= \oint \vec{f}_{\rm ext} \cdot d\vec{\ell} = \mathcal{E} \end{aligned}$$

where we used the fact that $f_{\rm ext} = 0$ outside the battery to add the extra (zero) term in line 2. See LB §26.1 for batteries with internal resistance.

2.2 Motional emf

Motional emf arises whenever a conductor moves through a magnetic field, and is the basis for simple generators. To see how it works, let's consider a simple rectangular loop with a resistance R on one side, as in Griffiths Figure 7.10.



On side ab, every electron in the wire experiences a magnetic force per unit charge

$$\vec{f}_{\rm mag} = \vec{v} \times \vec{B} = vB\,[\hat{x} \times (-\hat{z})] = vB\hat{y}$$

Thus we have an EMF at the instant shown of

$$\mathcal{E} = \oint \vec{f}_{\rm mag} \cdot d\vec{\ell} = vBh$$

where the integral was taken clockwise around the loop. There is no contribution from other segments of the loop because either $\vec{B} = 0$ or $\vec{v} \times \vec{B}$ is perpendicular to $d\vec{l}$. This emf drives a current clockwise around the loop. The segment ab then experiences a magnetic force due to the current of

$$\vec{F} = \int_a^b I d\vec{l} \times \vec{B} = IhB\,(-\hat{x})$$

Segments $bc$and da experience forces too, but they are equal and opposite, and sum to zero. Thus the net force on the loop is in the minus-x direction, and if nothing else is done the motion of the loop stops, the EMF$\rightarrow 0$, and the current$\rightarrow 0$. But if we pull the loop to the right with a balancing force

$$\vec{F}_{\rm pull} = -\vec{F}_{\rm mag} = IhB\hat{x}$$

we can keep the motion going. We have to do work at a rate

$$P = \vec{F}_{\rm pull} \cdot \vec{v} = IhBv$$

The electrical power expended in the circuit is

$$P = I^2 R = I\mathcal{E} = IvBh$$

The two powers are equal, as they must be. Remember: magnetic force does no work!

Motional emf occurs whenever the size, shape or orientation of a loop changes and a magnetic field is present. It is one example of *Faraday's law,* which may be expressed in the more general form

$$|\mathcal{E}| = \left| \frac{d\Phi_B}{dt} \right| \tag{9}$$

(Notice the absolute value signs!) The magnetic flux through the loop is

$$\Phi_B = \int \vec{B} \cdot \hat{n}\, dA$$

where the integral is over a surface spanning the loop and $\hat{n}$ is the usual normal to the surface element dA. In our case, with normal chosen into the paper (the $-\hat{z}$ direction, parallel to $\vec{B}$) we have

$$\Phi_B = Bh\,(-x)$$

(Notice that x is the location of side ab with respect to the edge of the magnetic field region, and is negative). Thus

$$\frac{d\Phi_B}{dt} = -Bh\frac{dx}{dt} = -Bhv$$

Thus we get the right value for $|\mathcal{E}|$. Putting the signs back, we have

$$\mathcal{E} = \oint \vec{f}_{\text{mag}} \cdot d\vec{\ell} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\int \vec{B} \cdot \hat{n}\, dA \tag{10}$$

where now the directions of $\hat{n}$ and $d\vec{\ell}$ are related through the usual right hand rule.

The time derivative of flux is non-zero if $\vec{B}$ changes, $\hat{n}$ changes or the area changes. In our case the area with non-zero B is decreasing. But when $\vec{B}$ changes we don't call it motional emf any more.

Equation (10) is the integral form of Faraday's law, and it includes the case of a static loop with changing $\vec{B}$ as well as the motional emf's we have already discussed. To use it correctly, you must choose a direction for $d\vec{\ell}$ (that is, a direction for going around the loop) and then choose the direction for the normal $\hat{n}$ according to the right hand rule. Equivalently, you can use equation (9) to relate the magnitudes of the emf and the rate of change of flux, and use Lenz's law to get the directions right.

Lenz's law: The induced emf opposes the change that creates it.

Lenz's law is a statement of energy conservation: without it we could build perpetual motion machines that would run forever without any energy input.

When the loop is stationary but the magnetic field changes, there is no magnetic force because $\vec{v} = 0$. So what causes the emf? It is an electric field: an induced electric field whose source is the changing $\vec{B}$. In this case the emf is

$$\mathcal{E} = \oint \vec{E}_{\text{ind}} \cdot d\vec{\ell}$$

Now for the static (or Coulomb) electric fields we have previously discussed, according to equation 3,

$$\oint \vec{E}_{\text{coulomb}} \cdot d\vec{\ell} = 0$$

So we can add $\vec{E}_{\text{coulomb}}$ to get

$$\mathcal{E} = \oint_C \left(\vec{E}_{\text{ind}} + \vec{E}_{\text{coulomb}}\right) \cdot d\vec{\ell} = \oint_C \vec{E}_{\text{total}} \cdot d\vec{\ell} \tag{11}$$

and then Faraday's law becomes

$$\mathcal{E} = \oint_C \vec{E}_{\text{total}} \cdot d\vec{\ell} = -\frac{d}{dt}\int \vec{B} \cdot \hat{n}\ dA \tag{12}$$

But here it is important to note that the electric field is measured in the rest frame of the line segment $d\vec{\ell}$. It is the electric field felt by an electron in the wire that would actually cause the electron to move.

It is important to note that equation (12) applies to any curve C whether or not there is actually a wire there. It there is a conducting wire coincident with the curve, the emf will cause a current to flow. If there is no conducting wire, the induced electric field still exists, but no current flows.

2.3 Calculating induced electric field

The method for calculating induced electric field is outlined in LB §30.4. As with the use of Gauss' Law and Ampere's law in integral form, we can only find the induced $\vec{E}$ when there is sufficient symmetry. Let's look at an example.

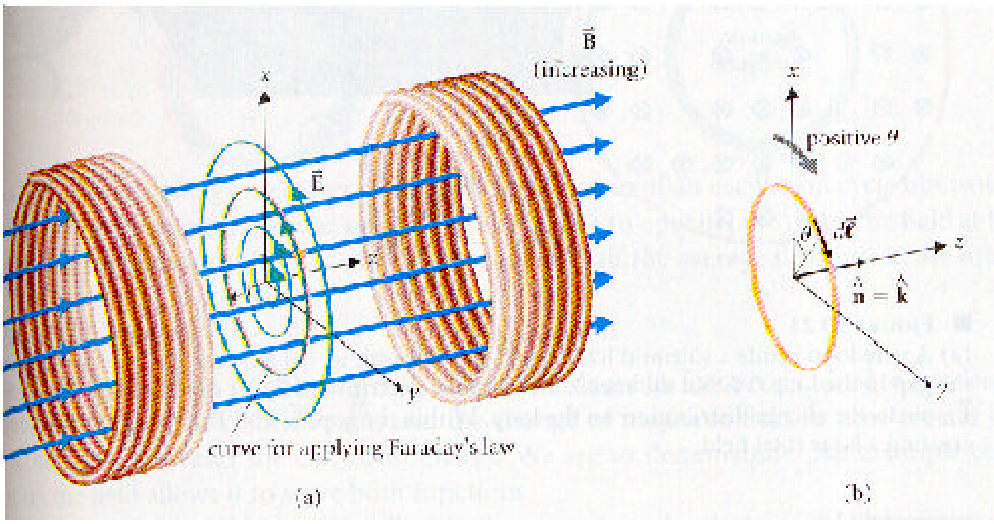
A solenoid with n turns per unit length carries a current I that is increasing at a constant rate dI/dt. The field inside a uniform solenoid is uniform and parallel to the solenoid axis, with magnitude $B = \mu_0 nI$. Thus as I increases, B increases too.

$$\frac{dB}{dt} = \mu_0 n \frac{dI}{dt}$$

Equation (12) shows that the induced field bears a similar relation to its source (changing magnetic flux) as magnetic field does to its source (current)

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 \int \vec{j} \cdot \hat{n}\ dA$$

There is a sign difference, which means that $\vec{E}_{\text{ind}}$ curls around the changing flux according to a left-hand rule. Thus our solenoid produces $\vec{E}$ in the same way that a wire with uniform $\vec{j}$ produces $\vec{B}$: the field lines form circles centered on the solenoid axis. If we rotate the solenoid about its axis, the picture doesn't change, so $\vec{E} = E_\theta\,(s)\,\hat{\theta}$



Then we place a circle with radius s centered on the axis of the solenoid, and go around it in the direction shown in (b).

$$\oint_C \vec{E} \cdot d\vec{\ell} = 2\pi s E_\theta$$

With this choice for going around C, $\hat{n} = \hat{z}$, and the flux is

$$\Phi_B = \pi s^2 B$$

then Faraday's law becomes

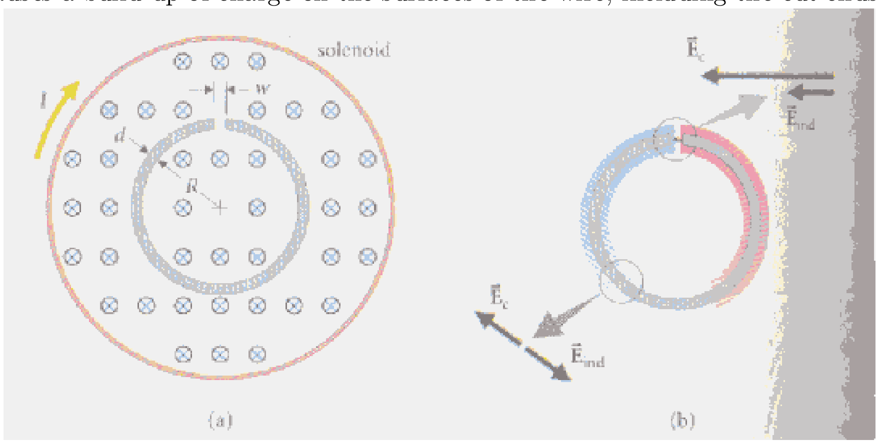
$$2\pi s E_\theta = -\pi s^2 \mu_0 n \frac{dI}{dt}$$

and

$$E_\theta = -\frac{s}{2}\mu_0 n \frac{dI}{dt}$$

The direction of $\vec{E}$ is $-\hat{\theta}$, as shown in the diagram. Notice that if we put a wire loop in the location of our curve, current would flow in the $-\hat{\theta}$ direction, and that current would produce $\vec{B}$ in the $-\hat{z}$ direction, thus reducing the rate at which $\vec{B}_{\text{inside}}$ increases. This is required by Lenz's law.

Something interesting happens if we use a wire loop with a small gap of width w in it (see LB pg 970). Now current cannot flow continuously because of the gap. There is a burst of current as we begin to increase I, and that current causes a build-up of charge on the surfaces of the wire, including the cut ends.



Once equilibrium is established, the net electric field inside the conducting material is zero:

$$\vec{E}_{\rm ind} + \vec{E}_{\rm coul} = 0$$

and thus

$$\vec{E}_{\rm coul} = -\vec{E}_{\rm induced}$$

The induced electric field is the same everywhere on the circle, but the Coulomb field changes direction in the gap, because

$$\oint_{\rm circle} \vec{E}_{\rm coul} \cdot d\vec{\ell} = 0$$

Thus the total electric field is very large in the gap. Applying Faraday's law:

$$\begin{aligned} \left| \oint_{\rm circle} \vec{E}_{\rm totall} \cdot d\vec{\ell} \right| &= \pi s^2 \mu_0 n \frac{dI}{dt} = \left| \int_{\rm gap} \vec{E}_{\rm total} \cdot d\vec{\ell} \right| \\ &\simeq E_{\rm total,gap}\, w \end{aligned}$$

and

$$E_{\rm total,gap} = \frac{\pi s^2 \mu_0 n}{w} \frac{dI}{dt}$$

Heinrich Hertz used a device like this as an antenna in his discovery of EM waves.

2.4 Differential form of Faraday's law

Now we want to get the differential equation that coresponds to equation (12).

$$\mathcal{E} = \oint_C \vec{E}_{\rm total} \cdot d\vec{\ell} = -\frac{d}{dt} \int \vec{B} \cdot \hat{n}\; dA$$

We start with a curve that is at rest, so the only thing that is changing is $\vec{B}$. Then we can move the time derivative inside the integral on the right, and apply Stokes' theorem to the integral on the left:

$$\oint_C \left(\vec{\nabla} \times \vec{E}_{\text{total}}\right) \cdot \hat{n}\ dA = -\int \frac{\partial}{\partial t}\vec{B} \cdot \hat{n}\ dA$$

Since this relation applies to *any curve* C, we must have

$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\vec{B} \tag{13}$$

This equation replaces equation (3), which is valid only for static fields.

To extend this to moving curves, let's start with a curve that has a uniform, constant, non-relativistic velocity $\vec{v}$. There is an additional contribution to the change in flux as the curve moves to a region with a differing value of $\vec{B}$. Using a Taylor series expansion, we have:

$$B\left(\vec{r} + \delta\vec{r}\right) = B\left(\vec{r} + \vec{v}\delta t\right) = \vec{B}\left(\vec{r}\right) + \left(\vec{v} \cdot \vec{\nabla}\right)\vec{B}\delta t + \cdots$$

Thus

$$B\left(\vec{r} + \delta\vec{r}\right) - \vec{B}\left(\vec{r}\right) = \delta\vec{B} = \left(\vec{v} \cdot \vec{\nabla}\right)\vec{B}\ \delta t$$

to first order in δt. Thus the change in flux due to motion of the curve is:

$$\delta\Phi_m = \int_S \delta\vec{B} \cdot \hat{n}\ dA = \int_S \left(\vec{v} \cdot \vec{\nabla}\right)\vec{B} \cdot \hat{n}\ dA\delta t$$

and hence

$$\frac{d}{dt}\Phi_m = \int_S \left(\vec{v} \cdot \vec{\nabla}\right)\vec{B} \cdot \hat{n}\ dA$$

due to motion of the curve. Adding the two contributions, we have

$$\left.\frac{d}{dt}\Phi_m\right|_{\text{total}} = \int_S \frac{\partial}{\partial t}\vec{B} \cdot \hat{n}\ dA + \int_S \left(\vec{v} \cdot \vec{\nabla}\right)\vec{B} \cdot \hat{n}\ dA$$

and applying Faraday's law:

$$\oint_C \vec{E}' \cdot d\vec{\ell} = -\left\{\int_S \frac{\partial}{\partial t}\vec{B} \cdot \hat{n}\ dA + \int_S \left(\vec{v} \cdot \vec{\nabla}\right)\vec{B} \cdot \hat{n}\ dA\right\}$$

Remember: the electric field $\vec{E}'$ is measured in the rest frame of $d\vec{\ell}$, i.e. the frame moving with velocity $\vec{v}$ with respect to the lab, and $\vec{B}$ is measured in the lab frame.

Now we want to convert the last term on the right to a line integral, so we use a result from the cover of Griffiths:

$$\vec{\nabla} \times \left(\vec{B} \times \vec{v}\right) = \left(\vec{v} \cdot \vec{\nabla}\right)\vec{B} - \left(\vec{B} \cdot \vec{\nabla}\right)\vec{v} + \vec{B}\left(\vec{\nabla} \cdot v\right) - \vec{v}\left(\vec{\nabla} \cdot \vec{B}\right)$$

But here we have chosen $\vec{v}$ to be constant, and from the second Maxwell equation, $\vec{\nabla}\cdot\vec{B}=0$, so only the first term on the right is non-zero:

$$\vec{\nabla}\times\left(\vec{B}\times\vec{v}\right)=\left(\vec{v}\cdot\vec{\nabla}\right)\vec{B}$$

Thus

$$\begin{aligned}\oint_C \vec{E}'\cdot d\vec{\ell} &= -\left\{\int_S \frac{\partial}{\partial t}\vec{B}\cdot\hat{n}\,dA+\int_S\left[\vec{\nabla}\times\left(\vec{B}\times\vec{v}\right)\right]\cdot\hat{n}\,dA\right\}\\ &= -\int_S \frac{\partial}{\partial t}\vec{B}\cdot\hat{n}\,dA-\int_C\left(\vec{B}\times\vec{v}\right)\cdot d\vec{\ell}\end{aligned}$$

where we used Stokes' theorem again in the second step. Comparing with equations (13), we may replace the integral of $\frac{\partial}{\partial t}\vec{B}$ with a line integral involving $\vec{E}$ in the lab frame[1]:

$$\oint_C \vec{E}'\cdot d\vec{\ell}=\oint_C \vec{E}\cdot d\vec{\ell}+\oint\left(\vec{v}\times\vec{B}\right)\cdot d\vec{\ell}=\oint \vec{f}\cdot d\vec{\ell}$$

where again the result is true for *any* curve C moving at constant velocity $\vec{v}$. Thus we obtain the transformation law:

$$\vec{E}'=\vec{E}+\vec{v}\times\vec{B}$$

The result is consistent with the Lorentz force law, and with our previous discussion of motional emf. Thus we have established that the constant of proportionality in Faraday's law is linked to the transformation properties of the electric field. We'll discuss this further later in the semester.

2.5 More on potential

Now that we have changed equation 3 to equation 13, we must rethink our ideas about potential. Since $\vec{\nabla}\times\vec{E}$ is no longer zero, we cannot conclude that $\vec{E}$ is the gradient of a scalar function. But equation (2) still holds, so we may still conclude that

$$\vec{B}=\vec{\nabla}\times\vec{A}$$

Now we insert this into Faraday's law:

$$\vec{\nabla}\times\vec{E}=-\frac{\partial}{\partial t}\vec{B}=\vec{\nabla}\times\vec{E}=-\frac{\partial}{\partial t}\vec{\nabla}\times\vec{A}=-\vec{\nabla}\times\frac{\partial}{\partial t}\vec{A}$$

Thus

$$\vec{\nabla}\times\left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right)=0$$

[1] Strictly, we must apply Faraday's law to a curve C' at rest in the lab that instantaneously coincides with the moving curve C.

Now we can conclude that the vector

$$\vec{E} + \frac{\partial \vec{A}}{\partial t}$$

is the gradient of a scalar function:

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} V$$

and thus

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} = \vec{E}_{\text{coulomb}} + \vec{E}_{\text{induced}}$$

We can confirm this decomposition of $\vec{E}$ by re-inserting this expression for $\vec{E}$ into Gauss' law:

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= -\nabla^2 V - \vec{\nabla} \cdot \frac{\partial \vec{A}}{\partial t} \\ &= -\nabla^2 V - \frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{A} = \frac{\rho}{\varepsilon_0} \end{aligned}$$

If we use the Coulomb gauge as we did in the static case, then $\vec{\nabla} \cdot \vec{A} = 0$, and we get

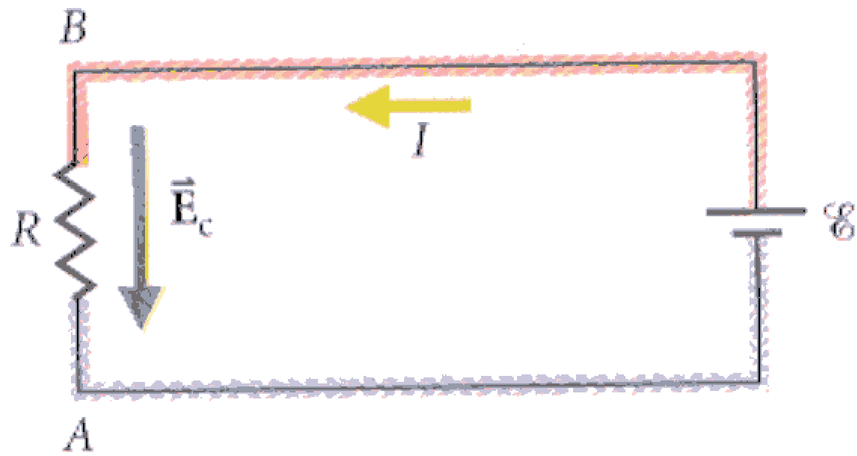
$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

This confirms that the source of V is the charge density, and thus $-\vec{\nabla} V$ is the Coulomb field. We may not always want to use this Gauge condition in time dependent cases, and we'll have more to say about this later, but the decomposition still holds.

This *mathematical* decomposition of $\vec{E}$ into induced and Coulomb fields is very useful, but of course if we put a test charge down and measure $\vec{E}$ as $\vec{F}/q_{\text{test}}$ we'll measure the whole $\vec{E}$: we won't be able to *measure* any difference between the two kinds of $\vec{E}$. It is important to remember that only Coulomb fields contribute to the scalar potential V :

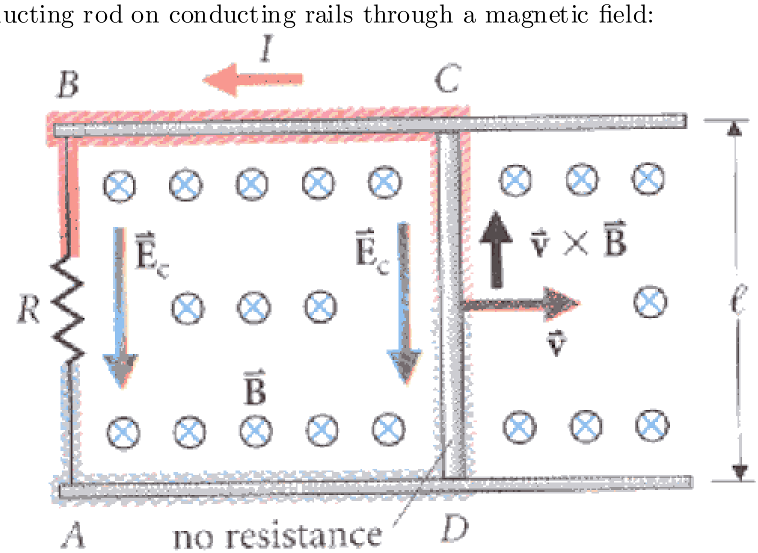
$$V_A - V_B = \int_A^B \vec{E}_{\text{Coulomb}} \cdot d\vec{\ell}$$

Now let's return to motional emf and see how this plays out.



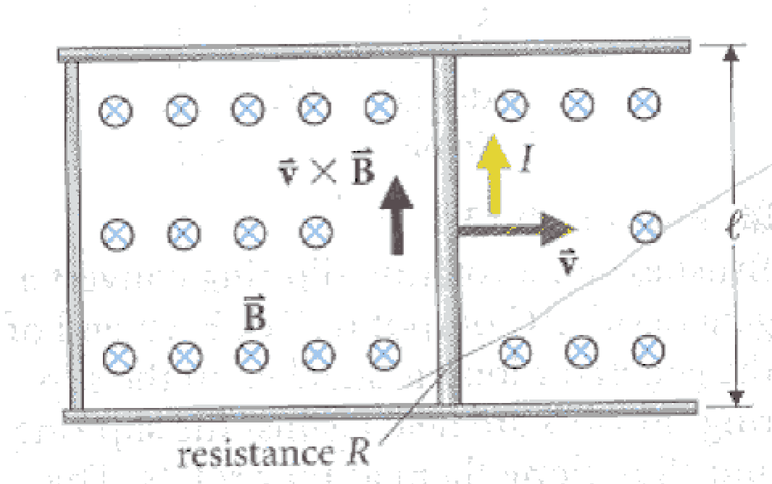
First look at a simple circuit with a battery and a resistor. In this circuit power flows from the battery and is used in the resistor (we'll worry about how it gets there later). There is a potential difference across the resistor: $\Delta V = IR = \mathcal{E}$.

Now look at a similar circuit in which the power source is a person pulling a conducting rod on conducting rails through a magnetic field:



There is a motional emf of magnitude $\mathcal{E} = \ell v B$. But the net electric field in the conducting rod on the right must be zero, so the magnetic force drives charges to the sides of the circuit as shown, positive on the top and negative on the bottom, leading to the production of a Coulomb electric field that in turn produces a potential difference across the resistor at the left. It is the electric field that drives current through the resistor.

Now we change the situation just a bit, by moving the resistance to the moving rod, as shown below:



The net force on a charge in the resistor is the magnetic force $\vec{F} = q\vec{v}\times\vec{B}$, and it is this force that drives current in the circuit. There are no charge distributions, no potentials, and no Coulomb electric fields. The net emf is of course the same as in the second circuit, but because the power is used in the *same place* as it is produced (in the moving rod), there is no need to transfer energy to another place in the circuit. The potential distribution in a circuit shows us how energy is stored and redistributed throughout the system.

3 Inductance

Capacitors are devices that store energy – electric field energy – in circuits. There are equivalent devices for storing magnetic field energy. They are *inductors.* Here's how it works. Suppose we have two current loops, one carrying current I_1 and one carrying current I_2. Each loop produces magnetic field, according to the Biot-Savart law. The magnetic field produced by loop 1 threads through both loop 1 and loop 2. B_1 is proportional to I_1 :

$$\vec{B}_1 = \frac{\mu_0}{4\pi} I_1 \oint \frac{d\vec{\ell}_1 \times \vec{R}}{R^2}$$

and thus the flux of $\vec{B}_1$ through loop 2 is also proportional to I_1 :

$$\Phi_{2\text{ due to }1} = \int_{S_2} \vec{B}_1 \cdot \hat{n}\, dA_2 = \frac{\mu_0}{4\pi} I_1 \int_{S_2} \left(\oint \frac{d\vec{\ell}_1 \times \vec{R}}{R^2} \right) \cdot \hat{n}\, dA_2 = M_{21} I_1$$

The constant of proportionality is called the mutual inductance M_{21}. We can express it more nicely using the vector potential and Stokes' theorem:

$$\Phi_{2\text{ due to }1} = \int_{S_2} \vec{B}_1 \cdot \hat{n}\, dA_2 = \int_{S_2} \left(\vec{\nabla} \times \vec{A}_1 \right) \cdot \hat{n}\, dA_2 = \oint_{C_2} \vec{A}_1 \cdot d\vec{\ell}_2 \qquad (14)$$

But we already have an expression for $\vec{A}$:

$$\vec{A}_1 = \frac{\mu_0}{4\pi} I_1 \oint_{C_1} \frac{d\vec{\ell}_1}{R}$$

and thus

$$M_{21} I_1 = \frac{\mu_0}{4\pi} I_1 \oint_{C_2} \oint_{C_1} \frac{d\vec{\ell}_1}{R} \cdot d\vec{\ell}_2$$

and thus

$$M_{21} = \frac{\mu_0}{4\pi} \oint_{C_2} \oint_{C_1} \frac{d\vec{\ell}_1 \cdot d\vec{\ell}_2}{R} \tag{15}$$

It is immediately clear from the symmetry of this expression that it doesn't matter which loop is labelled one and which is labelled two:

$$M_{21} = M_{12} = M$$

and it is also true that M is a purely geometrical property involving the size, shape and relative position and orientation of the two loops.

Now if we change I_1, we will change the flux of magnetic field through loop 2, and so there will be an emf induced in loop 2:

$$|\mathcal{E}_2| = \left|\frac{d\Phi_2}{dt}\right| = M \left|\frac{dI_1}{dt}\right|$$

Of course the flux through loop 1 also changes, and so there is also an induced emf in loop 1.

$$|\mathcal{E}_2| = \left|\frac{d\Phi_1}{dt}\right| = L \left|\frac{dI_1}{dt}\right|$$

where L is the self-inductance (or just inductance) of loop 1.

We can estimate the inductance of a very long solenoid of length ℓ, area A with N turns. The field inside is uniform and equals

$$B = \mu_0 n I$$

The flux through the solenoid is

$$\Phi = NBA = \mu_0 n N A I$$

and thus the inductance is

$$L = \frac{\Phi}{I} = \mu_0 n N A = \frac{\mu_0 N^2 A}{\ell} \tag{16}$$

4 Magnetic energy

Since the induced emf opposes the change that creates it (Lenz's law), the current in a circuit cannot jump instantly to its final value: it has to build up slowly. Let's look at a circuit with a resistor and an inductor (coil) and a battery with emf $\mathcal{E}_0$. If we want to apply Kirchhoff's loop rule, we can only use values of potential V, not induced emfs. But if we model the coil as made of perfectly conducting wire, then the total (Coulomb plus induced) field inside the wire is zero. Thus

$$\vec{E}_{\rm Coul} = -\vec{E}_{\rm induced}$$

and integrating along the wire we get

$$\int_a^b \vec{E}_{\rm Coul} \cdot d\vec{\ell} = -\int_a^b \vec{E}_{\rm induced} \cdot d\vec{\ell}$$

or

$$V_a - V_b = -\mathcal{E}_{\rm induced} = L\frac{dI}{dt}$$

To be sure we have the signs right, let's review the physics. The induced electric field tries to oppose the increase of I in the direction of $d\vec{\ell}$, so the Coulomb field points in the same direction as the line segment $d\vec{\ell}$, and since V decreases along field lines, the potential is higher at the end a of the coil at which the current enters, and is lower at b where the current leaves.

Now we are ready to use the loop rule:

$$\mathcal{E}_0 - IR - L\frac{dI}{dt} = 0$$

Choose a new variable $x = I - \mathcal{E}_0/R$. Then since $\mathcal{E}_0/R$ is constant,

$$\frac{L}{R}\frac{dx}{dt} = -x$$

which has solution

$$x = x_0 e^{-Rt/L}$$

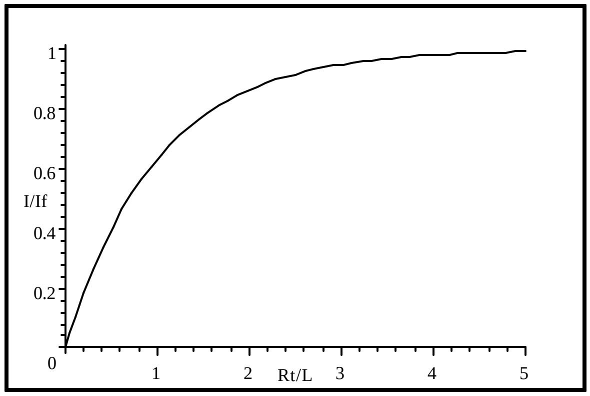
At time $t = 0$, $x = x_0 = 0 - \mathcal{E}_0/R$, so

$$I - \mathcal{E}_0/R = -\frac{\mathcal{E}_0}{R}\exp\left(-\frac{Rt}{L}\right)$$

and

$$I = \frac{\mathcal{E}_0}{R}\left[1 - \exp\left(-\frac{Rt}{L}\right)\right]$$

The current exponentially approaches its final value $I_f = \mathcal{E}_0/R$. The timescale $\tau = L/R$ governs how fast it approaches the final value. While it theoretically takes infinite time to get to I_f, within 3τ the current is within $e^{-3} = 5\%$ of the final value.



Now as the current is building up, the battery is pumping energy into the circuit at a rate

$$P_{\text{battery}} = \mathcal{E}_0 I = \frac{\mathcal{E}_0^2}{R}\left[1 - \exp\left(-\frac{Rt}{L}\right)\right]$$

and the resistor is using energy at a rate

$$P_{\text{resistor}} = I^2 R = \frac{\mathcal{E}_0^2}{R}\left[1 - \exp\left(-\frac{Rt}{L}\right)\right]^2$$

and these two rates are not the same. After a total time T, the energy put out by the battery is

$$\begin{aligned} U_{\text{battery}} &= \int_0^T P_{\text{battery}} dt = \frac{\mathcal{E}_0^2}{R}\int_0^T \left[1 - \exp\left(-\frac{Rt}{L}\right)\right] dt \\ &= \frac{\mathcal{E}_0^2}{R}\left[T + \frac{L}{R}\exp\left(-\frac{Rt}{L}\right)\Big|_0^T\right] \\ &= \frac{\mathcal{E}_0^2}{R}\left\{T + \frac{L}{R}\left[\exp\left(-\frac{RT}{L}\right) - 1\right]\right\} \end{aligned}$$

while the energy used by the resistor is

$$\begin{aligned} U_{\text{resistor}} &= \int_0^T P_{\text{resistor}} dt = \frac{\mathcal{E}_0^2}{R}\int_0^T \left[1 - \exp\left(-\frac{Rt}{L}\right)\right]^2 dt \\ &= \frac{\mathcal{E}_0^2}{R}\int_0^T \left[1 - 2\exp\left(-\frac{Rt}{L}\right) + \exp\left(-2\frac{Rt}{L}\right)\right] dt \\ &= \frac{\mathcal{E}_0^2}{R}\left\{T + 2\frac{L}{R}\left[\exp\left(-\frac{RT}{L}\right) - 1\right] - \frac{L}{2R}\left[\exp\left(-2\frac{RT}{L}\right) - 1\right]\right\} \end{aligned}$$

So we have

$$\begin{aligned} U_{\text{battery}} - U_{\text{resistor}} &= \frac{\mathcal{E}_0^2}{2R^2}L\left[1 - 2\exp\left(-\frac{RT}{L}\right) - \exp\left(-2\frac{RT}{L}\right)\right] \\ &= \frac{1}{2}LI^2 \end{aligned} \tag{17}$$

This energy is stored in the inductor in the magnetic field. Indeed using equation (16) for the inductance of a solenoid,

$$\begin{aligned}\Delta U &= \frac{1}{2}\frac{\mu_0 N^2 A}{\ell} I^2 \\ &= \frac{1}{2}\frac{\mu_0 N^2 A l}{\ell^2} I^2 = \frac{1}{2}\mu_0 n^2 I^2 V \\ &= \frac{1}{2}\frac{B^2}{\mu_0} V = u_B V\end{aligned}$$

where $V = Al$ is the volume of the solenoid. Thus the magnetic energy density is

$$u_B = \frac{1}{2}\frac{B^2}{\mu_0} \tag{18}$$

and thus the total energy ensity is

$$u = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu^0}\right) \tag{19}$$

We can get this result more generally, starting from (17) and using (14)

$$U = \frac{1}{2} L I^2 = \frac{1}{2}\Phi I = \frac{I}{2}\oint \vec{A}\cdot d\vec{\ell}$$

Now recall that we can get a more general expression by replacing $I d\vec{\ell}$ with $\vec{j}\, d\tau$. Then

$$U = \frac{1}{2}\int \vec{A}\cdot\vec{j}\; d\tau \tag{20}$$

and then from Ampere's law (4)

$$U = \frac{1}{2\mu_0}\int \vec{A}\cdot\left(\vec{\nabla}\times\vec{B}\right)\; d\tau$$

But

$$\vec{\nabla}\cdot\left(\vec{A}\times\vec{B}\right) = \vec{B}\cdot\left(\vec{\nabla}\times\vec{A}\right) - \vec{A}\cdot\left(\vec{\nabla}\times\vec{B}\right)$$

so

$$U = \frac{1}{2\mu_0}\int \left[\vec{B}\cdot\left(\vec{\nabla}\times\vec{A}\right) - \vec{\nabla}\cdot\left(\vec{A}\times\vec{B}\right)\right]\; d\tau$$

We use the divergence theorem on the second term to get

$$U = \frac{1}{2\mu_0}\left[\int_V \vec{B}\cdot\vec{B}\; d\tau - \int_S \left(\vec{A}\times\vec{B}\right)\cdot\hat{n}\; dA\right]$$

The integral is over all space, and as long as our current is confined to a finite region, then $\left|\vec{A}\right| \to 0$ at least as fast as $1/R^2$ and B goes as $1/R^3$ (Remember rule one for magnetic fields- the dominant term is a dipole!) Thus the surface integral is zero, and we have

$$U = \frac{1}{2\mu_0}\int_V B^2\; d\tau$$

5 Maxwell's equations

The equations we have derived so far are

$$\text{Coulomb's law (Gauss's Law):} \quad \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \tag{21}$$

$$\text{Gauss's Law for } \vec{B} \quad : \quad \vec{\nabla} \cdot \vec{B} = 0 \tag{22}$$

$$\text{Faraday's law} \quad : \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{23}$$

$$\text{Ampere's Law} \quad : \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{j} \tag{24}$$

There is a nice symmetry about equations (21) and (22), once we remember that there are no magnetic charges. But we seem to be missing something in equation (24) because there is no time derivative. In fact we can prove that something is missing. Take the divergence of both sides:

$$\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{B} \right) = \mu_0 \vec{\nabla} \cdot \vec{j}$$

The left hand side is zero, because the divergence of a curl is always zero. On the right hand side, we use the charge conservation relation (5) to get

$$0 = \mu_0 \left(-\frac{\partial \rho}{\partial t} \right)$$

which is clearly false if $\partial \rho / \partial t$ is not zero. We can see how to fix the problem by taking the time derivative of equation (21):

$$\frac{1}{\varepsilon_0} \frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t} \left(\vec{\nabla} \cdot \vec{E} \right)$$

Thus if we add a term $\mu_0 \varepsilon_0 \partial \vec{E} / \partial t$ we will have a fully self-consistent set of equations, and a nice symmetry in the two curl equations.

$$\text{Ampere-Maxwell Law:} \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{j} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{25}$$

The new term $\varepsilon_0 \partial \vec{E} / \partial t$ is called the displacement current. Maxwell was the first to notice the discrepancy and fix it, and so the law is now named for him as well as for Ampere.

5.1 Maxwell's equations in matter

We have already introduced the fields $\vec{D} = \varepsilon \vec{E} = \varepsilon_0 \vec{E} + \vec{P}$ and $\vec{H} = \vec{B}/\mu = \frac{\vec{B}}{\mu_0} - \vec{M}$ and showed how they can simplify the static form of Maxwell's equations by allowing us to ignore the explicit dependence on bound charges and currents. We obtained:

$$\vec{\nabla} \cdot \vec{D} = \rho_f$$

while the equivalent magnetic equation needs no change:

$$\vec{\nabla} \cdot \vec{B} = 0$$

Faraday's law does not involve charges and currents, so it is also unchanged:

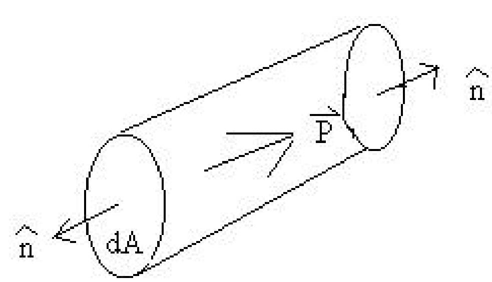
$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

But Ampere's law involves change of $\vec{E}$, which is related to change of bound as well as free charge.

Recall that

$$\rho_b = -\vec{\nabla} \cdot \vec{P}$$

If we look at a tiny volume of polarized material, it will have a surface bound charge density $\sigma_p = \vec{P} \cdot \hat{n}$ at each end. If we now let $\vec{P}$ change a little bit, the charge on each end also increases a little bit, as if a current flowed from one end to the other:



$$I\delta t = \vec{j}_B \cdot \hat{n}\ dA\delta t = \delta\sigma_b dA = \delta\vec{P} \cdot \hat{n}\ dA$$

Thus the "polarization" current is

$$\vec{j}_p = \frac{\partial \vec{P}}{\partial t}$$

This current satisfies the same charge conservation law as regular current:

$$\vec{\nabla} \cdot \vec{j}_p = \vec{\nabla} \cdot \frac{\partial \vec{P}}{\partial t} = \frac{\partial}{\partial t}(-\rho_b)$$

Thus we have four contributions to current:

conduction ("free") current due to moving charges: $\vec{j}_f$

magnetization current: $\vec{j}_{\text{mag}} = \vec{\nabla} \times \vec{M}$

$$\text{polarization current:} \quad \vec{j}_p = \frac{\partial \vec{P}}{\partial t}$$

$$\text{displacement current:} \ \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Putting all of these into Ampere's law, we have

$$\begin{aligned} \vec{\nabla} \times \vec{B} &= \mu_0 \left(\vec{j}_f + \vec{j}_{\text{mag}} + \vec{j}_p + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \\ &= \mu_0 \left(\vec{j}_f + \vec{\nabla} \times \vec{M} + \frac{\partial \vec{P}}{\partial t} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \end{aligned}$$

Now we combine the second term with the LHS and the third term with the last term, to get:

$$\begin{aligned} \vec{\nabla} \times \left(\frac{\vec{B}}{\mu_0} - \vec{M} \right) &= \vec{j}_f + \frac{\partial}{\partial t} \left(\vec{P} + \varepsilon_0 \vec{E} \right) \\ \vec{\nabla} \times \vec{H} &= \vec{j}_f + \frac{\partial}{\partial t} \vec{D} \end{aligned} \tag{26}$$

5.2 Boundary conditions for time-dependent fields

We do not need to rederive the boundary conditions for the divergence equations since they have no time-dependent terms. (Remember divergence→tuna-can→bc for normal component) With $\hat{n}$ pointing from medium 2 into medium 1, we had

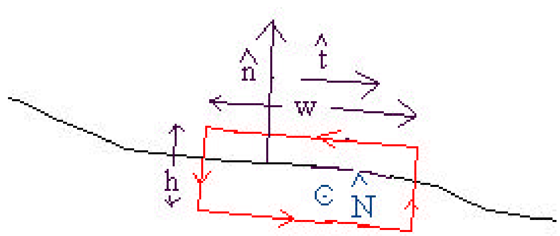
$$\left(\vec{D}_1 - \vec{D}_2 \right) \cdot \hat{n} = \sigma_f$$

and

$$\vec{B} \cdot \hat{n} \text{ is continuous}$$

Now we have to look at the curl equations. Once again the rule is curl→rectangle→bc for tangential components. Starting with Faraday's law:

$$\int_{\text{rectangle}} \left(\vec{\nabla} \times \vec{E} \right) \cdot \hat{N} dA = - \int \frac{\partial \vec{B}}{\partial t} \cdot \hat{N} dA$$



We use Stokes' theorem on the left, to get

$$\int_{\text{rectangle}} \vec{E}\cdot d\vec{\ell} = -\left(\vec{E}_1 - \vec{E}_2\right)\cdot \hat{t}\ell = -\frac{\partial \vec{B}}{\partial t}\cdot \hat{N}\ell w$$

Now as we let $w \to 0$, the RHS$\to 0$ because $\partial\vec{B}/\partial t$ must remain finite. Thus we obtain the same relation as before:

$$\left(\vec{E}_1 - \vec{E}_2\right)\times \hat{n} \text{ is continuous}$$

where $\hat{t} = \hat{n}\times\hat{N}$. Finally we look at the Ampere-Maxwell law:

$$\begin{aligned}
\int_{\text{rectangle}} \left(\vec{\nabla}\times\vec{H}\right)\cdot\hat{N}dA &= \int_{\text{rectangle}} \left(\vec{j}_f + \frac{\partial}{\partial t}\vec{D}\right)\cdot\hat{N}dA \\
-\left(\vec{H}_1 - \vec{H}_2\right)\cdot\hat{t}\ell &= \ell\int \vec{j}_f dw\cdot\hat{N} - \frac{\partial\vec{D}}{\partial t}\cdot\hat{N}\ell w \\
-\left(\vec{H}_1 - \vec{H}_2\right)\cdot\hat{t} &= \vec{K}_f\cdot\hat{N} - \frac{\partial\vec{D}}{\partial t}\cdot\hat{N}w
\end{aligned}$$

where

$$\vec{K}_f = \int \vec{j}_f \, dw$$

is the surface free current density, and again the second term on the right $\to 0$ as $w \to 0$, since the time derivative must remain finite. Thus

$$\begin{aligned}
-\left(\vec{H}_1 - \vec{H}_2\right)\cdot\hat{t} &= \vec{K}_f\cdot\left(\hat{t}\times\hat{n}\right) \\
&= -\left(\vec{K}_f\times\hat{n}\right)\cdot\hat{t}
\end{aligned}$$

So, since $\hat{t}$ is arbitrary,

$$\vec{H}_1 - \vec{H}_2 = \vec{K}_f\times\hat{n}$$