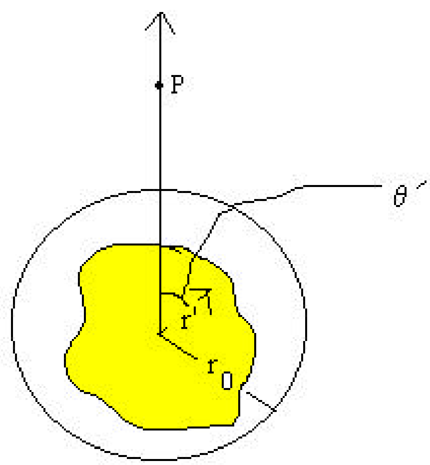
Multipole expansions

We have frequently referred to our RULE 1: far enough away from any charge distribution with net charge Q, the potential is approximately that due to a point charge Q located in the distribution. We also found from our very first example that the next correction is a dipole. The dipole potential falls off faster ($\propto 1/r^2$) than the point charge (or monopole) potential ($\propto 1/r$). Now we'd like to make these ideas more precise.

We have a charge distribution with charge density $\rho(\vec{r}')$. We put the origin somewhere inside the distribution, and we put the polar axis in a spherical coordinate system through a point P at which we want to find the potential. The entire charge distribution is located inside a sphere of radius r_0 and P is outside that sphere.



Then the potential at P is

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}d\tau' \tag{1}$$

where

$$|\vec{r}-\vec{r}'| = \sqrt{r^2 + (r')^2 - 2rr'\cos\theta'}$$

Since $r > r_0$ and $r' < r_0$, we factor out the r to get

$$|\vec{r}-\vec{r}'| = r\sqrt{1 + \left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\cos\theta'} = r\sqrt{1+\varepsilon}$$

where

$$|\varepsilon| = \left|\frac{r'}{r}\left(\frac{r'}{r} - 2\cos\theta'\right)\right| < 1$$

In fact, as P gets farther from the charge distribution, ε becomes much less than 1. So we expand the square root

$$\begin{aligned}\frac{1}{\sqrt{1+\varepsilon}} &= (1+\varepsilon)^{-1/2} = 1 - \frac{1}{2}\varepsilon + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}\varepsilon^2 + \cdots \\ &= 1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 - \frac{5}{16}\varepsilon^3 + \cdots\end{aligned}$$

So

$$\begin{aligned}\frac{1}{|\vec{r}-\vec{r}'|} &= \frac{1}{r}\left[1 - \frac{1}{2}\frac{r'}{r}\left(\frac{r'}{r} - 2\cos\theta'\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\theta'\right)^2 - \frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r} - 2\cos\theta'\right)^3 + \cdots\right] \\ &= \frac{1}{r}\left[\begin{array}{c} 1 + \frac{r'\cos\theta'}{r} - \frac{1}{2}\left(\frac{r'}{r}\right)^2 + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\cos\theta' + 4\cos^2\theta'\right) \\ -\frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r} - 2\cos\theta'\right)^3 + \cdots \end{array}\right] \\ &= \frac{1}{r}\left[\begin{array}{c} 1 + \frac{r'\cos\theta'}{r} + \frac{1}{2}\left(\frac{r'}{r}\right)^2\left(3\cos^2\theta' - 1\right) + \\ \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\cos\theta'\right) - \frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r} - 2\cos\theta'\right)^3 + \cdots \end{array}\right] \\ &= \frac{1}{r}\left[1 + \frac{r'}{r}P_1(\mu') + \left(\frac{r'}{r}\right)^2 P_2(\mu') + \cdots\right]\end{aligned}$$

I'll let you check the next few. In fact

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\mu')$$

Now we put this result into our integral (1) for the potential:

$$\begin{aligned}V(\vec{r}) &= \frac{1}{4\pi\varepsilon_0}\int \rho(\vec{r}')\frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\mu')\,d\tau' \\ &= \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\sum_{l=0}^{\infty}\frac{1}{r^l}\int \rho(\vec{r}')(r')^l P_l(\mu')\,d\tau' \qquad (2)\end{aligned}$$

This is the multipole expansion of the potential at P due to the charge distribution. The first few terms are:

$$l = 0: \quad \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\int \rho(\vec{r}')\,d\tau' = \frac{Q}{4\pi\varepsilon_0 r}$$

This is our RULE 1. The monople moment (the total charge Q) is indendent of our choice of origin. The potential does depend on the origin (because r does) but only weakly if $r \gg r_0$.

$$l = 1: \quad \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\int \rho(\vec{r}')\,r'\cos\theta'\;d\tau'$$

This is the dipole potential.

$$l=2: \quad \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\int \rho\left(\vec{r}'\right)\left(r'\right)^2 P_2\left(\mu'\right)d\tau'$$

This is the quadrupole potential.

Each succeeding term decreases faster with r and so becomes less important as P gets further from the origin.

These expressions depend on the particular coordinate system that we have chosen. Most importantly, P is on the polar axis. So let's see if we can write the results in a coordinate independent way. Note that

$$\cos\theta' = \frac{\hat{r}\cdot\vec{r}'}{r'}$$

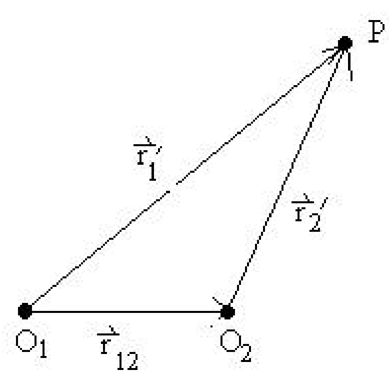
so the dipole potential is

$$\begin{aligned} V_{\text{dipole}}\left(\vec{r}\right) &= \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\int \rho\left(\vec{r}'\right)r'\frac{\hat{r}\cdot\vec{r}'}{r'}\,d\tau' \\ &= \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{r}\cdot\int \rho\left(\vec{r}'\right)\vec{r}'\,d\tau' \\ &= \frac{1}{4\pi\varepsilon_0 r^2}\hat{r}\cdot\vec{p} \end{aligned} \tag{3}$$

where

$$\vec{p} = \int \rho\left(\vec{r}'\right)\vec{r}'\,d\tau' \tag{4}$$

is the dipole moment of the charge distribution. Notice that this integral may depend on the choice of origin, because $\vec{r}'$ does. However, if the total charge Q is zero, then $\vec{p}$ is independent of origin. To see this, let $\vec{p}_1$ be the dipole moment with respect to origin 1, $\vec{p}_2$ with respect to origin 2, and let $\vec{r}_{12}$ be the position of origin 2 with respect to origin 1. Then



$$
\begin{aligned}
\vec{p}_1 &= \int \rho(\vec{r}')\,\vec{r}_1'\ d\tau' = \int \rho(\vec{r}')\,(\vec{r}_{12} + \vec{r}_2')\ d\tau' \\
&= \vec{r}_{12}\int \rho(\vec{r}')\ d\tau' + \int \rho(\vec{r}')\,\vec{r}_2'\ d\tau' \\
&= Q\vec{r}_{12} + \vec{p}_2 \qquad (5)
\end{aligned}
$$

So if $Q = 0$, then $\vec{p}_1 = \vec{p}_2$.

This is an example of a more general result:

The first non-zero multipole moment is independent of origin.

Result (5) also explains the results we obtained in our very first example for the dipole moment of the two point charges. A charge that is not at the origin but at position $\vec{r}_Q$ contributes a dipole moment $Q\vec{r}_Q$ with respect to that origin.

An ideal or "pure" dipole is located at a single point. That is, it is the dipole moment of two equal and opposite point charges separated by a distance d in the limit that $d \to 0$. In order that $\vec{p}$ not be zero, we have to let $q \to \infty$.

$$\vec{p} = \lim_{q\to\infty}\lim_{d\to 0} q\vec{d}$$

where, as we take the limit, we hold the product $qd = p$ constant. The vector $\vec{d}$ points from the negative charge to the positive charge in the pair. (Griffiths uses the term "pure", but I don't like it. I think "ideal" is more appropriate.)

Equation (2) gives the potential *at a point on the polar axis* as a series in powers of $1/r$. It does not give us the potential at other points. However, the dipole potential (3) is valid everywhere. It may be written

$$V_{\text{dipole}}(r,\theta) = \frac{1}{4\pi\varepsilon_0 r^2}\hat{r}\cdot\vec{p} = \frac{1}{4\pi\varepsilon_0 r^2} p\cos\theta$$

where θ is the angle between $\vec{p}$ and $\vec{r}$, that is, it is the polar angle in a coordinate system with polar axis along $\vec{p}$. Then we have

$$V_{\text{dipole}}(r,\theta) = \frac{p}{4\pi\varepsilon_0 r^2} P_1(\cos\theta) \qquad (6)$$

The dipole itself, remember, has $z-$component (from 2 with $l = 1$)

$$p_z = \int \rho(\vec{r}')\, r' P_1(\mu')\, d\tau'$$

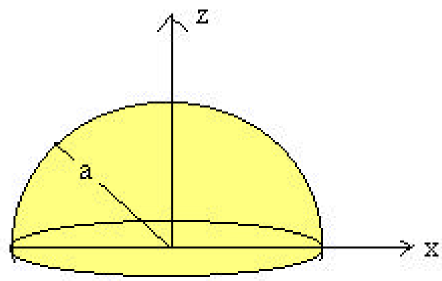
and it is not coincidental that P_1 shows up again in (6). In fact, *if our charge distribution has azimuthal symmetry* about an axis that we choose as our polar axis, then we may write:

$$V(r,\theta) = \frac{1}{4\pi\varepsilon_0 r}\sum_{l=0}^{\infty}\frac{P_l(\cos\theta)}{r^l}\int \rho(\vec{r}')\,(r')^l\, P_l(\mu')\, d\tau' \qquad (7)$$

We still do not have a completely general result, and that will have to wait until Physics 704.

Example:

A hemisphere of radius a contains charge density $\rho = \rho_0 \frac{z}{a} + \rho_1 \frac{r^2}{a^2}$. Find the monopole, dipole and quadrupole moments of this charge distribution, and hence find the potential at distance $r \gg a$ from the hemisphere.



The monopole is the total charge.

$$\begin{aligned}
Q &= \int \rho(r,\theta)\, 2\pi r^2 \sin\theta d\theta dr \\
&= 2\pi \int_0^a r^2 dr \int_0^1 \left(\rho_0 \frac{r\mu}{a} + \rho_1 \frac{r^2}{a^2}\right) d\mu \\
&= 2\pi \int_0^a r^2 dr \left(\rho_0 \frac{r\mu^2}{2a} + \rho_1 \frac{r^2}{a^2}\mu\right)\Big|_0^1 \\
&= 2\pi \left(\rho_0 \frac{r^4}{8a} + \rho_1 \frac{r^5}{5a^2}\right) \\
&= 2\pi a^3 \left(\frac{\rho_0}{8} + \frac{\rho_1}{5}\right)
\end{aligned}$$

The dipole moment is

$$\vec{p} = \int \rho \vec{r} d\tau = \int \rho(r,\theta)(z\hat{z} + x\hat{x} + y\hat{y})\, r^2 \sin\theta d\theta d\phi dr$$

Only the $z-$component is non-zero, because

$$x = r \sin\cos\phi$$

and

$$\int_0^{2\pi} \cos\phi d\phi = 0$$

The $y-$component vanishes similarly.

Then

$$\begin{aligned}
p_z &= 2\pi \int_0^a \int_0^1 \left(\rho_0 \frac{r\mu}{a} + \rho_1 \frac{r^2}{a^2}\right)(r\mu)\, r^2 d\mu dr \\
&= 2\pi \int_0^a \int_0^1 \left(\rho_0 \frac{r^4\mu^2}{a} + \rho_1 \frac{r^5\mu}{a^2}\right) d\mu dr \\
&= 2\pi \int_0^a \left(\rho_0 \frac{r^4}{3a} + \rho_1 \frac{r^5}{2a^2}\right) dr \\
&= 2\pi a^4 \left(\frac{\rho_0}{15} + \frac{\rho_1}{12}\right) \\
&= \frac{2\pi a^4}{3}\left(\frac{\rho_0}{5} + \frac{\rho_1}{4}\right)
\end{aligned}$$

Finally the quadrupole is

$$\begin{aligned}
q_{zz} &= \int \rho(\vec{r})\, r^2 P_2(\mu)\, d\tau \\
&= 2\pi \int_0^a \int_0^1 \left(\rho_0 \frac{r\mu}{a} + \rho_1 \frac{r^2}{a^2}\right) r^2 \frac{1}{2}\left(3\mu^2 - 1\right) r^2 dr d\mu \\
&= \pi \int_0^a \int_0^1 \left(\rho_0 \frac{r^5}{a}\left(3\mu^3 - \mu\right) + \rho_1 \frac{r^6}{a^2}\left(3\mu^2 - 1\right)\right) dr d\mu \\
&= \pi \int_0^a \left(\rho_0 \frac{r^5}{a}\left(\frac{3}{4} - \frac{1}{2}\right) + \rho_1 \frac{r^6}{a^2}(1-1)\right) dr \\
&= \pi a^5 \frac{\rho_0}{6}\left(\frac{1}{4}\right) = \pi a^5 \frac{\rho_0}{24}
\end{aligned}$$

Thus the potential is

$$\begin{aligned}
V(r,\theta) &= \frac{1}{4\pi\varepsilon_0}\left\{\frac{Q}{r} + \frac{p_z \cos\theta}{r^2} + \frac{q_{zz}}{r^3} P_2(\cos\theta) + \cdots\right\} \\
&= \frac{\pi a^3}{4\pi\varepsilon_0 r}\left\{2\left(\frac{\rho_0}{8} + \frac{\rho_1}{5}\right) + \frac{2a}{3r}\left(\frac{\rho_0}{5} + \frac{\rho_1}{4}\right)\cos\theta + \frac{\rho_0}{24}\frac{a^2}{r^2}\frac{1}{2}\left(3\cos^2\theta - 1\right) + \cdots\right\} \\
&= \frac{\pi a^3}{4\pi\varepsilon_0 r}\left\{\frac{\rho_0}{4} + 2\frac{\rho_1}{5} + \frac{2a}{3r}\left(\frac{\rho_0}{5} + \frac{\rho_1}{4}\right)\cos\theta + \frac{\rho_0}{48}\frac{a^2}{r^2}\left(3\cos^2\theta - 1\right) + \cdots\right\}
\end{aligned}$$

Are there more terms? Yes there are.

$$\int \left(\rho_0 \frac{r\mu}{a} + \rho_1 \frac{r^2}{a^2}\right) r^l P_l(\mu)\, d\tau = \frac{2\pi}{a}\int_0^a r^{l+3} dr \int_0^1 \left[\rho_0 \mu P_l(\mu) + \rho_1 \frac{r}{a} P_l(\mu)\right] d\mu$$

Now if l is even, then $P_l(\mu)$ is an even function of μ, but $\mu P_l(\mu)$ is odd. But if l is odd, then $\mu P_l(\mu)$ is even. For an even function

$$\int_0^1 f(\mu)\, d\mu = \frac{1}{2}\int_{-1}^1 f(\mu)\, d\mu$$

Thus for l even

$$\int_0^1 P_l(\mu)\, d\mu = \frac{1}{2}\int_{-1}^1 P_0(\mu) P_l(\mu)\, d\mu = 0 \text{ unless } l = 0$$

We already found that this is true for $l = 2$ above. Then for l odd

$$\int_0^1 \mu P_l(\mu)\, d\mu = \frac{1}{2}\int_{-1}^1 P_1(\mu) P_l(\mu)\, d\mu = 0 \text{ unless } l = 1$$

Thus there are higher multipoles, but for $l > 1$, all even multipoles involve only ρ_0 but all odd multipoles involve only ρ_1.

Plot

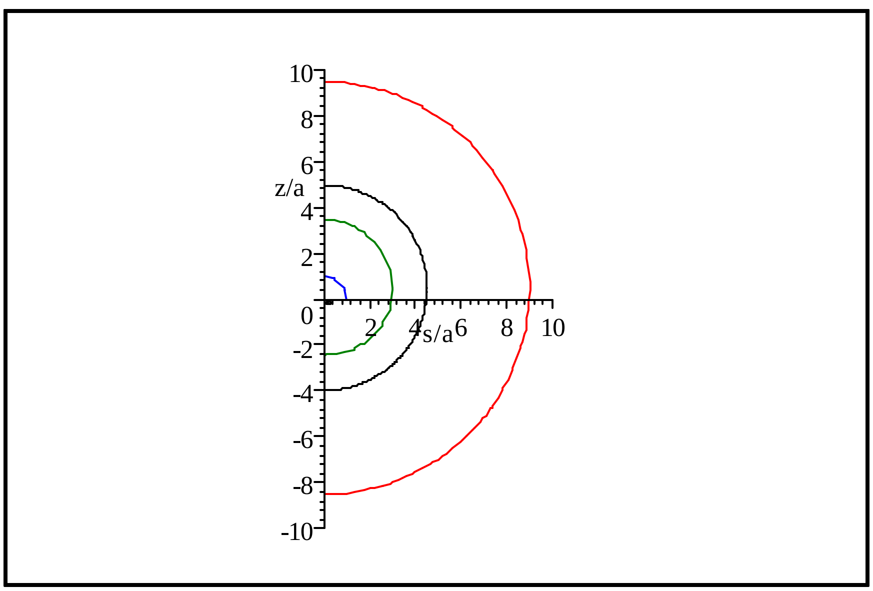
$$V(r,\theta) = \frac{\pi a^3 \rho_0 a}{4\pi\varepsilon_0 a r}\left\{\frac{1}{4} + 2\frac{\rho_1}{5\rho_0} + \frac{2a}{3r}\left(\frac{1}{5} + \frac{\rho_1}{4\rho_0}\right)\cos\theta + \frac{1}{48}\frac{a^2}{r^2}\left(3\cos^2\theta - 1\right) + \cdots\right\}$$

Thus

$$\frac{V(r,\theta)}{\frac{\pi a^3\rho_0}{4\pi\varepsilon_0 a}} = \left(\frac{1}{4} + 2\frac{\rho_1}{5\rho_0}\right)\frac{a}{r} + \frac{2a^2}{3r^2}\left(\frac{1}{5} + \frac{\rho_1}{4\rho_0}\right)\cos\theta + \frac{1}{48}\frac{a^3}{r^3}\left(3\cos^2\theta - 1\right) + \cdots$$

Now suppose $\rho_1 = \rho_0/2$. Then

$$\begin{aligned}\frac{V(r,\theta)}{\frac{\pi a^3\rho_0}{4\pi\varepsilon_0 a}} &= \left(\frac{1}{4} + \frac{1}{5}\right)\frac{a}{r} + \frac{2a^2}{3r^2}\left(\frac{1}{5} + \frac{1}{8}\right)\cos\theta + \frac{1}{48}\frac{a^3}{r^3}\left(3\cos^2\theta - 1\right) + \cdots \\ &= \frac{9}{20}\frac{a}{r} + \frac{13}{60}\frac{a^2}{r^2}\cos\theta + \frac{1}{48}\frac{a^3}{r^3}\left(3\cos^2\theta - 1\right) + \cdots\end{aligned}$$



Green 0.15, Black 0.1 red 0.05

Notice how the equipotential surfaces get more spherical as distance from the hemisphere (in blue) increases.