Separation of variables in spherical coordinates

The previous solutions worked well because we were concerned with a region whose boundaries were easily described in Cartesian coordinates. The boundaries were flat and corresponded to constant values of x, y or z. If our boundary is a sphere $r =$ constant, the region is most easily described in spherical coordinates.

Laplace's equation in spherical coordinates is

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

This looks like a mess, because the three coordinates r, θ, ϕ are all mixed up in each term but the first. We are going to make life a bit easier for ourselves by restircting attention to problems that have azimuthal symmetry: that is, there is one axis that we can rotate about without changing our system at all. We make this axis the polar axis in our spherical coordinate system. Then the potential is independent of ϕ and the last term in our equation is identically zero, leaving:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) = 0$$

or, multiplying by r^2, we have

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) = 0$$

Now we can look for a separable solution of the form

$$V(r,\theta) = R(r)\, P(\theta)$$

Stuffing in, we get

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) P + \frac{R}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial P}{\partial \theta}\right) = 0$$

and dividing by $V = RP$, we have the separated equation

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + \frac{1}{P \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial P}{\partial \theta}\right) = 0 \tag{1}$$

Now we make an argument similar to the one we made in Cartesian coordinates. Suppose we move around the surface of a sphere at constant radius, changing θ but not r. We could change the second term without changing the first. Thus both terms must be constants:

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) = k \tag{2}$$

$$\frac{1}{P \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial P}{\partial \theta}\right) = -k \tag{3}$$

The two constants are equal and opposite because they have to sum to zero. But what do we choose for k? Referring to our standard method, we have to first find the solutions to these ordinary differential equations. It turns out that the key to choosing k comes from the $\theta-$equation.

In spherical coordinates if it often convenient to use the variable $\mu = \cos\theta$. Then our volume element $r^2 \sin\theta d\theta d\phi dr = -r^2 d\mu d\phi dr$. This change of variable also simplifies the differential equation since

$$\frac{d}{d\mu} = \frac{1}{-\sin\theta}\frac{d}{\partial\theta}$$

Then equation (3) becomes

$$\frac{d}{d\mu}\left[\left(1-\mu^2\right)\frac{dP}{d\mu}\right] + kP = 0$$

We can solve this equation using a series method. We express the solution P as a power series in μ.

$$P = \sum_{n=0}^{\infty} a_n \mu^n$$

Then

$$\begin{aligned}
\frac{dP}{d\mu} &= \sum_{n=0}^{\infty} n a_n \mu^{n-1} \\
\frac{d^2P}{d\mu^2} &= \sum_{n=0}^{\infty} n(n-1) a_n \mu^{n-2}
\end{aligned}$$

and the differential equation is

$$\begin{aligned}
\left(1-\mu^2\right)\frac{d^2P}{d\mu^2} - 2\mu\frac{dP}{d\mu} + kP &= 0 \\
\left(1-\mu^2\right)\sum_{n=0}^{\infty} n(n-1) a_n\mu^{n-2} - 2\mu\sum_{n=0}^{\infty} n a_n\mu^{n-1} + k\sum_{n=0}^{\infty} a_n\mu^n &= 0 \\
\sum_{n=0}^{\infty} n(n-1) a_n\mu^{n-2} - \sum_{n=0}^{\infty} n(n-1) a_n\mu^{n} - 2\sum_{n=0}^{\infty} n a_n\mu^{n} + k\sum_{n=0}^{\infty} a_n\mu^n &= 0
\end{aligned}$$

With an equation like this, the coefficient of every power of μ must separately equal zero. So we start with μ^0 and work our way up. We get μ^0 in the first term if $n = 2$, in the second and third terms if $n = 0$ (but these term are identically zero), and in the last terms if $n = 0$. So the coefficient of μ^0 is

$$2 \times 1 a_2 - 0 - 0 + k a_0 = 0$$

So

$$a_2 = \frac{-k a_0}{2}$$

Next we look at $\mu^1 = \mu$

$$\begin{aligned} 3 \times 2a_3 - 0 - 2a_1 + ka_1 &= 0 \\ a_3 &= \frac{(2-k)\,a_1}{6} \end{aligned}$$

And for μ^2

$$\begin{aligned} 4 \times 3a_4 - 2 \times 1a_2 - 2 \times 2a_2 + ka_2 &= 0 \\ a_4 &= \frac{a_2\,(2 \times 3 - k)}{4 \times 3} = -\frac{(6-k)}{12}\frac{k}{2}a_0 \end{aligned}$$

and in general for μ^p

$$\begin{aligned} (p+2)\,(p+1)\,a_{p+2} - p\,(p-1)\,a_p - 2pa_p + ka_p &= 0 \\ a_{p+2} &= a_p\frac{p\,(p+1) - k}{(p+2)\,(p+1)} \qquad (4) \end{aligned}$$

If we start our solution witth a non-zero a_0, we will get a solution with only even powers of μ, but if we start with a non-zero a_1, we will get a solution with odd powers. These are the two solutions to our second order equation. One solution is even in μ and one is odd in μ.

The ratio of successive terms in the series is

$$\frac{a_{p+2}\mu^{p+2}}{a^p\mu^p} = \frac{p\,(p+1) - k}{(p+2)\,(p+1)}\mu^2$$

For $\mu < 1$,

$$\frac{p\,(p+1) - k}{(p+2)\,(p+1)}\mu^2 \to \mu^2 < 1 \text{ as } p \to \infty$$

so the series converges nicely. But if $\mu = \pm 1$,

$$\frac{p\,(p+1) - k}{(p+2)\,(p+1)}\mu^2 \to \mu^2 = 1$$

While the ratio test is not definitive in this case, it turns out that this series does *not* converge for $\mu = \pm 1$, that is for $\theta = 0$ and $\theta = \pi$. But there is no physical reason for the potenial to blow up along this line (the polar axis). So what do we do? We force the series to terminate after a finite number of terms by choosing our constant k to equal $l\,(l+1)$ for some integer l. Then equation (4) gives

$$a_{l+2} = a_l\frac{l\,(l+1) - l\,(l+1)}{(l+2)\,(l+1)} = 0$$

and since a_{l+4} is proportional to a_{l+2} it is zero too. In fact, all the a_p with $p > l$ are zero. So the series stops with the term in μ^l. The resulting solutions are the Legendre polynomials. The constant a_0 (or a_1) is not determined from the differential equation, so we *define* each polynomial to have the value 1 when its argument is 1, and that fixes a_0. The first few polynomials are:

$$\begin{aligned}
P_0(\mu) &= 1 \\
P_1(\mu) &= \mu \\
P_2(\mu) &= \frac{1}{2}\left(3\mu^2 - 1\right) \\
P_3(\mu) &= \frac{1}{2}\left(5\mu^3 - 3\mu\right)
\end{aligned}$$

and so on. You should verify that these polynomials satsify the differential equation.

If l is even, we get a polynomial in even powers. The second solution that starts with a_1 will never terminate since $p(p+1)$ with p odd can never equal $l(l+1)$ with l even. Thus this solution diverges on the polar axis and we will not want to do use this solution.

Ok so now we know that $k = l(l+1)$, so the equation (2) for R is

$$\begin{aligned}
\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) &= k = l(l+1) \\
\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) &= l(l+1)R
\end{aligned}$$

This equation is satisfied by a single power of r, $R = r^p$, for then

$$\begin{aligned}
\frac{dR}{dr} &= pr^{p-1} \\
r^2\frac{dR}{dr} &= pr^{p+1} \\
\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) &= (p+1)pr^p = p(p+1)R = l(l+1)R
\end{aligned}$$

So clearly $R = r^l$ is one solution. The second solution has $p = -(l+1)$, $p+1 = -l$ so that $p(p+1) = -(l+1)(-l) = (l+1)l$. Thus the azimuthally symmetric solution to Laplace's equation is

$$\left(Ar^l + \frac{B}{r^{l+1}}\right)P_l(\cos\theta)$$

where l may be any integer. Then as in the Cartesian case we form the general solution as a linear combination of these terms with different l :

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta) \tag{5}$$

Special cases:

$l = 0$ The solution looks like

$$A + \frac{B}{r}$$

The first term is a constant potential, and the second is the potential due to a point charge at the origin.

$l = 1$

$$Ar\cos\theta + \frac{B\cos\theta}{r^2}$$

The first term is Az and this gives us a uniform field $\vec{E} = -\vec{\nabla}V = -A\hat{z}$. The second term is a dipole potential with the dipole aligned along the $z-$axis.

The positive powers of r diverge as $r \to \infty$ while the negative powers diverge as $r \to 0$. Thus if our region R is the inside of a sphere, we'll want to use the positive power solutions and not the negative powers. The exception is if there is some charge at the origin that causes the potential to diverge there. On the other hand, if our region R is outside a spherical boundary, we will have a solution with negative powers and no positive powers. The exception would be the $l = 1$ term that allows for a uniform field outside the sphere.

If we know the potential $V_0(\theta)$ on a spherical surface $r = a$, the potential outside the sphere is

$$V(r > a, \theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

with

$$V_0(\theta) = V(a, \theta) = \sum_{l=0}^{\infty} \frac{B_l}{a^{l+1}} P_l(\cos\theta)$$

We can find the coefficients B_l because the Legendre Polynomials also have the properties of completeness and orthogonality that the sine functions have. The integral

$$\int_{-1}^{+1} P_l(\mu) P_m(\mu) d\mu = \int_0^{\infty} P_l(\cos\theta) P_m(\cos\theta)\sin\theta d\theta = 0 \text{ if } l \neq m$$

If $l = m$ the integral is:

$$\int_{-1}^{+1} [P_l(\mu)]^2 d\mu = \int_0^{\pi} [P_l(\cos\theta)]^2 \sin\theta d\theta = \frac{2}{2l+1}$$

Thus:

$$\begin{aligned}\int_{-1}^{+1} V_0(\theta) P_m(\mu) d\mu &= \int_{-1}^{+1} \sum_{l=0}^{\infty} \frac{B_l}{a^{l+1}} P_l(\mu) P_m(\mu) d\mu = \sum_{l=0}^{\infty} \frac{B_l}{a^{l+1}} \int_{-1}^{+1} P_l(\mu) P_m(\mu) d\mu \\ &= \frac{B_m}{a^{m+1}} \frac{2}{2m+1}\end{aligned}$$

So in principle we have a solution.

If the potential on the surface is $V_0(\theta) = V_0\cos^2\theta$, we have

$$B_m = a^{m+1}\frac{2m+1}{2} V_0 \int_{-1}^{+1} \mu^2 P_m(\mu) d\mu$$

Now note that

$$P_2(\mu) = \frac{1}{2}\left(3\mu^2 - 1\right)$$

so

$$\mu^2 = \frac{2}{3}P_2(\mu) + \frac{1}{3} = \frac{2}{3}P_2(\mu) + \frac{1}{3}P_0(\mu)$$

Then

$$\int_{-1}^{+1} \mu^2 P_m(\mu)\, d\mu = \int_{-1}^{+1} \left[\frac{2}{3}P_2(\mu) + \frac{1}{3}P_0(\mu)\right] P_m(\mu)\, d\mu$$

The integral is zero unless $m = 0$ or $m = 2$.

For $m = 0$:

$$\int_{-1}^{+1} \left[\frac{2}{3}P_2(\mu) + \frac{1}{3}P_0(\mu)\right] P_0(\mu)\, d\mu = 0 + \frac{1}{3}\frac{2}{2\times 0 + 1} = \frac{2}{3}$$

giving

$$B_0 = \frac{V_0 a}{2}\frac{2}{3} = \frac{V_0 a}{3}$$

and for $m = 2$

$$\int_{-1}^{+1} \left[\frac{2}{3}P_2(\mu) + \frac{1}{3}P_0(\mu)\right] P_2(\mu)\, d\mu = \frac{2}{3}\left(\frac{2}{2\times 2 + 1}\right) + 0 = \frac{4}{15}$$

giving

$$B_2 = V_0 a^3 \frac{5}{2}\frac{4}{15} = \frac{2}{3}V_0 a^3$$

So the potential function is:

$$\begin{aligned} V(r,\theta) &= \frac{V_0}{3}\left(\frac{a}{r} + 2\frac{a^3}{r^3}P_2(\cos\theta)\right) \\ &= \frac{V_0}{3}\left(\frac{a}{r} + 2\frac{a^3}{r^3}\frac{1}{2}\left(3\cos^2\theta - 1\right)\right) \\ &= \frac{V_0}{3}\left[\frac{a}{r} + \frac{a^3}{r^3}\left(3\cos^2\theta - 1\right)\right] \end{aligned}$$

Check that this solution does have the correct value at $r = a$.

Another example

An uncharged, conducting sphere is placed in a region where the electric field is uniform, $\vec{E} = \vec{E}_0$. Find the electric field in the region after the sphere is put in place.

First let's see if we can figure out what happens. We know that the field inside the conducting sphere must be zero, and that charges will move to the surface of the sphere to make this happen. The electric field lines outside the sphere must begin or end on these surface charges, and the field lines meet the sphere at right angles. The charge distribution is odd in θ, so we expect l odd.

We place the polar axis along the direction of the uniform field. Then the system has azimuthal symmetry about this line, and the potential outside the sphere may be expressed as (5)

$$V(r > a, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

There are only two non-zero A_l. The $l = 0$ term gives us a constant potential A_0. With $l = 1$ we get the uniform field.

$$A_1 r P_1(\cos\theta) = -E_0 z \Rightarrow A_1 = -E_0$$

The other A_l are all zero so that the field approaches $\vec{E}_0$ as $r \to \infty$. Now we have

$$V(r > a, \theta) = A_0 - E_0 r \cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

The next boundary condition is that V =constant at $r = a$. We may put our reference point on the sphere and choose that constant to be zero. Then

$$\begin{aligned} 0 &= A_0 - E_0 a P_1(\theta) + \sum_{l=0}^{\infty} \frac{B_l}{a^{l+1}} P_l(\cos\theta) \\ &= A_0 + \frac{B_0}{a} + \left(\frac{B_1}{a^2} - E_0 a \right) P_1(\cos\theta) + \sum_{l=2}^{\infty} \frac{B_l}{a^{l+1}} P_l(\cos\theta) \end{aligned}$$

Now since the P_l are orthogonal functions, the coefficient of each one must separately equal zero. We can show this by muliplying the whole equation by $P_m(\mu)$ and integrating from -1 to $+1$. Only the terms in P_m survive the integration.

Uisng this principle, we can see right away that

$$\begin{aligned} B_l &= 0 \text{ for } l > 1 \\ B_1 &= E_0 a^3 \\ B_0 &= -a A_0 \end{aligned}$$

Note here that the B_0 term gives us the "point charge" potential, but using RULE 1 and the fact that this sphere is uncharged, we must have $B_0 = A_0 = 0$. Thus our solution is

$$V(r > a, \theta) = -E_0 r \cos\theta + E_0 a \left(\frac{a}{r} \right)^2 \cos\theta$$

The second term is a dipole potential, which is not surprising because our qualitative picture shows the charge separating with positive charge on one hemisphere of the sphere and negative on the other.

Cylindrical coordinates- 2 dimensions

If our system is 2-dimensional (planar, or infinite in z) with boundaries that are circles (r =constant) or planes (ϕ =constant) we should use polar coordinates in a plane:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2}=0$$

Using the usual method, we look for a solution of the form

$$V=R(r)\,W(\phi)$$

so that

$$\frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)+\frac{1}{W}\frac{\partial^2 W}{\partial \phi^2}=0$$

The first term is a function of r only and the second a function of ϕ only, so

$$\begin{aligned}\frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) &= k\\ \frac{1}{W}\frac{\partial^2 W}{\partial \phi^2} &= -k\end{aligned}$$

If our region includes a full circle, then each 2π range of the angle ϕ gives the same region of physical space. The coordinates (r,ϕ) and $(r,\phi+2\pi)$ describe the same point. So our function W must be periodic with period 2π. So we choose k to be m^2 where m is an integer. Then the solution for W is

$$W=A_m\sin m\phi+B_m\cos m\phi$$

Then the equation for R is

$$r\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)=m^2R$$

The solution is a power $R=r^p$ where

$$r\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)=r\frac{\partial}{\partial r}\left(rpr^{p-1}\right)=rp^2r^{p-1}=p^2R$$

so

$$p=\pm m$$

So one term in our solution looks like

$$\left(A_m\sin m\phi+B_m\cos m\phi\right)\left(r^m+C_mr^{-m}\right)$$

But wait a minute– a line charge potential $V=-\frac{\lambda}{2\pi\varepsilon_0}\ln r$ should be one of our solutions! Oh, yes, we forgot the special case $m=0$. For this case

$$\begin{aligned}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) &= 0\\ r\frac{\partial R}{\partial r} &= C_0\\ R &= C_0\ln\rho+K_0\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 W}{\partial \phi^2} &= 0 \\ W &= \alpha_0 + \beta_0 \phi\end{aligned}$$

Since the function ϕ is not periodic we can use it only if our region is an angular wedge of space, with boundaries at $\phi = \theta_1$ and $\phi = \theta_2$.