Separation of variables

The idea here is to try to find a solution to Laplace's equation that is a product of functions, each of which depends on only one of the variables. We start with a 2-D problem in Cartesian coordinates. Then we look for a solution of the form

$$V\left(x,y\right)=X\left(x\right)Y\left(y\right)$$

where

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=\frac{\partial^2 X}{\partial x^2}Y+X\frac{\partial^2 Y}{\partial y^2}=0$$

Now divide the whole equation by $V=XY$, and we have

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}+\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}=0 \tag{1}$$

Here the first term is a function of x but not y and the second is a function of y but not x. The equation must be satisfied for all values of x and y in our region. Suppose we have satisfied the equation at some point x_0, y_0. If we move along a line at constant $x=x_0$ while changing y, we could change the value of the second term but leave the first unchanged. Thus the equation would not be satisfied at $x=x_0$, $y\neq y_0$ unless the second term were constant. Thus we must have

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}=k$$

where k is a constant, and then we must also have

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}=-k$$

so that equation (1) remains satisfied. We have replaced one, linear, second order partial differential equation with two, coupled, linear, ordinary, second order differential equations. Both ordinary differential equations are of the form

$$\frac{d^2 w}{du^2}=Cw \tag{2}$$

where $C=\pm k$. If C is positive, $C=\beta^2$, then the solution is an exponential function

$$w=Ae^{\beta u}+Be^{-\beta u} \tag{3}$$

while if C is negative, $C=-\alpha^2$, the solution is a combination of sine and cosine:

$$w=A\sin\left(\alpha u\right)+B\cos\left(\alpha u\right) \tag{4}$$

Here is where the boundary conditions become important. There are several differences between the solutions (3) and (4). The sines and cosines in (4) are periodic and take the value zero twice every period. On the other hand the exponential functions in (3) are not periodic and do not take the value zero anywhere. (We can form the linear combination $w=\sinh\alpha u=\frac{1}{2}\left(e^{\alpha u}-e^{-\alpha u}\right)$

that is zero at one, and only one, point $u = 0$.) One exponential function ($e^{\beta u}$) is unbounded as $u \to \infty$; all the other functions remain bounded. Further, we know that any reasonably well-behaved function on the range $0 \le u \le 2\pi$ may be expanded in a Fourier series

$$f(u) = \sum_{n=0}^{\infty} a_n \sin nu + b_n \cos nu$$

(This property of sines and cosines is called "completeness".) We can use these properties to figure out what the solution must be.

We start with a region that is a rectangular box measuring $a \times b$. Its length in the third dimension is either infinite or zero. We put the x and $y-$axes along the two finite sides of the box. Then the potential is independent of z. Now suppose the sides at $x = 0$ and $x = a$ and at $y = 0$ are all grounded, but the potential on the side at $y = b$ is non-zero $V(x, b) = V_0(x)$.

Since the potential is zero at *two* values of x, we must choose the periodic functions as the apropriate solutions for X. That means that we need k to be positive, $k = \alpha^2$. Further, if the solution is zero at $x = 0$, the correct solution is the sine. Then we have

$$X(x) = \sin(\alpha x) = 0 \text{ at } x = 0$$

Now to make the solution zero at $x = a$, we need

$$\sin \alpha a = 0 = \sin n\pi \Rightarrow \alpha = \frac{n\pi}{a}$$

for any integer n. Then the equation for Y becomes

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = k = \alpha^2$$

with solution

$$Y = Ae^{\alpha y} + Be^{-\alpha y}$$

Notice that since we chose k to be positive, one set of solutions (here X) are sines and cosines while the other set (Y) are exponentials. This is forced upon us because the sum of the two constants *must be zero* so that the partial differential equation is satisfied. It is never possible for both sets of functions to be sines and cosines.

Now we take the other surface on which $V = 0$ and make our solution satisfy that boundary condition. At $y = 0$ we have

$$A + B = 0 \Rightarrow B = -A$$

and then

$$Y = A\left(e^{\alpha y} - e^{-\alpha y}\right) = 2A \sinh \alpha y = 2A \sinh \frac{n\pi y}{a}$$

So far we have a solution

$$XY = \sinh \frac{n\pi y}{a} \sin \frac{n\pi x}{a}$$

This function satisfies the differential equation and the boundary conditions on 3 of the four sides of the box. This is a solution to the differential equation, not *the* solution to the problem, because it does not satisfy the remaining boundary condition at $y = b$. But we can have any integer value for n. Thus the solution we need is a linear combination of such functions with different values of n :

$$V(x, y) = \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi y}{a} \sin \frac{n\pi x}{a}$$

We have one boundary condition to go. On the side at $y = b$:

$$V(x, b) = \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi x}{a} = V_0(x) \tag{5}$$

Thus

$$A_n \sinh \frac{n\pi b}{a}$$

is the coefficient in the Fourier sine series for $V_0(x)$. Now we proceed using the usual method for finding the coefficients in a Fourier series. We use a property of the sines called orthogonality:

$$\int_0^a \sin \frac{n\pi x}{a} \sin \frac{m\pi x}{a} dx = \begin{cases} 0 & if \quad n \neq m \\ \frac{a}{2} & if \quad n = m \end{cases} \tag{6}$$

So we multiply both sides of equation (5) by $\sin m\pi x/a$ and integrate from 0 to a.

$$\int_0^a \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi x}{a} \sin \frac{m\pi x}{a} dx = \int_0^a V_0(x) \sin \frac{m\pi x}{a} dx$$

We interchange the sum and the integral on the left. (This is legal. See Lea Ch 6 for the reasons why.)

$$\sum_{n=1}^{\infty} A_n \sinh \frac{n\pi b}{a} \int_0^a \sin \frac{n\pi x}{a} \sin \frac{m\pi x}{a} dx = \int_0^a V_0(x) \sin \frac{m\pi x}{a} dx$$

and use result (6). The integral on the left is zero unless $n = m$. But since the sum is over all n, one of the values will be m, and that is the only value that gives a non-zero result. Thus the infinite sum reduces to one term:

$$\begin{aligned} \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi b}{a} \int_0^a \sin \frac{n\pi x}{a} \sin \frac{m\pi x}{a} dx &= \sum_{n=1}^{m-1} A_n \sinh \frac{n\pi b}{a} \times 0 + A_m \sinh \frac{m\pi b}{a} \times \frac{a}{2} + \sum_{n=m+1}^{\infty} A_n \sinh n \\ &= A_m \sinh \frac{m\pi b}{a} \times \frac{a}{2} \end{aligned}$$

Thus

$$\left(A_m \sinh \frac{m\pi b}{a} \right) \frac{a}{2} = \int_0^a V_0(x) \sin \frac{m\pi x}{a} dx \tag{7}$$

In principle we are done, but to get values for the coefficients A_m we'll need a specific function $V_0(x)$.

Suppose our box has insulating strips at $x = 0$, $y = b$ and at $x = a/2$, $y = b$ with the potential $V_0(x) = V_0$ for $0 < x < a/2$ and zero for $a/2 < x < a$. then

$$
\begin{aligned}
\int_0^a V_0(x) \sin \frac{m\pi x}{a} dx &= \int_0^{a/2} V_0 \sin \frac{m\pi x}{a} dx + \int_{a/2}^a 0 \times \sin \frac{m\pi x}{a} dx \\
&= V_0 \frac{-a}{m\pi} \cos \frac{m\pi x}{a} \bigg|_0^{a/2} + 0 \\
&= V_0 \frac{a}{m\pi} \left(1 - \cos \frac{m\pi}{2}\right) \qquad (8)
\end{aligned}
$$

Then, combining (8) and (7), we have

$$
A_m = 2V_0 \frac{1 - \cos m\pi/2}{m\pi \sinh m\pi b/a}
$$

So

$$
V(x, y) = 2V_0 \sum_{m=1}^{\infty} \frac{1 - \cos m\pi/2}{m\pi \sinh m\pi b/a} \sin \frac{m\pi x}{a} \sinh \frac{m\pi y}{a}
$$

Now

$$
\cos \frac{m\pi}{2} = \begin{cases} 0 & \text{if} \quad m \text{ is odd} \\ 1 & \text{if} \quad m = 2p \text{ and } p \text{ is even} \\ -1 & \text{if} \quad m = 2p \text{ and } p \text{ is odd} \end{cases}
$$

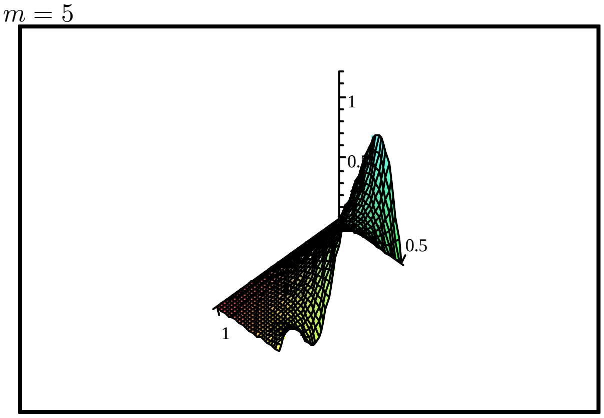
So the first few terms are

$$
V(x, y) = \frac{2V_0}{\pi} \left(\sin \frac{\pi x}{a} \frac{\sinh \frac{\pi y}{a}}{\sinh \frac{\pi b}{a}} + \sin \frac{2\pi x}{a} \frac{\sinh \frac{2\pi y}{a}}{\sinh \frac{2\pi b}{a}} + \frac{1}{3} \sin \frac{3\pi x}{a} \frac{\sinh \frac{3\pi y}{a}}{\sinh \frac{3\pi b}{a}} + \frac{1}{5} \sin \frac{5\pi x}{a} \frac{\sinh \frac{5\pi y}{a}}{\sinh \frac{5\pi b}{a}} + \cdots \right)
$$

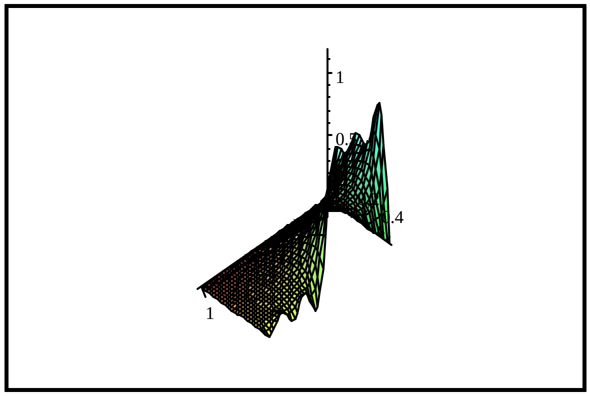
With $b = a/2$:

$$
V(x, y) = \frac{2V_0}{\pi} \left(\sin \frac{\pi x}{a} \frac{\sinh \frac{\pi y}{a}}{\sinh \frac{\pi}{2}} + \sin \frac{2\pi x}{a} \frac{\sinh \frac{2\pi y}{a}}{\sinh \pi} + \frac{1}{3} \sin \frac{3\pi x}{a} \frac{\sinh \frac{3\pi y}{a}}{\sinh \frac{3\pi}{2}} + \frac{1}{5} \sin \frac{5\pi x}{a} \frac{\sinh \frac{5\pi y}{a}}{\sinh \frac{5\pi}{2}} + \cdots \right)
$$

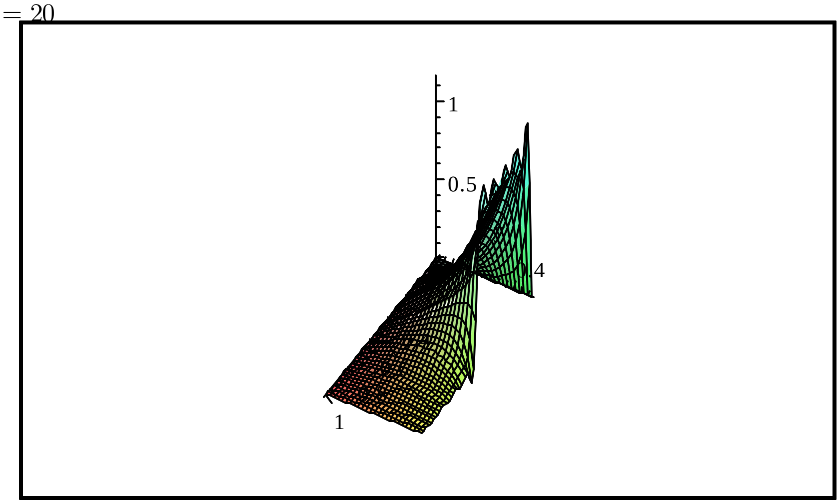
Terms up to $m = 5$



$m = 10$



$m = 20$



We can see the correct boundary conditions appearing as we increase m. The series converges well if $y \ll b$, but convergence gets slower as $y \to b$. This is a characterisic of these types of solutions.

Solution in three dimensions.

In 3-d Laplace's equation is

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

and now we look for a separated solution of the form

$$V = X(x) Y(y) Z(z)$$

Stuffing in, we get

$$\frac{\partial^2 X}{\partial x^2} YZ + X \frac{\partial^2 Y}{\partial y^2} Z + XY \frac{\partial^2 Z}{\partial z^2} = 0$$

and dividing by XYZ, we have

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}+\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}+\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}=0 \tag{9}$$

We have succeeded in separating the equation into three terms, each of which depends on only one of the three variables x,y and z. So again we can imagine moving along a line with constant x and y, letting only z change. This would change the third term Z''/Z without changing the other two, disturbing the equality. We must prohibit this possibility by requiring this term to be a constant.

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}=k_1$$

We can make the same argument about the second term by moving along a line with x and z constant and only y varying. Thus

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}=k_2$$

Inserting these values back into the differential equation (9), we have

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}=-k_1-k_2=k_3$$

All three of our equations are now of the form (2) with solutions of the form (3) or (4).

To see how the solution goes, let's consider an infinite slot that extends from $x=0$ out to infinity, from $y=0$ to $y=a$, and from $z=0$ to $z=b$. Suppose that the surface at $x=0$ is a uniformly charged sheet, with charge density σ. There are narrow insulating strips at the edges, and the other four surfaces at $y=0,a$ and $z=0,b$ are grounded conductors.

As before we start with one of the coordinates that has two boundary conditions that are zeros. So let's start with y. We need a function that is zero at two places: $y=0$ and $y=a$. So we must choose a sine (because $\sin\alpha y=0$ when $y=0$) and then we must force another zero of the sine function to be at $y=a$ by picking $\alpha=n\pi/a$. That makes our constant $k_2=-\alpha^2=-\left(n\pi/a\right)^2$.

$$Y=\sin\frac{n\pi y}{a}$$

Now we do something similar with the function of z. Again we need the sine, with the constant chosen to make the function zero again at $z=b$. Thus $k_1=-\left(m\pi/b\right)^2$ and

$$Z=\sin\left(\frac{m\pi z}{b}\right)$$

Now we have the equation for X : it is

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}=-k_1-k_2=-\left[-\left(\frac{n\pi}{a}\right)^2-\left(\frac{m\pi}{b}\right)^2\right]=+\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2$$

The constant k_3 must be positive because both k_1 and k_2 are negative. The sum of all three constants has to be zero. That means that our solutions for x have to be exponentials. We have two more boundary conditions to satisfy. The first is that $X \to 0$ as $x \to \infty$, and this means we need the negative exponential:

$$X = \exp\left[-\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}\, x\right]$$

Finally, we have to match our solution to the charge density σ at $x = 0$. The charge sits on a thin insulator, on top of a conductor. That allows us to make the charge density anything we like, with zero field on the other $(x < 0)$ side in the conductor. Then we need

$$E_\perp = \frac{\sigma}{\varepsilon_0} = \left.\frac{\partial V}{\partial X}\right|_{x=0} = \left.\frac{\partial X}{\partial X}\right|_{x=0} YZ$$

As before, single values of n and m will not do this for us, and we will need a linear combination. This time it is a double sum over both n and m.

$$V(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} \exp\left[-\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}\, x\right] \sin\frac{n\pi y}{a} \sin\left(\frac{m\pi z}{b}\right)$$

with the boundary condition

$$\begin{aligned}
\frac{\sigma}{\varepsilon_0} &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} -\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2} A_{nm} \left.\exp\left[-\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}\, x\right]\right|_{x=0} \sin\frac{n\pi y}{a} \sin\left(\frac{m\pi z}{b}\right) \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} -\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2} A_{nm} \sin\frac{n\pi y}{a} \sin\left(\frac{m\pi z}{b}\right)
\end{aligned}$$

This is a Fourier series for σ and we find the coefficients in the usual way. This time we have to use the orthogonality of the sines twice: once in y and once in z. First multiply both sides of the equation by $\sin(p\pi y/a)$ and integrate over y from 0 to a.

$$\int_0^a \frac{\sigma}{\varepsilon_0} \sin\frac{p\pi y}{a} dy = \int_0^a \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} -\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2} A_{nm} \sin\frac{n\pi y}{a} \sin\left(\frac{m\pi z}{b}\right) \sin\frac{p\pi y}{a} dy$$

Interchange the sum and the integral on the right, and move everything that does not depend on y out of the integral:

$$\frac{\sigma}{\varepsilon_0}\int_0^a \sin\frac{p\pi y}{a} dy = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} -\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2} A_{nm} \sin\left(\frac{m\pi z}{b}\right) \int_0^a \sin\frac{n\pi y}{a} \sin\frac{p\pi y}{a} dy$$

The integral on the right is zero except for the one term with $n = p$. So we have

$$\begin{aligned}
\frac{\sigma}{\varepsilon_0}\frac{a}{p\pi} \left. -\cos\frac{p\pi y}{a}\right|_0^a &= \sum_{m=1}^{\infty} -\sqrt{\left(\frac{p\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2} A_{pm} \sin\left(\frac{m\pi z}{b}\right) \frac{a}{2} \\
\frac{\sigma}{\varepsilon_0}\frac{2}{p\pi}(1 - \cos p\pi) &= \sum_{m=1}^{\infty} -\sqrt{\left(\frac{p\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2} A_{pm} \sin\left(\frac{m\pi z}{b}\right)
\end{aligned}$$

Now we do it again, this time multiplying by $\sin q\pi z/b$

$$\begin{aligned}
\frac{\sigma}{\varepsilon_0}\frac{2}{p\pi}\left(1-\cos p\pi\right)\int_0^b \sin\frac{q\pi z}{b}dz &= \sum_{m=1}^{\infty}-\sqrt{\left(\frac{p\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2}A_{pm}\int_0^b \sin\frac{q\pi z}{b}\sin\left(\frac{m\pi z}{b}\right)dz \\
\frac{\sigma}{\varepsilon_0}\frac{2}{p\pi}\left(1-\cos p\pi\right)\frac{b}{q\pi}\left(1-\cos q\pi\right) &= -\sqrt{\left(\frac{p\pi}{a}\right)^2+\left(\frac{q\pi}{b}\right)^2}A_{pq}\frac{b}{2}
\end{aligned}$$

Aha! We have isolated the constant A_{pq}.

$$A_{pq}=-\frac{\sigma}{\varepsilon_0}\frac{4}{pq\pi^2}\frac{\left(1-\cos p\pi\right)\left(1-\cos q\pi\right)}{\sqrt{\left(\frac{p\pi}{a}\right)^2+\left(\frac{q\pi}{b}\right)^2}}$$

But $\cos p\pi=+1$ if p is even, so $1-\cos p\pi=0$, and $\cos p\pi=-1$ if p is odd, and in that case $1-\cos p\pi=2$. Thus A_{pq} is zero unless both p and q are odd, and then

$$A_{pq}=-\frac{\sigma}{\varepsilon_0}\frac{16}{pq\pi^2}\frac{1}{\sqrt{\left(\frac{p\pi}{a}\right)^2+\left(\frac{q\pi}{b}\right)^2}}$$

and then the potential is

$$V\left(x,y,z\right)=-\frac{16\sigma}{\pi^2\varepsilon_0}\sum_{\substack{n=1\\ \text{odd}}}^{\infty}\sum_{\substack{m=1\\ \text{odd}}}^{\infty}\frac{1}{nm\sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2}}\exp\left[-\sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2}\,x\right]\sin\frac{n\pi y}{a}\sin\left(\frac{m\pi z}{b}\right)$$

The result has the correct physical dimensions of charge/($\varepsilon_0\times$length) (remember that $\sigma=$charge/length^2) and the series converges very rapidly, both because of the denominator $nm\sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2}$, and, for $x>0$, the exponential function. As we saw in the 2-d case, convergence is worst right *at* the boundary where V is non-zero.

Let's review the method step by step.

Separation of variables method for solving Laplace's equation.

- 1. Separate the PDE into 2 (or 3) coupled ODEs. Note that the separation constants must sum to zero. Start with a coordinate that has zero potential or zero $E_\perp$ on the constant coordinate surfaces (for example $V=0$ at $x=0$ and $x=a$).

  2. Determine the possible set of solutions of the ODE in your chosen coordinate. Since it is a second order equation, it will have two possible solutions.

  3. Find the correct function (of the two possible functions you found in step 2) using one of the two boundary conditions.

  4. Find the separation constant using the second boundary condition.

  5. Repeat steps 2-4 for the second coordinate, if V depends on 3 coordinates.

6. At this point you know the complete equation for the third function. Solve it, using the last zero boundary condition to determine the correct function (of the two possible functions you found in step 2).
7. Form a linear combination of the solutions you have identified using steps 1-6.
8. Use orthogonality together with your final boundary condition to determine the constants in your solution.