Magnetic force

The electric field is defined in terms of the electric force acting on a charged particle. $\vec{E} = \frac{\vec{F}}{q}$. Similarly, to understand magnetic field we start with the force. There is no magnetic force on stationary charged particles, and the force on a moving particle is perpendicular to the particle's velocity. The relation between force and field is

$$\vec{F} = q\vec{v} \times \vec{B}$$

This force causes circular motion in the plane perpendicular to $\vec{B}$.

Let's review uniform circular motion. The speed is

$$v = \omega r$$

and the velocity is

$$\vec{v} = \vec{\omega} \times \vec{r}$$

where $\vec{\omega}$ lies along the axis of rotation. Take the derivative:

$$\vec{a} = \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt}$$

Uniform circular motion means $\vec{\omega}$ is constant, so

$$\vec{a} = \vec{\omega} \times \vec{v} \tag{1}$$

Now compare with the magnetic force:

$$\vec{a} = \frac{\vec{F}}{m} = \frac{q}{m}\vec{v} \times \vec{B} = -\frac{q}{m}\vec{B} \times \vec{v} \tag{2}$$

These relations (1 and 2) are the same if

$$\vec{\omega} = -\frac{q}{m}\vec{B}$$

The frequency

$$\omega_c = \frac{|q|\, B}{m} \tag{3}$$

is called the cyclotron frequency. The important fact here is that the magnetic force is always perpendicular to the velocity, as is the centripetal force.

The radius of the circle is

$$r = \frac{v}{\omega} = \frac{mv}{qB} \tag{4}$$

This is the Larmor radius.

If $\vec{v}$ is not initially in the plane perpendicular to $\vec{B}$, the component of $\vec{v}$ parallel to $\vec{B}$ does not contribute to the force, and v in the above formulae should be replaced with $v_\perp$. $v_\parallel$ remains unchanged, and the path of the particle is a helix.

Motion in $\vec{E}$ and $\vec{B}$ fields.

When both $\vec{E}$ and $\vec{B}$ are present, the total force is the Lorentz force

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \tag{5}$$

- 1. *Electric and magnetic fields parallel.* In this case The motions are decoupled (in the non-relativistic case). The electric force changes the component of $\vec{v}$ along $\vec{B}$ while the magnetic force changes the component of $\vec{v}$ perpendicular to $\vec{B}$, causing circular motion.
  2. *Electric field perpendicular to magnetic field.* In this case it is possible for the total force to be zero, if we start with exactly the right velocity. Let's see how that can be. Suppose

  $$\vec{E} + \vec{v} \times \vec{B} = 0$$

  Notice that only the component of $\vec{v}$ perpendicular to $\vec{B}$ contributes, so let's write that explicitly: Then

  $$\vec{E} = -\vec{v}_\perp \times \vec{B}$$

  Now take the cross product with $\vec{B}$:

  $$\vec{E} \times \vec{B} = -\left(\vec{v}_\perp \times \vec{B}\right) \times \vec{B}$$

  Expand the product on the right, using the BAC-CAB rule:

  $$\vec{E} \times \vec{B} = -\left[\vec{B}\left(\vec{v}_\perp \cdot \vec{B}\right) - \vec{v}_\perp\left(\vec{B} \cdot \vec{B}\right)\right] = \vec{v}_\perp B^2$$

  since $\vec{v}_\perp \cdot \vec{B} = 0$. Rearrange:

  $$\vec{v} = \frac{\vec{E} \times \vec{B}}{B^2} \tag{6}$$

  So if a particle starts out with velocity (6), it will experience no acceleration and it will continue to move with that velocity.

If a charged particle starts out with zero velocity, the electric field will accelerate it, and then it will also experience a magnetic force. To simplify the discussion, let's take q to be positive. The magnetic force bends the trajectory around. As the speed increases, (due to $\vec{E}$) the radius of the circle increases (eqn 4). But as the particle reaches the top of the trajectory and starts to move opposite $\vec{E}$, it slows down, the Larmor radius decreases, and ultimately the particle comes momentarily to rest again. Then the motion repeats. The path is called a cycloid. It is also the path traced out by a particle on the rim of a wheel rotating with constant ω.

To analyze this mathematically, let $\vec{B} = B\hat{x}$, $\vec{E} = E\hat{y}$, and let the particle start at the origin. The force is

$$\vec{F} = q\left(E\hat{y} + \vec{v} \times B\hat{x}\right) = q\left[\left(E + v_z B\right)\hat{y} - v_y B\hat{z}\right]$$

Thus

$$\frac{dv_y}{dt} = \frac{q}{m}\left(E + v_z B\right) \tag{7}$$

$$\frac{dv_z}{dt} = -\frac{q}{m} v_y B \tag{8}$$

Differentiating the first equation, and inserting the second, we have

$$\frac{d^2 v_y}{dt^2} = -\left(\frac{qB}{m}\right)^2 v_y = -\omega^2 v_y$$

with solution

$$v_y = A \sin \omega t + B \cos \omega t$$

Since $v_y = 0$ at $t = 0$, $B = 0$, and so

$$v_y = A \sin \omega t \tag{9}$$

Then

$$\frac{dv_z}{dt} = -\omega A \sin \omega t \Rightarrow v_z = A \cos \omega t + C$$

Since $v_z = 0$ at $t = 0$, $C = -A$. Finally, from (7) and (9),

$$\frac{dv_y}{dt} = \omega A \cos \omega t = \omega \left[\frac{E}{B} + A\left(\cos \omega t - 1\right)\right]$$

So

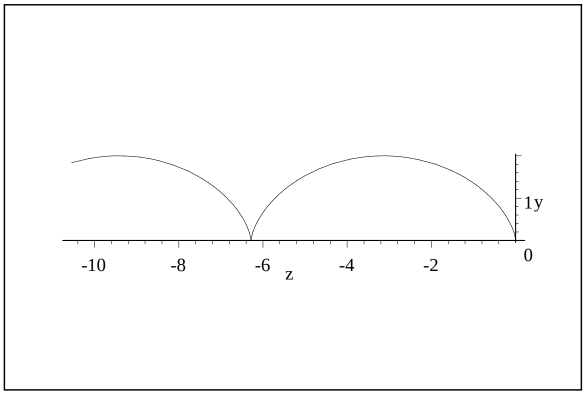
$$\frac{E}{B} = A$$

Thus the solution is

$$\begin{aligned} v_y &= \frac{E}{B} \sin \omega t \\ v_z &= \frac{E}{B} \left(\cos \omega t - 1\right) \end{aligned}$$

and integrating once, we get the position

$$\begin{aligned} y &= \frac{E}{\omega B} \left(1 - \cos \omega t\right) \\ z &= \frac{E}{\omega B} \left(\sin \omega t - \omega t\right) \end{aligned}$$



The particle undergoes circular motion superposed on a uniform drift

$$\vec{v} = -\frac{E}{B}\hat{z} = \frac{\vec{E}\times\vec{B}}{B^2}$$

This velocity is the same as (6). This is no coincidence. In fact it is an example of a general result:

> With $\vec{E}$ perpendicular to $\vec{B}$, the particle undergoes circular motion plus a constant drift with velocity $\vec{E}\times\vec{B}/B^2$.

If we work in a frame moving with the drift velocity $\vec{v}_D = -(E/B)\,\hat{z}$, the particle's position in that frame is

$$\begin{aligned} y &= \frac{E}{\omega B}(1-\cos\omega t) \\ z &= \frac{E}{\omega B}\sin\omega t \end{aligned}$$

We can combine these by writing

$$\cos\omega t = 1 - \frac{\omega B y}{E}; \qquad \sin\omega t = \frac{\omega B z}{E}$$

So

$$\begin{aligned} \cos^2\omega t + \sin^2\omega t &= 1 \\ \left(1-\frac{\omega B y}{E}\right)^2 + \left(\frac{\omega B z}{E}\right)^2 &= 1 \end{aligned}$$

or

$$\left(y - \frac{E}{\omega B}\right)^2 + z^2 = \left(\frac{E}{\omega B}\right)^2$$

This is the equation of a circle with center at $y = E/\omega B$ and radius $E/\omega B$.

$$r = \frac{E}{\omega B} = \frac{v_D}{\omega}$$

This the Larmor radius for a particle moving at v_D. This is expected, of course, because a particle starting at rest in the lab frame is moving at speed v_D in the moving frame.

In the plane perpendicular to $\vec{B}$, the particle moves perpendicular to the applied force. This should remind you of the motion of a spinning body undergoing precession: it also moves perpendicular to $\vec{F}$. The particle gyrating around the magnetic field lines acts like a spinning body.

Finally, note that if $\vec{E}$ is at some angle θ to $\vec{B}$, the particle accelerates along $\vec{B}$ with acceleration $(qE/m)\cos\theta$ and has a conbination of gyration plus drift in the plane perpendicular to $\vec{B}$, the drift speed being $(E\sin\theta)/B$.

A significant feature of the magnetic force is that it *does no work*, because it is always perpendicular to $\vec{v}$.

$$dW = \vec{F} \cdot d\vec{s} = \vec{F} \cdot \vec{v}\ dt = q\left(\vec{v} \times \vec{B}\right) \cdot \vec{v}\ dt \equiv 0$$

As we saw in the examples above, the magnetic force changes the direction but not the magnitude of $\vec{v}$.

Current

Current in a wire is defined as the amount of charge passing a point in the wire per unit time. As discussed in LB Ch 26 §2, the quantity that is directly influenced by the fields is the current density $\vec{j}$. For a wire with uniform cross section, A

$$j = \frac{I}{A}$$

with direction parallel to the wire, so the units of $\vec{j}$ are amps/m^2. (Yes, that is m^2 not m^3.) In fact current is the flux of current density:

$$I = \int \vec{j} \cdot \hat{n}\ dA \tag{10}$$

(Griffiths says that I is a vector, but that's not right. $\vec{j}$ is a vector. Since I is the flux of $\vec{j}$, it has a sign that is associated with its direction, but it is a scalar none the less.)

To understand the relation between $\vec{j}$ and the properties of the material, we look at a wire segment of length $d\ell$. If there are n_e moving electrons per unit volume, and each has a net drift velocity $\vec{v}_d$, the number of electrons in the segment at any given moment is

$$dN = n_e A\ d\ell$$

and they all move out of the segment in a time

$$dt = \frac{d\ell}{v}$$

Thus the current is

$$I = e\frac{dN}{dt} = e\frac{n_e A\ d\ell}{d\ell/v} = n_e evA$$

and thus the current density is

$$j = \frac{I}{A} = n_e ev \tag{11}$$

We can use these definitions to find the magnetic force on a current distribution. The magnetic field exerts force on all the moving charges (usually electrons) in the wire. The magnetic force on a segment of length $d\ell$ is

$$\begin{aligned} d\vec{F} &= dNq\vec{v} \times \vec{B} = qn_e A\ d\ell\vec{v} \times \vec{B} \\ &= qn_e Av\ d\vec{\ell} \times \vec{B} = jA\ d\vec{\ell} \times \vec{B} \end{aligned}$$

since $\vec{v}$ and $d\vec{\ell}$ are parallel. Thus

$$d\vec{F} = I\ d\vec{\ell} \times \vec{B}$$

and the force on the whole wire is

$$\vec{F} = \int I\ d\vec{\ell} \times \vec{B} \tag{12}$$

We may also write the expression for $d\vec{F}$ as

$$d\vec{F} = \vec{j} \times \vec{B}\ dAd\ell = \vec{j} \times \vec{B}\ d\tau$$

giving us a more general expression

$$\vec{F} = \int \vec{j} \times \vec{B}\ d\tau \tag{13}$$

that is valid whether or not the current is confined in wires.

Aside: Notice that we replaced the quantity $Id\vec{\ell}$ in equation (12) with $\vec{j}d\tau$ in (13). We will want to remember this for future applications.

Note that the net force on a closed current loop immersed in a uniform field is always zero.

$$\vec{F} = \oint Id\vec{\ell} \times \vec{B} = I\left(\oint d\vec{\ell}\right) \times \vec{B} = 0$$

But the torque is not zero. For a rectangular loop measuring $a \times c$ in the $x-y-$plane with magnetic field $\vec{B} = B_0\hat{x}$, we have zero force on the two sides parallel to the $x-$axis. The forces on the other two sides are equal and opposite (because I reverses) and have magnitude

$$F = IcB$$

Thus the net force is zero (as expected) and since we have a couple (LB §11.1.3), the torque is the same about any point and equals

$$\tau = IcBa = IBA = mB$$

where A is the area of the loop and $m = IA$ is the magnetic moment. Putting in the directions, we get

$$\vec{\tau} = \vec{m} \times \vec{B}$$

See, eg, LB Example 29.8. Grad students: show that this result is true for any planar loop.

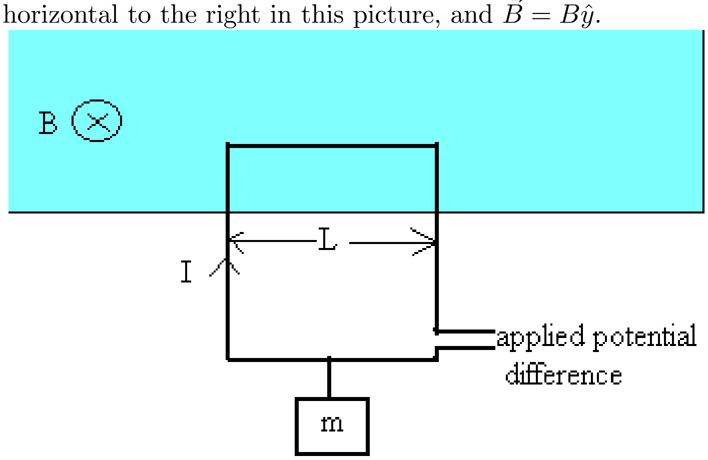
The energy of an electric dipole in an electric field is $U = -\vec{p} \cdot \vec{E}$. Similarly, the energy of a magnetic dipole in a magnetic field is $U = -\vec{m} \cdot \vec{B}$.

Magnetic forces (and magnetic fields in general) are enormously important in our technological society. That is one reason why it is worthwhile struggling with the occasionally complex mathematical relations that arise in trying to understand the physics of magnetic fields. Magnetic fields are also ubiquitous in space, and are essential in making the earth a place that can support life. See LB Ch 29 for many applications of magnetic fields.

How can magnetic fields be so important if they do no work? Let's look at one example to see what happens.

A current balance is a device that uses currents to measure forces (or sometimes forces to measure currents- it works either way.) Griffiths has an idealized case in Example 5.3. There is a somewhat more realistic example in LB example 29.7. Let's look at the Griffiths case.

A current flows in a rectangular loop of wire that supports a mass m. A uniform magnetic field exists in the region $z > 0$, as shown in the diagram. The x-axis is horizontal to the right in this picture, and $\vec{B} = B\hat{y}$.



The magnetic forces on the two vertical segments of the loop are equal and opposite, and exactly cancel. The magnetic force on the horizontal segment is

$$\vec{F}_{\text{mag}} = IL\hat{x} \times B\hat{y} = ILB\hat{z}$$

This upward force will balance the weight of the block if we adjust the current so that

$$ILB = mg$$

So far so good. Now if we increase the current (by turning up the applied potential difference) we can make the magnetic force greater than the weight and thus lift the block. In fact all we need is for F_{mag} to be infinitesimally greater than mg. Now we let the loop and attached block move upward at speed v. That means that the electrons in the wire have an additional component of velocity, and thus an additional force acts on them.

$$\vec{F}_{\text{new}} = qv\hat{z} \times B\hat{y} = qvB\,(-\hat{x})$$

This force is opposing the flow of charge that constitutes the current in the loop. Thus we must exert some opposing force to maintain the current. That force is an electric force. The battery distributes charge around the circuit to establish the necessary fields. Thus

$$\begin{aligned} F_E &= qE = qvB \\ \Delta V &= \int \vec{E} \cdot d\vec{\ell} = vBL \end{aligned}$$

The work done to lift the mass a height $dy = vdt$ is

$$dW_{\text{lift}} = mg\ dy = mgv\ dt$$

The electron drift speed v_d carries each electron a distance $dx = v_d dt$ in time dt.So the work done by our electric force on all the electrons is

$$\begin{aligned} dW_{\text{elec}} &= (n_e LA)\, qvB\ dx = n_e LAqvBv_d\ dt \\ &= (n_e qv_d A)\, LBv\ dt = ILBv\ dt = mgv\ dt = dW_{\text{lift}} \end{aligned}$$

The two amounts of work are equal. So it is actually the battery that is doing the work to lift the block. The magnetic force acts as a facilitator to make it all fit together.

There is another way to think about this situation using Faraday's law. See if you can figure it out.

Charge conservation

We may use the definition of current and equation (10) to compute the total charge entering a fixed volume V in time dt:

$$dQ = -\oint_S \vec{j} \cdot \hat{n}\ dA\ dt$$

We need the minus sign because $\hat{n}$ is outward, so if $\vec{j} \cdot \hat{n}$ is positive, it means charge is leaving the volume and thus dQ is negative. Now we use the divergence theorem to rewrite the RHS, and express Q in terms of the charge density:

$$\frac{dQ}{dt} = \frac{d}{dt} \int \rho\ d\tau = -\int \vec{\nabla} \cdot \vec{j}\ d\tau$$

Now since the volume is fixed, we may move the time derivative inside:

$$\int \frac{\partial}{\partial t} \rho\ d\tau = -\int \vec{\nabla} \cdot \vec{j}\ d\tau$$

It's a partial time derivative of ρ because ρ may depend on position as well. This relation must be true for absolutely any volume V, so we may express the local equation for charge conservation as

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0 \tag{14}$$

In the static field situations we are considering, ρ is independent of time and this relation simplifies to

$$\vec{\nabla} \cdot \vec{j} = 0$$

Thus $\vec{j}$ is a solenoidal field. That means that current lines form closed loops. This will be another useful result to remember.