Electric displacement

Now let's see how to investigate the fields that are produced by a collection of charges, both bound and free. (Free charges are those not bound up in atoms and molecules of the material. This includes charges on the surface of conductors.)

We have Gauss' law

$$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\varepsilon_0}=\frac{\rho_f+\rho_b}{\varepsilon_0}$$

But we also discovered that

$$\rho_b=-\vec{\nabla}\cdot\vec{P}$$

so

$$\vec{\nabla}\cdot\vec{E}=\frac{\rho_f-\vec{\nabla}\cdot\vec{P}}{\varepsilon_0}$$

Now it is convenient to gather the two divegences together:

$$\varepsilon_0\vec{\nabla}\cdot\vec{E}+\vec{\nabla}\cdot\vec{P}=\rho_f=\vec{\nabla}\cdot\vec{D} \tag{1}$$

where the electric displacement $\vec{D}$ is defined to be

$$\vec{D}=\varepsilon_0\vec{E}+\vec{P} \tag{2}$$

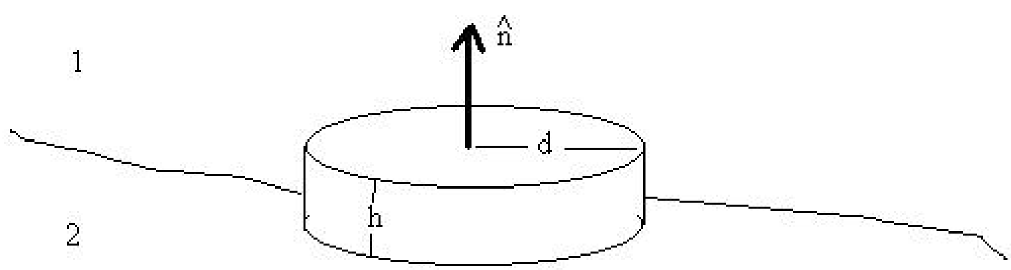
For LIH materials

$$\vec{D}=\varepsilon_0\vec{E}+\varepsilon\chi_e\vec{E}=\varepsilon_0\left(1+\chi_e\right)\vec{E}=\varepsilon\vec{E} \tag{3}$$

ε is the permittivity of the material and the dimensionless ratio $\varepsilon/\varepsilon_0=\kappa$ is the dielectric constant. Note that the $\vec{E}$ here is the net $\vec{E}$ at position $\vec{r}$ due to all sources, including $\vec{P}$ itself. So it is often easier to work with $\vec{D}$, which satisfies Poisson's equation in the form (1).

Boundary conditions for $\vec{D}$.

We find the boundary conditions for $\vec{D}$, as we did for $\vec{E}$, by putting a tuna-can across the boundary, and integrating equation (1) across the boundary.



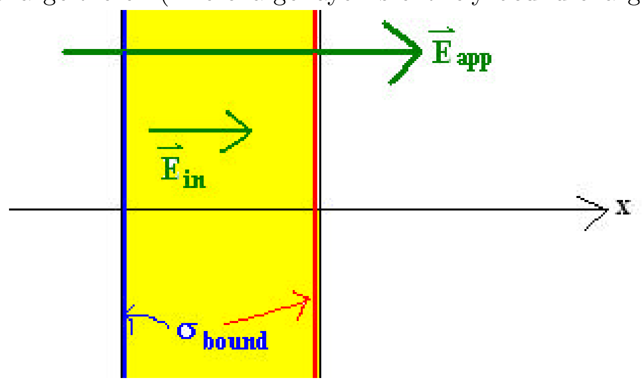
$$
\begin{aligned}
\int \vec{\nabla} \cdot \vec{D}\ dV &\neq \int \rho_f\ dV \\
\oint \vec{D} \cdot d\vec{A} &= \int \rho_f\ dh dA = \sigma_f\ dA
\end{aligned}
$$

The contribution from the sides of the box is negligible, because $h \ll d$. So

$$
\left(\vec{D}_1 - \vec{D}_2\right) \cdot \hat{n} = \sigma_f \tag{4}
$$

Previuosly we used the differential equation $\vec{\nabla}\times\vec{E} = 0$ to derive the boundary condition that $\vec{E}_{\tan}$ is continuous across a boundary. If $\vec{D} = \varepsilon\vec{E}$ in an LIH material, then $\vec{\nabla} \times \vec{D}$ is also zero. But $\vec{\nabla} \times \vec{D}$ is not zero at the boundary because ε changes abruptly there. In fact, $\vec{\nabla} \times \vec{D}$ is infinite at the boundary. So we have one boundary condition (4) for $\vec{D}$ and one for $\vec{E}$.

To see how this works, let's look at our slab again. With the slab's sides parallel to the $y-z-$plane, and normals in the $\pm x$ direction, the normal component of $\vec{D}$ is D_x. Thus D_x is continuous as we cross the boundary since there is no *free* charge there. (The charge layer is entirely bound charge.)



$$
D_{x,\text{out}} = \varepsilon_0 E_{x,\text{app}} = \varepsilon E_{x\text{in}} = D_{x,\text{in}}
$$

The tangential component of $\vec{E}$ is zero both inside and outside. So

$$
\vec{E}_{\text{in}} = \frac{\varepsilon_0}{\varepsilon}\vec{E}_{\text{app}}
$$

The boundary condition on $\vec{E}$ gives us the bound charge density. On the right hand side

$$
E_{x,\text{out}} - E_{x,\text{in}} = \frac{\sigma}{\varepsilon_0} = E_{x,\text{app}}\left(1 - \frac{\varepsilon_0}{\varepsilon}\right)
$$

Thus

$$
\sigma_b = (\varepsilon - \varepsilon_0)\, E_{x,\text{app}}
$$

Using the same method, convince yourself that the charge on the left side is the exact negative of this.

Boundary value problems

If we have a boundary between two uniform regions (two uniform dielectrics, or a dielectric and a conductor, or a dielectric and vacuum) then we have the following set of equations:

$$\vec{\nabla} \cdot \vec{D} = \rho_f \tag{5}$$

and

$$\vec{\nabla} \times \vec{E} = 0$$

within each region. Since $\vec{\nabla} \times \vec{E} = 0$, we may write $\vec{E} = -\vec{\nabla} V$, and equation (5) may be written

$$-\vec{\nabla} \cdot \left(\varepsilon \vec{\nabla} V \right) = \rho_f = -\varepsilon \nabla^2 V - \vec{\nabla} \varepsilon \cdot \vec{\nabla} V$$

Since we are assuming our medium is uniform, $\vec{\nabla} \varepsilon = 0$, and the equation for the potential is

$$\nabla^2 V = -\frac{\rho_f}{\varepsilon} \tag{6}$$

Be careful! $\vec{\nabla} \varepsilon \neq 0$ right *at* the boundary. We may use equation (6) within each region, but not across the boundary.

The boundary conditions are:

$$\left(\vec{D}_1 - \vec{D}_2 \right) \cdot \hat{n} = \sigma_f$$

$$\begin{array}{ll} \vec{E}_{\tan} & \text{is continuous} \\ V & \text{is continuous} \end{array}$$

Dielectric sphere in a uniform field

To see how to use these relations, consider a dielectric sphere of radius a placed in a uniform field $\vec{E}_0$. Because the boundary is spherical, we use spherical coordinates with origin at the center of the sphere and polar axis parallel to $\vec{E}_0$. There is no free charge, so V satisfies Laplace's equation both inside and outside the sphere (but not ON the boundary), and we know the potential may be written in terms of Legendre polynomials.

Inside the sphere, the potential must not blow up at $r = 0$, so we have only positive powers of r :

$$V\left(r < a\right) = \sum_{l=0}^{\infty} A_l r^l P_l\left(\cos\theta\right)$$

Outside, we must have a potential that gives us the uniform field $\vec{E}_0$. That potential is $V_{\text{unif}} = -E_0 z = -E_0 r \cos\theta = -E_0 r P_1\left(\cos\theta\right)$. Other than that, all the other terms must $\rightarrow 0$ as $r \rightarrow \infty$, so

$$V\left(r > a\right) = -E_0 r P_1\left(\cos\theta\right) + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l\left(\cos\theta\right)$$

Now we apply all our conditions, starting with the easiest one. V is continuous.

$$\sum_{l=0}^{\infty} A_l a^l P_l(\cos\theta) = -E_0 a P_1(\cos\theta) + \sum_{l=0}^{\infty} \frac{B_l}{a^{l+1}} P_l(\cos\theta)$$

Because the P_l are orthogonal, the coefficient of each P_l must separately equal zero. Thus

$$l = 1: \quad A_1 a = -E_0 a + \frac{B_1}{a^2} \tag{7}$$

$$l \neq 1: \quad A_l a^l = \frac{B_l}{a^{l+1}} \tag{8}$$

Now we use the boundary condition for $\vec{D}$. The normal component is the radial component, and there is no free charge on the surface, so we have

$$\begin{aligned} \varepsilon \left.\frac{\partial V(r<a)}{\partial r}\right|_{r=a} &= \left.\frac{\partial V(r>a)}{\partial r}\right|_{r=a} \\ \varepsilon \sum_{l=0}^{\infty} l A_l a^{l-1} P_l(\cos\theta) &= -E_0 P_1(\cos\theta) + \sum_{l=0}^{\infty} -(l+1)\frac{B_l}{a^{l+2}} P_l(\cos\theta) \end{aligned}$$

Again the coefficient of each P_l must separately equal zero.

$$l = 1: \quad \varepsilon A_1 = -E_0 - 2\frac{B_l}{a^3} \tag{9}$$

$$l \neq 1: \quad \varepsilon l A_l a^{l-1} = -(l+1)\frac{B_l}{a^{l+2}} \tag{10}$$

For $l \neq 1$, we use equation (8) to eliminate A_l from equation (10), and get

$$l \neq 1: \quad \varepsilon l \frac{B_l}{a^{l+2}} = -(l+1)\frac{B_l}{a^{l+2}}$$

which could only be true if $\varepsilon = -(l+1)/l$. This is impossible because ε is not negative for ordinary materials, and cannot equal $-(l+1)/l$ for more than one l, if any, in any case. Thus we must conclude that $B_l = A_l = 0$.

It is a different story for $l = 1$. We use equation (7) to eliminate A_1 from equation (9).

$$\begin{aligned} \varepsilon\left(-E_0 + \frac{B_1}{a^3}\right) &= -E_0 - 2\frac{B_l}{a^3} \\ \frac{B_1}{a^3}(\varepsilon + 2) &= E_0(\varepsilon - 1) \end{aligned}$$

So

$$B_1 = E_0 a^3 \frac{\varepsilon - 1}{\varepsilon + 2}$$

and then

$$A_1 = -E_0 + E_0 \frac{\varepsilon - 1}{\varepsilon + 2} = -3\frac{E_0}{\varepsilon + 2}$$

The potentials are

$$\begin{aligned} V\,(r<a) &= -\frac{3}{\varepsilon+2}E_0 r\cos\theta \\ V\,(r>a) &= -E_0 r\cos\theta + E_0\frac{\varepsilon-1}{\varepsilon+2}\frac{a^3}{r^2}\cos\theta \end{aligned}$$

The field inside is uniform:

$$\vec{E}\,(r<a) = \frac{3}{\varepsilon+2}\vec{E}_0$$

and outside we have the applied, uniform field, plus a dipole field with dipole moment

$$\vec{p} = 4\pi\varepsilon_0\vec{E}_0 a^3\frac{\varepsilon-1}{\varepsilon+2}$$

The dipole moment is zero if $\varepsilon = 1$ (the sphere is vacuum, ie there is no sphere!). The field inside the sphere is less than the applied field for $\varepsilon > 1$, as expected.

Since ε measures the degree to which the material can reduce an applied field in its interior, we might imagine that a conductor, which reduces an applied field to zero, coresponds to a very large value of ε. In fact some properties of conductors may be retrieved in the limit $\varepsilon \to \infty$. This limit applied to our sphere gives us zero field inside and a dipole moment $\vec{p} = 4\pi\varepsilon_0\vec{E}_0 a^3$. These are the results we found previously for a conducting sphere.

Energy

We have learned several things about electric energy that are relevant to our present discussion.

- The energy density stored in the electric field in vacuum is $u = \frac{1}{2}\varepsilon_0 E^2$
- The energy stored in a capacitor is $U = \frac{1}{2}C\,(\Delta V)^2$
- A dipole in an electric field has potential energy $U = -\vec{p}\cdot\vec{E}$

Let's look at the second of these first. If we fill a capacitor with dieletric we increase the capacitance by ε. To see why, compute the capacitance by putting a charge Q on the capacitor. The field inside the dielectric is a factor $\kappa = \varepsilon/\varepsilon_0$ smaller than with no dielectric, so the potential difference is also a factor of κ smaller, and thus the capacitance $C = Q/\Delta V$ is a factor κ larger. Thus the stored energy also increases by a factor of κ (if the potential difference is held fixed).

But we know that the energy stored in a capacitor is stored in the electric field, and, without dielectric

$$U = \frac{1}{2}C_0\,(\Delta V)^2 = \int \frac{1}{2}\varepsilon_0 E^2 dV$$

where the integral is over the volume of the capacitor. With dielectric:

$$\begin{aligned} U &= \frac{1}{2}C(\Delta V)^2 = \frac{1}{2}\kappa C_0 (\Delta V)^2 = \int \frac{1}{2}\kappa\varepsilon_0 E^2 dV \\ &= \int \frac{1}{2}\varepsilon E^2 dV = \int \frac{1}{2}\vec{E}\cdot\vec{D}\ dV \end{aligned}$$

So it looks as if the energy density is

$$u = \frac{1}{2}\vec{E}\cdot\vec{D}$$

which is greater than $\frac{1}{2}\varepsilon_0 E^2$.(for the same E.) Why is that? Well, as we increase the field by bringing up free charge, that field acts to create and/or align the atomic dipoles in the material. The work done by the field is negative (3rd point above) and thus the stored energy in the field is increased by the work done.

We can confirm these conjectures by imagining a process to put the system together. So we bring up the free charges, bit by bit. We don't have control over the bound charges– they just do what they must do. So when the potential has the value V and we bring up the next piece of free charge, we have to do work

$$\begin{aligned} \Delta W &= \int (\Delta\rho_f) V\ d\tau \\ &= \int \left(\vec{\nabla}\cdot\Delta\vec{D}\right) V\ d\tau \end{aligned}$$

Now we do the usual "integration by parts" trick.

$$\vec{\nabla}\cdot\left(\Delta\vec{D}V\right) = \left(\vec{\nabla}\cdot\Delta\vec{D}\right)V + \Delta\vec{D}\cdot\vec{\nabla}V$$

So

$$\begin{aligned} \Delta W &= \int \left[\vec{\nabla}\cdot\left(\Delta\vec{D}V\right) - \Delta\vec{D}\cdot\vec{\nabla}V\right]\ d\tau \\ &= \int_{S_\infty} \Delta\vec{D}V\cdot\hat{n}\ dA + \int \left(\Delta\vec{D}\right)\cdot\vec{E}\ d\tau \end{aligned}$$

The first term is zero, for the usual reasons. Now if our material is LIH, then $\Delta\vec{D} = \varepsilon\Delta\vec{E}$ and

$$\begin{aligned} \left(\Delta\vec{D}\right)\cdot\vec{E} &= \varepsilon\left(\Delta\vec{E}\right)\cdot\vec{E} = \frac{1}{2}\varepsilon\Delta\left(\vec{E}\cdot\vec{E}\right) \\ &= \frac{1}{2}\Delta\left(\vec{E}\cdot\vec{D}\right) \end{aligned}$$

Thus

$$\Delta W = \Delta\int \frac{1}{2}\vec{E}\cdot\vec{D}\ d\tau$$

and hence

$$u = \frac{1}{2}\vec{E} \cdot \vec{D}$$

as we conjectured above.

Note: I found Griffith's discussion somewhat misleading. *All* the energy is electric, including his so-called "spring" energy. For a more complete discussion, see Jackson §4.7.

Forces:

We have already noted from energy arguments that dielectrics are sucked into higher field regions. Here we'll investigate the force that does the sucking. We'll use as our example system a parallel plate capacitor. The plates measure $w \times \ell$ and the plate separation is d. Let's charge up the capacitor, and then disconnect it from the battery. This gives us a nice, clean *isolated* system with the charge on each plate *fixed* at $Q = Q_0$.. Now we insert a dielectric slab that fills the space between the capacitor plates. The initial energy is

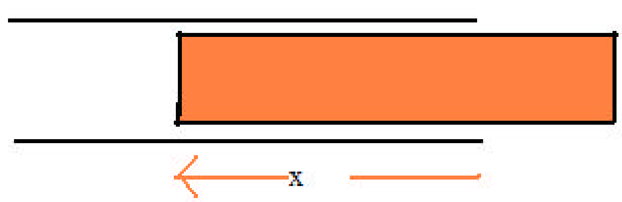
$$U_i = \frac{1}{2}\frac{Q_0^2}{C_i} = \frac{1}{2}C_i (\Delta V_0)^2$$

The final energy is

$$U_f = \frac{1}{2}\frac{Q_0^2}{C_f} = \frac{1}{2}\frac{Q_0^2}{\kappa C_i} = \frac{U_i}{\kappa}$$

The energy decreases because the capacitor does work on the slab as it sucks it in.

To find the force, we look at the system when the slab is part way in, as shown:



We may model this system as two capacitors in parallel: one with plate area $A_1 = w(\ell - x)$ and one with plate area $A_2 = wx$. The two capacitances are

$$C_1 = \frac{\varepsilon_0 A_1}{d} \quad \text{and} \quad C_2 = \frac{\varepsilon A_2}{d}$$

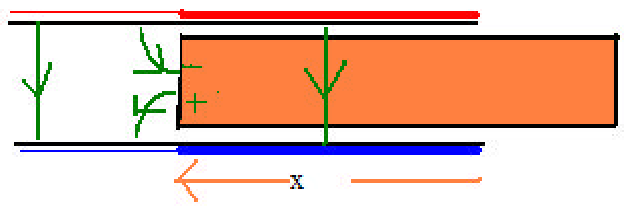
The total capacitance for the two in parallel is

$$C(x) = C_1 + C_2 = \varepsilon_0 \frac{w}{d}[\ell - x + \kappa x] = \varepsilon_0 \frac{w}{d}[\ell + (\kappa - 1)x]$$

and the stored energy is

$$U(x) = \frac{1}{2}\frac{Q_0^2}{C(x)} = \frac{1}{2}\frac{Q_0^2 d}{\varepsilon_0 w [\ell + (\kappa - 1)x]}$$

The potential difference between the plates is $\Delta V(x) = Q/C(x)$ and may be measured along any path between any two points on the plates. Thus the electric field between the plates ($\vec{E} = -\vec{\nabla}V$) is the same in the part with the dielectric and the part without. But $\vec{D} = \varepsilon\vec{E}$ is greater in the part with the dielectric, and thus the free charge density on the conducting plates is greater where the dielectric is. As the slab moves in, charge migrates across the plates.



Now in the ideal models we have been using, the field lines, both in the vacuum and in the dielectric, are everywhere perpendicular to the plates, and the charge density, both bound and free, forms layers parallel to the plates. But that's not quite right. At the very end of the slab there is some bound charge density and some field lines connect from the plates to the end of the slab. The volume occupied by these curved field lines is very small, and consequently our calculation of the capacitance is extremely accurate, as is the energy. This is just as well, because it is very hard to calulate this "fringing field" accurately. But it is precisely this field that gives rise to the sucking force! Fortunately we can calculate the force using energy arguments, without having to find the field.

$$\begin{aligned} & \text{Work done by the fields reduces the stored energy} \\ dW &= \vec{F}\cdot d\vec{s} = U(x) - U(x+dx) \\ F_x dx &= -\frac{dU}{dx}dx \end{aligned}$$

and so

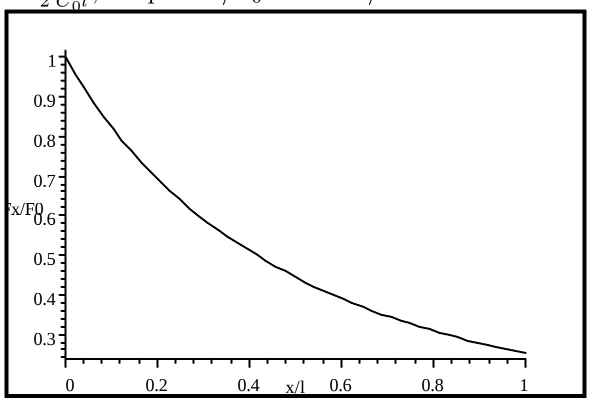
$$\begin{aligned} F_x &= -\frac{dU}{dx} \quad (\text{constant } Q) && (11) \\ &= -\frac{d}{dx}\frac{1}{2}\frac{Q_0^2}{C(x)} && (12) \\ &= -\frac{d}{dx}\frac{1}{2}\frac{Q_0^2 d}{\varepsilon_0 w\left[\ell + (\kappa-1)x\right]} \\ &= \left(-\frac{1}{2}\frac{Q_0^2 d}{\varepsilon_0 w}\right)\left(\frac{-(\kappa-1)}{\left[\ell+(\kappa-1)x\right]^2}\right) \\ &= \frac{1}{2}\frac{Q_0^2 d}{\varepsilon_0 w l^2}\frac{(\kappa-1)}{\left[1+(\kappa-1)x/\ell\right]^2} \\ F_x(x) &= \frac{1}{2}\frac{Q_0^2}{C_0 l}\frac{(\kappa-1)}{\left[1+(\kappa-1)x/\ell\right]^2} && (13) \end{aligned}$$

Dimensionally this is correct, since $Q/C_0 = \Delta V_0$ and $Q\Delta V_0/l$ is dimensionally charge×electric field = force. F_x is positive, so the force does pull the slab in, as we already concluded. The force decreases as x increases, and is maximum just as the slab enters the capacitor.

We can aslo write the force (13) as

$$F_x = \frac{1}{2}\frac{C_0}{\ell}\frac{Q_0^2}{C_0^2}\frac{(\kappa-1)}{[1+(\kappa-1)\,x/\ell]^2} = \frac{1}{2}(\kappa-1)\frac{C_0}{\ell}\frac{Q_0^2}{C(x)^2} = \frac{1}{2}(\kappa-1)\frac{C_0}{\ell}[\Delta V(x)]^2 \tag{14}$$

Letting $F_0 = \frac{1}{2}\frac{Q_0^2}{C_0 l}$, we plot F/F_0 versus x/ℓ.



Now what happens if we leave the capacitor hooked up to the battery? The capacitor is no longer isolated, so the discussion is a bit more complicated. This time the potential difference stays fixed at $\Delta V = \Delta V_0$, and the battery pumps extra charge onto the plates as the slab goes in. The final energy is then

$$U_f = \frac{1}{2}C_f\,(\Delta V_0)^2 = \kappa U_i$$

The energy increases! So the battery adds energy as well as charge. Yet the slab still gets sucked in. How can this be? Energy balance looks like this:

Work done by battery - work done by fields = change in stored energy.

The work done by the battery to add charge dQ is

$$\begin{aligned} dW_{\text{battery}} &= dQ\,(\Delta V_0) \\ &= dC\,(\Delta V_0)^2 \\ &= \frac{dC}{dx}dx\,(\Delta V_0)^2 \end{aligned}$$

The work done by the fields on the slab is thus

$$dW_{\text{fields}} = dW_{\text{battery}} - dU = \frac{dC}{dx}(\Delta V_0)^2\,dx - \frac{dU}{dx}dx$$

But the energy now is

$$U(x) = \frac{1}{2} C(x) (\Delta V_0)^2 = \frac{1}{2} \varepsilon_0 \frac{w}{d} [\ell + (\kappa - 1) x] (\Delta V_0)^2$$

and

$$\frac{dU}{dx} = \frac{1}{2} \frac{dC}{dx} (\Delta V_0)^2$$

So

$$\begin{aligned} dW_{\text{fields}} &= F_x dx = \frac{dC}{dx} (\Delta V_0)^2 \left[1 - \frac{1}{2}\right] dx = \frac{1}{2} \frac{dC}{dx} (\Delta V_0)^2 dx \\ F_x &= \frac{1}{2} \frac{dC}{dx} (\Delta V_0)^2 = \frac{dU}{dx} \quad \text{(constant } \Delta V\text{)} && (15) \\ F_x &= \frac{1}{2} (\kappa - 1) \frac{C_0}{l} (\Delta V_0)^2 && (16) \end{aligned}$$

If we compare (11) and (15) it appears that the force has changed sign. But because the energy U depends on C, Q and ΔV, it is not that simple. When we actually *calculate* the force in the two cases we find that it is in the *same direction*, although with the battery connected, the force remains constant as the slab is inserted. Comparing (14) and (16), we see that the force in the second case equals the initial force in the first case.