Physics 360

Notes on Griffths - pluses and minuses

No textbook is perfect, and Griffiths is no exception. The major plus is that it is pretty readable. For minuses, see below.

Much of what G says about the del operator applies only in Cartesian coordinates. For example, we may only think of del as a vector such that $\vec{\nabla} \cdot \vec{u}$ is the dot product of $\vec{\nabla}$ and $\vec{u}$ if $\vec{\nabla}$ is expressed in Cartesian coordinates.

Page 27, The unit vectors in Cartesian coordinates are constants and may be pulled out of the integral, but unit vectors in other coordinate systems are not constants! For example, $\hat{r}$ may not be pulled out of an integral.

Unit vectors are written as $\hat{x}$, $\hat{y}$, $\hat{z}$ not $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$. This is consistent with what we do in every other coordinate system, such as $\hat{r}$, $\hat{\theta}$, $\hat{\phi}$ in spherical coordinates. (I regard this as a plus.)

G is often sloppy in writing vectors– experts can get away with things that beginners cannot because they know where they are going. But it's a bad habit. Don't be sloppy! Always write the vector sign, and be sure that there is a vector on both sides of an equals sign or neither. By convention, writing a vector without its vector sign means the magnitude of the vector: $v \equiv |\vec{v}|$. If you want to write the component, label it precisely, eg v_x or v_r, for example. Components have signs that indicate the direction of the vector; magnitudes are always positive.

G is also sloppy about explaining where results come from. Results of complex integrals are sometimes just thrown down with no explanation. In your homework and exams, you should either (a) do the integral yourself, showing all the steps, or (b) look it up, and give the complete reference to where you found the result, and what substitutions have to be made. (For example, if you have $\int \frac{1}{r} dr$ but the table has $\int \frac{1}{x} dx$, you will need to say that you are using result N.mm from Book X with $x = r$.) (a) is definitely preferred, because you will get better and better at doing integrals if you practice them, and in the long run it will save you time as well has help to develop your intuition. But go with (b) if the integral is really hard, and the homework is due in 10 minutes!

There are also a few Physics errors. I will point these out as we go along.

So let's get going!

Physics is a set of conceptual ideas that we express in mathematical form in order to solve problems. To understand physics well we have to use three different and equally important languages: English, pictures, and mathematics. Your solutions should contain adequate amounts of all three.

The subject of this course is electricity and magnetism. In the 20th century we learned that the fundamental forces of physics are:

1. Gravitational force
2. Strong or color force
3. Electroweak force

(1) is studied using Newton's laws, or, for strong fields, Einstein's equations.

(2) is studied using advanced mathematical concepts such as groups.

(3) is the unification of electromagnetic theory and the weak nuclear force. Electromagnetic theory is the first unified field theory: the unification of electric and magnetic field theory, and is the subject of our study this semester and next (in 460).

In Physics 230, we studied electric fields first, then magnetic fields. This is traditional, because the mathematics is a bit easier for electric fields, and we shall follow that plan in 360 too. But we have to remember that both are different aspects of *one* electromagnetic field. In Physics 230, we started with the fields produced by a single source element (eg a point charge) and built up the field due due a distribution of sources using *the principle of superposition.* We can do this because the relation of the fields to their sources is *linear.* This is an *experimental* fact. The electric field is a vector so when we add the fields due to different sources we are always doing *vector addition.*

Fundamental principles and definitions.

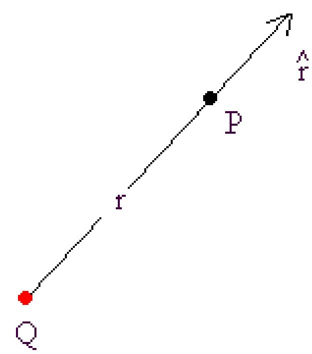
Charge is conserved. We believe that the total charge of the universe is exactly zero. Thus the only way to get a net charge Q in one place is to have a corresponding net charge $-Q$ somewhere else. Opposite charges attract, so these separated charges are always trying to get back together. This is the primary reason why electrostatic forces are not very obvious in everyday life, or on the large scales of astronomy..

Charge is quantized and appears in units of $e/3$, but we are doing *classical* theory this semester, so for the most part our charges will consist of so many fundamental units that we can consider charge to be a continuous quantity.

We start with the definition of electric field. We place a test charge q at point P and measure the electric force $\vec{F}$ acting on q. Then the electric field at P is

$$\vec{E} = \lim_{q \to 0} \frac{\vec{F}}{q}$$

From *Coulomb's law* (another experimental fact) we find that;



The electric field produced by a point charge Q is

$$\vec{E}\,(P) = \frac{kQ}{r^2}\hat{r}$$

where $\vec{r}$ is the vector with its tail on Q and its head at point P.

A simple example.

Suppose we have a point charge Q =1μC at the origin and a second charge $-2Q$ at point P_1 with coordinates (1 mm, 2 mm, 3 mm). What is the electric field at an arbitrary point P with coordinates (x, y, z)?

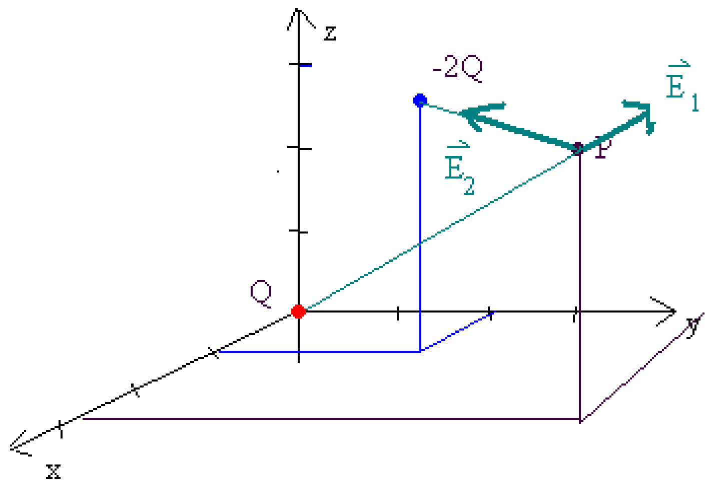
Given any problem, the solution consists of the following steps.

MODEL First determine the important physicical principles that govern the behavior of the system.

Here the principle that we need is Coulomb's Law. We are also going to use the geometry of flat space, as expressed in the Pythagorean theorem. That seems so obvious that you probably wouldn't think of it as a separate principal. The more physics you learn, the more the first bits you learned seem this "obvious", but we must be careful not to get complacent! We must be especially careful not to think that things are obvious when they are actually not even true!

A sketch of the field line diagram will help us understand the field. The net charge of the system is $Q - 2Q = -Q$ Using 4 lines per Q, 4 lines leave the $+Q$ charge and g to the $-Q$ charge, while 4 come in from infinity to the $-Q$ charge. We have to have spherical symmetry at a very great distance from both charges, or very close to either one. (You can download a program from my Physics 230 web site that will allow you to draw a pretty accurate diagram in a plane for this system.).

SETUP Now we are going to decide how to solve the problem, and get to the point where all we have left to do is some math. Let's start with a diagram.



The diagram shows the two charges and the field produced by each. Notice that $\vec{E}_2$ points toward the negative charge. Now we can calculate the field using Coulomb's law:

$$\vec{E}\,(P) = \frac{kq}{r^2}\hat{r} = kq\frac{\vec{r}}{r^3}$$

The second form will be easiest to use here, since we will have to calculate $\hat{r}$ as $\vec{r}/r$. We must be very clear as to what the symbol "r" means in this formula. It is the vector with its tail at the charge and its head at point P. For the first charge, Q, which is at the origin, $\vec{r}$ is the position vector of point P : $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ and $r = \sqrt{x^2 + y^2 + z^2}$. For the second charge, $-2Q$, the vector $\vec{r}_2$ has components $(x-1, y-2, z-3)$ where all lengths are measured in mm. , Then $r_2 = \sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2}$. Thus

$$\vec{E}_1 = \frac{kQ}{(x^2 + y^2 + z^2)^{3/2}}\,(x\hat{x} + y\hat{y} + z\hat{z})$$

and

$$\vec{E}_2 = -\frac{2kQ}{\left[(x-1)^2 + (y-2)^2 + (z-3)^2\right]^{3/2}}\,[(x-1)\,\hat{x} + (y-2)\,\hat{y} + (z-3)\,\hat{z}]$$

Finally, using the principle of superposition, we have

$$\vec{E} = \vec{E}_1 + \vec{E}_2$$

SOLVE Now we are ready to do the addition.

$$\vec{E} = \frac{kQ}{\left(x^2+y^2+z^2\right)^{3/2}}\left(x\hat{x}+y\hat{y}+z\hat{z}\right) - \frac{2kQ}{\left[(x-1)^2+(y-2)^2+(z-3)^2\right]^{3/2}}\left((x-1)\,\hat{x}+(y-2)\,\hat{y}+(z-3)\,\hat{z}\right)$$

This is pretty ugly. Let's do it one component at a time:

$$E_x = kQ\left\{\frac{x}{\left(x^2+y^2+z^2\right)^{3/2}} - \frac{2\,(x-1)}{\left[(x-1)^2+(y-2)^2+(z-3)^2\right]^{3/2}}\right\}$$

It's not going to be possible to make this look much nicer. We should always try to simplify our answer as much as possible, but here I don't see how we can do much more. The other components will look similar.

ANALYZE. This is a critical step.

Does the answer have the right physical dimensions? Let's check. The quantity in the curly brackets is length/(length)^3 = 1/(length)^2, so the answer is of the form $k\left(\text{charge/length}^2\right)$ which is correct.

Does the answer have the right value in special cases? Well, if we let x get very big, let's see what we get. The relevant physical value to compare with is the 1 mm separation (along the $x-$axis) of the two charges. So let $x \gg 1$ mm. Then we rewrite the second fraction like this:

$$\begin{aligned}
\frac{2\,(x-1)}{\left[(x-1)^2+(y-2)^2+(z-3)^2\right]^{3/2}} &= \frac{2x\left(1-\frac{1}{x}\right)}{x^3\left[\left(1-\frac{1}{x}\right)^2+\left(\frac{y-2}{x}\right)^2+\left(\frac{z-3}{x}\right)^2\right]^{3/2}} \\
&= \frac{2}{x^2}\frac{\left(1-\frac{1}{x}\right)}{\left[\left(1-\frac{1}{x}\right)^2+\left(\frac{y-2}{x}\right)^2+\left(\frac{z-3}{x}\right)^2\right]^{3/2}}
\end{aligned}$$

Now let's also suppose that $x \gg y, z$. In this limit the quantity multiplying $2/x^2 \to 1$ and we get

$$E_x = kQ\left\{\frac{1}{x^2} - \frac{2}{x^2}\right\} = -\frac{kQ}{x^2}$$

The y and z components are much smaller:

$$\frac{E_y}{E_x} \sim \frac{y}{x} \ll 1$$

Our result is the electric field on the $x-$axis due to a point charge $-Q$ at the origin. This is a particular example of a general rule:

RULE 1

> The electric field due to any charge distribution with net charge Q, measured at a very great distance from it, equals the electric field due to a point charge Q located within the charge distribution.

Now let's set $y = z = 0$, keeping $x \gg 1$ mm, and see what the first order correction is.

$$\begin{aligned} E_x &= kQ \left\{ \frac{1}{x^2} - \frac{2}{x^2} \frac{\left(1-\frac{1}{x}\right)}{\left[\left(1-\frac{1}{x}\right)^2 + \left(\frac{-2}{x}\right)^2 + \left(\frac{-3}{x}\right)^2\right]^{3/2}} \right\} \\ &= kQ \left\{ \frac{1}{x^2} - \frac{2}{x^2} f \right\} \end{aligned}$$

We're going to keep terms of order $1/x^3$ but drop terms of order $1/x^4$ and higher powers of $1/x$. So in the term f multiplying $2/x^2$ we keep terms of order $1/x$.

$$f = \frac{\left(1-\frac{1}{x}\right)}{\left[\left(1-\frac{1}{x}\right)^2 + \left(\frac{-2}{x}\right)^2 + \left(\frac{-3}{x}\right)^2\right]^{3/2}} = \frac{\left(1-\frac{1}{x}\right)}{\left(1-\frac{2}{x}\right)^{3/2}} + \mathcal{O}\left(1/x^2\right)$$

Now we use the binomial theorem to expand the denominator

$$\left(1-\frac{2}{x}\right)^{-3/2} = 1 + \left(-\frac{3}{2}\right)\left(\frac{-2}{x}\right) + \mathcal{O}\left(\frac{1}{x}\right)^2$$

Thus

$$f = \left(1-\frac{1}{x}\right)\left(1+\frac{3}{x}\right) = 1 + \frac{2}{x} - \frac{3}{x^2}$$

But we have already dropped terms of order $1/x^2$ in f, so here we *must* drop the $3/x^2$ term to be consistent. (You can make some very serious errors by using inconsistent levels of approximation!) Putting it all together, we have

$$\begin{aligned} E_x &= kQ \left\{ \frac{1}{x^2} - \frac{2}{x^2}\left(1+\frac{2}{x}\right) \right\} \\ &= kQ \left\{ -\frac{1}{x^2} - \frac{4}{x^3} \right\} \end{aligned}$$

The first term is the "point source" term that we had before. The second term results from the fact that there are actually two separate charges. It is a dipole field. (See LB $24.5 The field on the axis of the dipole is

$$\vec{E} = \frac{2k\vec{p}}{x^3})$$

We model our charge distribution as a point charge $-Q$ plus a dipole with charges $-Q$ at $x = 1$ mm and Q at the origin. The dipole moment $\vec{p}$ points from the negative charge to the positive charge, so in this case its $x-$component is negative.

$$p_x = Q \times (-1 \text{ mm})$$

This gives a dipole electric field on the x-axis of

$$E_x = -\frac{2kQ}{x^3}$$

But wait! We are off by a factor 2! That's because our point charge $-Q$ is not exactly *at* the origin, but displaced by a distance of 1 mm to the right. So it contributes an extra dipole moment of $p = -Q\,(1\text{ mm})$. We'll learn more about how to compute dipole moments in Chapter 3 (§3.4).

Just to round off the discussion, let's find the field on the $y-$axis ($x = z = 0$) with $y \gg 3$ mm.

$$\begin{aligned}
\vec{E}\,(0,y,0) &= \frac{kQy}{\left(y^2\right)^{3/2}}\hat{y} - \frac{2kQ}{\left[(-1)^2 + (y-2)^2 + (-3)^2\right]^{3/2}}\left(-\hat{x} + (y-2)\,\hat{y} - 3\hat{z}\right) \\
&= \frac{kQ}{y^2}\left(\hat{y} - 2\frac{\left(-\frac{1}{y}\hat{x} + \left(1-\frac{2}{y}\right)\hat{y} - \frac{3}{y}\hat{z}\right)}{\left[\left(-\frac{1}{y}\right)^2 + \left(1-\frac{2}{y}\right)^2 + \left(-\frac{3}{y}\right)^2\right]^{3/2}}\right)
\end{aligned}$$

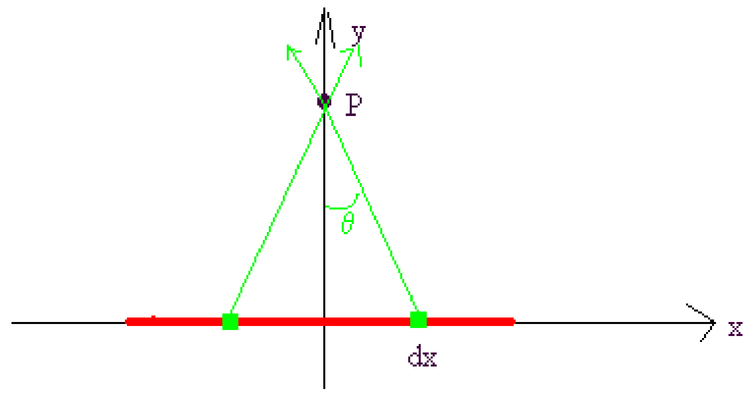
As before, let's drop terms in $1/y^2$ and higher inside the parentheses.

$$\begin{aligned}
\vec{E}\,(0,y,0) &= \frac{kQ}{y^2}\left(\hat{y} - 2\frac{\left(-\frac{1}{y}\hat{x} + \left(1-\frac{2}{y}\right)\hat{y} - \frac{3}{y}\hat{z}\right)}{\left(1-\frac{4}{y}\right)^{3/2}}\right) \\
&= \frac{kQ}{y^2}\left\{\hat{y} - 2\left[\left(-\frac{1}{y}\hat{x} + \left(1-\frac{2}{y}\right)\hat{y} - \frac{3}{y}\hat{z}\right)\right]\left(1+\frac{3}{y}\right)\right\} \\
&= \frac{kQ}{y^2}\left\{\frac{2}{y}\hat{x} + \hat{y}\left(1-2-\frac{2}{y}\right) + \frac{6}{y}\hat{z}\right\} \\
&= -\frac{kQ}{y^2}\hat{y} + \frac{kQ}{y^3}\left\{2\hat{x} - 2\hat{y} + 6\hat{z}\right\}
\end{aligned}$$

The leading term is the point charge field, as expected, and the correction is a dipole field, due to all three components of our dipole.. We'll come back to the details of this when we study chapter 3.

Example 2. Find the electric field at a point on the $y-$axis due to a filament of length 2ℓ that stretches from $x = -\ell$ to $x = \ell$ and has a uniform linear charge density λ.

Again we use *superposition*, but we start by *modelling* the line as a collection of differential elements, each of which we treat as a point charge.to which we can apply Coulomb's law.



Setup: A typical differential element is at coordinate x, with length dx and charge $\lambda dx = dq$ It prduces an electric field at P

$$d\vec{E} = k\frac{dq}{r^2}\hat{r}$$

that has both x and y components, as shown in the diagram. Now we can simplify our calculation by noting that our filament has mirror symmetry about the $y - z$ plane. Thus we can find another element at coordinate $-x$, the "mirror image" of our first element, that produces an electric field of the same magnitude at P. This second field has an identical $y-$component but the exact opposite $x-$component, so when we add them together the net result is a field in the $y-$direction. Thus as we sum up the contributions from all our elements, we find that the total electric field must be in the $y-$direction. It is important that we express the distance r in terms of the variable x. The electric field due to the pair of elements shown is

$$dE_y\left(P\right) = 2\frac{k\lambda dx}{\left(x^2 + y^2\right)}\cos\theta$$

Be careful here! x is the x-coordinate of our differential element (the one on the right) while y is the $y-$coordinate of the point P. It would be better to give them labels that emphasize this difference, so we'll use x' rather than x. From the diagram

$$\cos\theta = \frac{y}{r}$$

Thus

$$dE_y\left(P\right) = 2\frac{k\lambda y\ dx'}{\left[\left(x'\right)^2 + y^2\right]^{3/2}}$$

Now we sum the elements. The differential dx' tells us what our integration variable will be. The limits of the integral are 0 to l, since we are summing over pairs of elements, labelled by the $x-$coordinate of the element on the right in each pair.

Solve:

$$E_y(0,y,0)=\int_0^{\ell}2\frac{k\lambda y\ dx'}{\left[(x')^2+y^2\right]^{3/2}}$$

Now k, λ and y are all independent of x', so we may remove them from the integral.

$$E_y(0,y,0)=2k\lambda y\int_0^{\ell}\frac{dx'}{\left[(x')^2+y^2\right]^{3/2}}$$

To do the integral, we use the tangent substitution. Let $x'=y\tan\theta$. (Note that this angle is actually the same θ marked in the diagram.) Then $(x')^2+y^2=y^2\left(1+\tan^2\theta\right)=y^2\sec^2\theta$, and so

$$\begin{aligned}E_y &= 2k\lambda y\int_0^{\tan^{-1}(\ell/y)}\frac{y\sec^2\theta d\theta}{y^3\sec^3\theta}\\ &= \frac{2k\lambda}{y}\int_0^{\tan^{-1}(\ell/y)}\cos\theta\ d\theta\\ &= \frac{2k\lambda}{y}\sin\theta|_0^{\tan^{-1}(\ell/y)} \qquad (1)\end{aligned}$$

To evaluate the sine, note that

$$\sin\theta=\cos\theta\tan\theta=\frac{\tan\theta}{\sqrt{\sec^2\theta}}=\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$$

Thus

$$\begin{aligned}E_y(0,y,0) &= \frac{2k\lambda}{y}\frac{\ell/y}{\sqrt{1+(\ell/y)^2}} \qquad (2)\\ &= \frac{2k\lambda}{y}\frac{\ell}{\sqrt{y^2+\ell^2}} \qquad (3)\end{aligned}$$

Analyze: First note that our result has the right dimensions. Looking at the result in the form labelled (2), we see that the factor multiplying $2k\lambda/y$ is dimensionless, because ℓ/y is a dimensionless number. The linear charge density λ is charge/length, so $k\lambda/y$ is $k\times$charge/length^2, as required.

Now let's look at the field a long way from the line, so that $y\gg\ell$. Then in (2) we neglect the term $(\ell/y)^2$ compared with the one in the denominator. The result is

$$E_y=k\frac{2\lambda\ell}{y^2\sqrt{1+\ell^2/y^2}}\simeq k\frac{2\lambda\ell}{y^2}=k\frac{Q}{y^2}$$

This is the electric field due to a point charge $Q=2\lambda\ell$ at the origin. Here is another example of RULE 1 that we found above.

Now let's see what happens if we let our line become infinitely long. The easiest way to get the result is to go back to result (1) and let $l \to \infty$. We note that $\lim_{w\to\infty} \tan^{-1}(w) = \pi/2$, and $\sin \pi/2 = 1$, so we have immediately

$$E_y = \frac{2k\lambda}{y} \qquad \text{(infinite line)}$$

From (3) we can see that this is also the result when we are very close to a finite line ($y \ll \ell$).

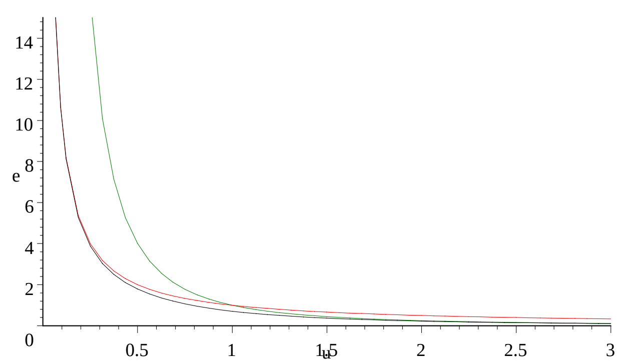
We have found more than you might imagine. Notice that our system has rotational symmetry about the $x-$axis in addition to the mirror symmetry we already exploited. Thus we have found the electric field anywhere in the $y - z$ plane, and we may write the result as

$$\vec{E} = \frac{2k\lambda}{s} \frac{\ell/s}{\sqrt{1 + (\ell/s)^2}} \hat{r}$$

where s is the radial coordinate in a cylindrical coordinate system with z-axis along the line of the charge.

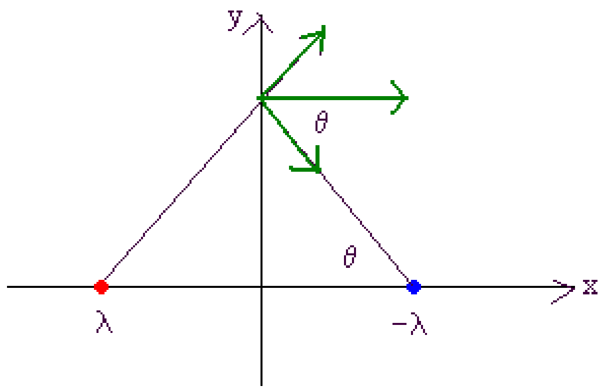
Now let's compare the three results (3), (**??**) and (**??**). We plot the dimensionless field variable $e = E_y/(2k\lambda/\ell)$ versus the dimensionless distance variable $u = y/\ell$. Then the three expressions are

$$\begin{aligned} &\text{Exact (black line):} && e = \frac{1}{u\sqrt{1+u^2}} \\ &\text{Near (red line)} \quad : && e = \frac{1}{u} \\ &\text{Far (green line)} \quad : && e = \frac{1}{u^2} \end{aligned}$$



The "near" result works very well for $u \lesssim 0.3$ while the "far" result works well for $u \gtrsim 1.5$.

Example 3. Now what if we have two infinite lines, with linear charge densities λ and $-\lambda$?. What is the electric field produced by these lines at a point in the plane midway between them?



MODEL We use the resut we have already found for an infinite line, and add the two fields, as shown in the diagram.

SETUP Putting the $z-$axis along the line joining the two infinite filaments, and y axis along the bisector, with point P having coordinates $(0, y)$ and the separation of the lines being $2L$, we can see that the $y-$components cancel while the $x-$components add, to give

SOLVE

$$\begin{aligned}\vec{E} &= 2\frac{2k\lambda}{y}\cos\theta\ \hat{x} \\ &= \frac{4k\lambda}{y}\frac{L}{\sqrt{L^2+y^2}}\hat{x}\end{aligned}$$

ANALYZE for $y \gg L$, we get

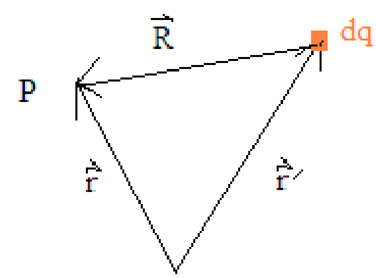
$$\vec{E} \to \frac{4k\lambda}{y^2}\hat{x}$$

This is the result for a line dipole. Let's summarize what we have learned so far about how electric fields vary with distance from the source.

source	one	two (dipole)
point	1/distance^2	1/distance^3
line	1/distance	1/distance^2

We'll finish here by writing the expression for $\vec{E}$ as an integral in its most general form. We label the point P at which we want to find $\vec{E}$ with the position vector $\vec{r}$ with respect to some origin O, and our differential element with position vector $\vec{r}'$. Then the vector pointing from the element $dq = \rho(\vec{r}')\,d\tau'$ to P is

$$\vec{R} = \vec{r} - \vec{r}'$$



and the electric field at P produced by the element is

$$d\vec{E}\left(\vec{r}\right) = k\frac{dq}{R^2}\hat{R} = k\frac{\rho\left(\vec{r}'\right)d\tau'}{\left|\vec{r}-\vec{r}'\right|^3}\left(\vec{r}-\vec{r}'\right)$$

and then we sum over all the elements by integrating over the primed coordinates:

$$\vec{E}\left(\vec{r}\right) = \int d\vec{E} = k\int_V \frac{\rho\left(\vec{r}'\right)\left(\vec{r}-\vec{r}'\right)}{\left|\vec{r}-\vec{r}'\right|^3}d\tau'$$

The volume V is all space, but in practice we can limit it to the volume where ρ is not zero.