Green's function for the wave equation

Non-relativistic case

January 2012

1 The wave equations

In the **Lorentz Gauge**, the wave equations for the potentials are (notes 1 eqns 35 and 36):

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{j} \tag{1}$$

and

$$\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\varepsilon_0} \tag{2}$$

The Gauge condition is (Notes 1 eqn 34):

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0 \tag{3}$$

In **Coulomb Gauge** we have the Gauge condition:

$$\vec{\nabla} \cdot \vec{A} = 0 \tag{4}$$

which leads to the wave equations (Notes 1, eqn below 35, with $\vec{\nabla} \cdot \vec{A} = 0$)

$$\nabla^2 \Phi = -\frac{\rho}{\varepsilon_0} \tag{5}$$

and (Notes 1 eqn 33 with $\vec{\nabla} \cdot \vec{A} = 0$)

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{j} + \frac{1}{c^2}\vec{\nabla}\frac{\partial \Phi}{\partial t} \tag{6}$$

2 Longitudinal and transverse currents

The Coulomb Gauge wave equation for $\vec{A}$ (6) is awkward because it contains Φ. We can eliminate the potential Φ to obtain an equation for $\vec{A}$ alone. First we separate the current into two pieces, called the *longitudinal current* $\vec{J}_\ell$ and the *transverse current* $\vec{J}_t$:

$$\vec{J} = \vec{J}_\ell + \vec{J}_t$$

where

$$\vec{\nabla} \cdot \vec{J}_t = 0 \tag{7}$$

and

$$\vec{\nabla} \times \vec{J}_\ell = 0 \tag{8}$$

(Basically we are applying the Helmholtz theorem. We can always do this: See Lea Appendix II). From charge conservation:

$$\vec{\nabla}\cdot\vec{J}+\frac{\partial\rho}{\partial t}=0$$

and using equation 5 to eliminate ρ, this becomes:

$$\vec{\nabla}\cdot\vec{J}_\ell=-\frac{\partial}{\partial t}\left(-\varepsilon_0\nabla^2\Phi\right)=\vec{\nabla}\cdot\left[\varepsilon_0\frac{\partial}{\partial t}\left(\vec{\nabla}\Phi\right)\right]$$

Thus, up to an irrelevant constant,

$$\vec{J}_\ell=\varepsilon_0\vec{\nabla}\frac{\partial}{\partial t}\Phi+\vec{\nabla}\times\vec{u}$$

But from (8)

$$\vec{\nabla}\times\vec{J}_\ell=\vec{\nabla}\times\left(\vec{\nabla}\times\vec{u}\right)=0=\vec{\nabla}\left(\vec{\nabla}\cdot\vec{u}\right)-\nabla^2\vec{u}$$

So we take $\vec{u}=0$, and then

$$\vec{J}_\ell=\varepsilon_0\vec{\nabla}\frac{\partial}{\partial t}\Phi \tag{9}$$

Using equation 9 in equation 6, we have:

$$\begin{aligned}\nabla^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} &= -\mu_0\left(\vec{J}-\varepsilon_0\vec{\nabla}\frac{\partial\Phi}{\partial t}\right)\\ &= -\mu_0\left(\vec{J}-\vec{J}_\ell\right)=-\mu_0\vec{J}_t \qquad (10)\end{aligned}$$

In the Coulomb Gauge, the transverse current $\vec{J}_t$ is the source of $\vec{A}$.

We can also use result (9) to express $\vec{J}_l$ in terms of $\vec{J}$. Since equation (5) is the same as in the static case, the solution is also the same (Notes 1 eqn 22). Thus

$$\begin{aligned}\vec{J}_\ell\left(\vec{x},t\right) &= \frac{1}{4\pi}\vec{\nabla}\frac{\partial}{\partial t}\int\frac{\rho\left(\vec{x}',t\right)}{\left|\vec{x}-\vec{x}'\right|}d^3x'\\ &= \frac{1}{4\pi}\vec{\nabla}\int\frac{\frac{\partial}{\partial t}\rho\left(\vec{x}',t\right)}{\left|\vec{x}-\vec{x}'\right|}d^3x'\\ &= -\frac{1}{4\pi}\vec{\nabla}\int\frac{\vec{\nabla}'\cdot\vec{J}\left(\vec{x}'\right)}{\left|\vec{x}-\vec{x}'\right|}d^3x' \qquad (11)\end{aligned}$$

where we used charge conservation in the last step. We can also express $\vec{J}_t$ in terms of $\vec{J}$ as follows, using (11),

$$\vec{J}_t=\vec{J}-\vec{J}_\ell=\vec{J}+\frac{1}{4\pi}\vec{\nabla}\int\frac{\vec{\nabla}'\cdot\vec{J}}{\left|\vec{x}-\vec{x}'\right|}d^3x'$$

Let's work on the integral:

$$\begin{aligned}\int \frac{\vec{\nabla}' \cdot \vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x' &= \int \vec{\nabla}' \left(\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right) d^3x' - \int \vec{J} \cdot \vec{\nabla}' \frac{1}{|\vec{x}-\vec{x}'|} d^3x' \\ &= \int_{S_\infty} \frac{\vec{J} \cdot \hat{n}}{|\vec{x}-\vec{x}'|} d^2x' + \int \vec{J} \cdot \vec{\nabla} \frac{1}{|\vec{x}-\vec{x}'|} d^3x' \\ &= 0 + \vec{\nabla} \cdot \int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x'\end{aligned}$$

The surface integral is zero provided that $\vec{J}$ is localized. Then

$$\begin{aligned}\vec{J}_t(\vec{x},t) &= \vec{J} + \frac{1}{4\pi}\vec{\nabla}\left(\vec{\nabla} \cdot \int \frac{\vec{J}(\vec{x}',t)}{|\vec{x}-\vec{x}'|} d^3x'\right) \\ &= \vec{J} + \frac{1}{4\pi}\left[\vec{\nabla} \times \left(\vec{\nabla} \times \int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x'\right) + \nabla^2 \int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x'\right] \\ &= \vec{J} + \frac{1}{4\pi}\left[\vec{\nabla} \times \left(\vec{\nabla} \times \int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x'\right) + \int \vec{J}\, \nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} d^3x'\right] \\ &= \vec{J} + \frac{1}{4\pi}\left[\vec{\nabla} \times \left(\vec{\nabla} \times \int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x'\right) + \int \vec{J}\left[-4\pi\delta(\vec{x}-\vec{x}')\right] d^3x'\right] \\ \vec{J}_t &= \frac{1}{4\pi}\vec{\nabla} \times \left(\vec{\nabla} \times \int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x'\right) \qquad (12)\end{aligned}$$

We still have the rather unphysical result from equation (5) that Φ changes instantaneously everywhere as ρ changes. In classical physics the potential is just a mathematical construct that we use to find the fields, and it can be shown (eg prob 6.20) that $\vec{E}$ is causal even though Φ is not.

3 The Green's function

With either gauge we have a wave equation of the form

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -\,(\text{source})$$

where Φ can be either the scalar potential (in Lorentz Gauge) or a Cartesian component of $\vec{A}$. (In Coulomb Gauge the scalar potential is found using the methods we have already developed for the static case.) The corresponding Green's function problem is:

$$\nabla^2 G(\vec{x},t;\vec{x}',t') - \frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} = -4\pi\delta(\vec{x}-\vec{x}')\,\delta(t-t') \qquad (13)$$

where the source is now a unit *event* located at position $\vec{x} = \vec{x}'$ and happening at time $t = t'$. To solve this equation we first Fourier transform in time (see, eg, Lea pg 503):

$$G(\vec{x},t;\vec{x}',t') = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G(\vec{x},\omega;\vec{x}',t')\, e^{-i\omega t} d\omega$$

and the transformed equation (13) becomes

$$\begin{aligned}\left(\nabla^2+\frac{\omega^2}{c^2}\right)G\left(\vec{x},\omega;\vec{x}',t'\right) &= -4\pi\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta\left(\vec{x}-\vec{x}'\right)\delta\left(t-t'\right)e^{i\omega t}dt \\ &= -\frac{4\pi}{\sqrt{2\pi}}\delta\left(\vec{x}-\vec{x}'\right)e^{i\omega t'}\end{aligned}$$

So let $G\left(\vec{x},\omega;\vec{x}',t'\right)=g\left(\vec{x},\vec{x}'\right)e^{i\omega t'}/\sqrt{2\pi}$ and then g satisfies the equation

$$\left(\nabla^2+k^2\right)g=-4\pi\delta\left(\vec{x}-\vec{x}'\right) \tag{14}$$

where $k=\omega/c$. In free space without boundaries, g must be a function only of $R=\left|\vec{x}-\vec{x}'\right|$ and must posess spherical symmetry about the source point. Thus in spherical coordinates, we can write:

$$\frac{1}{R}\frac{d^2}{dR^2}\left(Rg\right)+k^2g=-4\pi\delta\left(\vec{R}\right) \tag{15}$$

For $\vec{R}\neq 0$, the right hand side is zero. Then the function Rg satisfies the exponential equation, and the solution is:

$$\begin{aligned}Rg &= Ae^{ikR}+Be^{-ikR} \\ g &= \frac{1}{R}\left(Ae^{ikR}+Be^{-ikR}\right)\end{aligned} \tag{16}$$

Near the origin, where the delta-function contributes, the second term on the LHS of (14) is negligible compared with the first, and equation (14) becomes:

$$\nabla^2 g=-4\pi\delta\left(\vec{x}-\vec{x}'\right)$$

and we recognize that this has solution

$$g=\frac{1}{R}$$

This is consistent with equation (16) as $R\to 0$, provided that

$$A+B=1$$

(You should convince yourself that this solution is correct by differentiating and stuffing back into equation 15.)

Thus we have the solution

$$G\left(\vec{x},\omega;\vec{x}',t'\right)=\frac{1}{\sqrt{2\pi}R}\left(Ae^{ikR}+Be^{-ikR}\right)e^{i\omega t'}$$

Now we do the inverse transform:

$$\begin{aligned}G\left(\vec{x},t;\vec{x}',t'\right) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}R}\left(Ae^{ikR}+Be^{-ikR}\right)e^{i\omega t'}e^{-i\omega t}d\omega \\ &= \frac{1}{2\pi R}\int_{-\infty}^{\infty}\left\{A\exp\left[i\omega\left(R/c+t'-t\right)\right]+B\exp\left[i\omega\left(-R/c+t'-t\right)\right]\right\}d\omega \\ &= \frac{A}{R}\,\delta\left[t'-\left(t-R/c\right)\right]+\frac{B}{R}\,\delta\left[t'-\left(t+R/c\right)\right]\end{aligned} \tag{17}$$

The second term is usually rejected (take $B\equiv 0$ and thus $A=1$) because it predicts a response to an event occurring in the future. However, Feynman and Wheeler[1] have

[1] Reviews of Modern Physics, 1949, **21**, 425

proposed a theory in which both terms are kept. They show that this theory can be consistent with observed causality provided that the universe is perfectly absorbing in the infinite future. (This now appears unlikely.) The time $t - R/c$ that appears in the first term is called the *retarded time* $t_{\text{ret.}}$ Thus we take

$$G(\vec{x}, t; \vec{x}', t') = \frac{1}{R}\,\delta\left[t' - (t - R/c)\right] = \frac{1}{R}\,\delta\left(t' - t_{\text{ret}}\right) \tag{18}$$

Causality (an event cannot precede its cause) requires that the symmetry of this Green's function is:

$$G(\vec{x}, t; \vec{x}', t') = G(\vec{x}', -t'; \vec{x}, -t)$$

(See Morse and Feshbach Ch 7 pg 834-835). and also

$$G(\vec{x}, -\infty; \vec{x}', t') = 0$$

and

$$G(\vec{x}, t; \vec{x}', t') = 0 \text{ for } t < t'$$

4 The potentials

Now that we have the Green's function (18), we can solve our original equations. Modifying eqn 1.44 in Jackson to include time dependence, and with $S \to \infty$, we get an integral over a volume in space-time rather than just space:

$$\Phi(\vec{x}, t) = \frac{1}{4\pi\varepsilon_0}\int \rho(\vec{x}', t')\, G(\vec{x}, t; \vec{x}', t')\; dt' d^3\vec{x}'$$

Thus, inserting (18), we have

$$\Phi(\vec{x}, t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{x}', t')}{R}\delta\left(t' - t_{\text{ret}}\right)\; dt' d^3\vec{x}' \tag{19}$$

$$= \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{x}', t_{\text{ret}})}{R(t_{\text{ret}})}\; d^3\vec{x}' \tag{20}$$

in Lorentz Gauge, and similarly:

$$\vec{A}(\vec{x}, t) = \frac{\mu_0}{4\pi}\int \frac{\vec{j}(\vec{x}', t_{\text{ret}})}{R(t_{\text{ret}})}\; d^3x' \tag{21}$$

Notice that these equations have the same form as the static potentials (equations 1.17 and 5.32 in Jackson), but we must evaluate the source at the retarded time. This allows for the time for a signal to travel from the source to the observer at speed c.

Similar equations hold in Coulomb Gauge. The scalar potential changes instantaneously as ρ changes, and the vector potential involves the transverse current only in equation (21). In spite of this peculiarity, the fields $\vec{E}$ and $\vec{B}$ are causal. (See problem 6.20 which demonstrates this in a relatively simple case. This problem may be "relatively" simple, but it is not easy.)

When spatial boundaries are present the analysis is more complicated. We must use the same kind of techniques that we used in the static case, expanding G in eigenfunctions. However, the vector nature of $\vec{A}$ makes the problem much harder. (See Chapter 9 sections 6-12.)

5 Radiation from a moving point charge

5.1 The Lienard-Wiechert potentials

Our source is a point charge moving with velocity $\vec{v}(t)$. Then the charge and current densities are

$$\rho(\vec{x},t) = q\delta(\vec{x} - \vec{r}(t))$$

and

$$\vec{j}(\vec{x},t) = q\vec{v}\delta(\vec{x} - \vec{r}(t))$$

Because the source terms are delta-functions, it turns out to be easier to back up one step. Then from equation 19, we have:

$$\Phi(\vec{x},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{q\delta(\vec{x}' - \vec{r}(t'))}{R}\delta(t' - t_{\text{ret}})\ dt'd^3x'$$

We do the integral over the spatial coordinates first. Then

$$\Phi(\vec{x},t) = \frac{1}{4\pi\varepsilon_0}\int q\frac{\delta(t' + R(t')/c - t)}{R(t')}dt'$$

where $R(t') = |\vec{x} - \vec{r}(t')|$. To do the t' integral, we must re-express the delta-function. Recall (Lea eqn 6.10)

$$\delta[f(x)] = \sum_i \frac{1}{|f'(x_i)|}\delta(x - x_i)$$

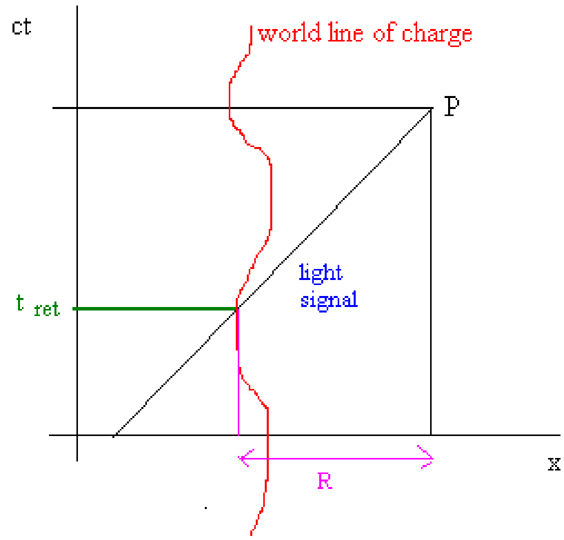
where $f(x_i) = 0$. In this case:

$$f(t') = t' + \frac{R(t')}{c} - t$$

and, since $\vec{v} = d\vec{r}/dt$,

$$\begin{aligned} f'(t') &= 1 + \frac{1}{c}\frac{dR}{dt'} = 1 + \frac{1}{c}\frac{d}{dt'}\sqrt{[\vec{x} - \vec{r}(t')]\cdot[\vec{x} - \vec{r}(t')]} \\ &= 1 + \frac{1}{c}\frac{[\vec{x} - \vec{r}(t')]}{|\vec{x} - \vec{r}(t')|}\cdot\frac{d}{dt'}(\vec{x} - \vec{r}(t')) \\ &= 1 - \frac{\vec{v}\cdot[\vec{x} - \vec{r}(t')]}{c|\vec{x} - \vec{r}(t')|} = 1 - \frac{\vec{v}\cdot\vec{R}}{cR} \end{aligned}$$

The function f is zero when t' equals the solution of the equation $t_{\text{ret}} = t - R(t_{\text{ret}})/c$. A space-time diagram shows this most easily.



Thus, evaluating the integral, we get:

$$\Phi\left(\vec{x},t\right) = \frac{1}{4\pi\varepsilon_0}\int q\frac{\delta\left(t'-t_{\text{ret}}\right)}{R\left(t'\right)\left(1-\frac{\vec{v}\cdot\vec{R}}{cR}\right)}dt' = \frac{1}{4\pi\varepsilon_0}\left.\frac{q}{R\left(1-\frac{\vec{v}\cdot\vec{R}}{cR}\right)}\right|_{t_{\text{ret}}} \tag{22}$$

and similarly

$$\vec{A}\left(\vec{x},t\right) = \frac{\mu_0}{4\pi}\left.\frac{q\vec{v}}{R\left(1-\frac{\vec{v}\cdot\vec{R}}{cR}\right)}\right|_{t_{\text{ret}}} \tag{23}$$

These are the Lienhard-Wiechert potentials. It is convenient to use the shorthand

$$r_v = R\left(1-\frac{\vec{v}\cdot\vec{R}}{cR}\right) = R - \frac{\vec{v}\cdot\vec{R}}{c} \tag{24}$$

so that

$$\vec{A}\left(\vec{x},t\right) = \frac{\mu_0}{4\pi}\left.\frac{q\vec{v}}{r_v}\right|_{t_{\text{ret}}} = \frac{\vec{v}}{c^2}\Phi$$

5.2 Calculating the fields

The fields are found using

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}$$

and

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

But our expressions for the potentials are in terms of $\vec{x}$ and t_{ret}, not $\vec{x}$ and t, so we have to be very careful in taking the partial derivatives. We can put the origin at the instantaneous position of the charge to simplify things. Then $R = r$. Our potential may be written:

$$\Phi(\vec{x}, t) \equiv \Psi(\vec{x}, t_{\text{ret}})$$

Then

$$d\Phi = \vec{\nabla}\Phi\Big|_{\text{const t}} \cdot d\vec{x} + \frac{\partial \Phi}{\partial t} dt \equiv \vec{\nabla}\Psi\Big|_{\text{const } t_{\text{ret}}} \cdot d\vec{x} + \frac{\partial \Psi}{\partial t_{\text{ret}}} dt_{\text{ret}} = d\Psi$$

But $dt_{\text{ret}} = dt - dr/c$, so

$$\vec{\nabla}\Phi\Big|_{\text{const t}} \cdot d\vec{x} + \frac{\partial \Phi}{\partial t} dt \equiv \vec{\nabla}\Psi\Big|_{\text{const } t_{\text{ret}}} \cdot d\vec{x} - \frac{\partial \Psi}{\partial t_{\text{ret}}} \frac{dr}{c} + \frac{\partial \Psi}{\partial t_{\text{ret}}} dt$$

Thus the $r-$component of $\vec{\nabla}\Phi$ must be modified:

$$\frac{\partial \Phi}{\partial r}\Big|_{\text{const t}} = \frac{\partial \Psi}{\partial r}\Big|_{\text{const } t_{\text{ret}}} - \frac{1}{c}\frac{\partial \Psi}{\partial t_{\text{ret}}} \tag{25}$$

Now we can calculate the fields:

$$\vec{\nabla}\Phi = \frac{1}{4\pi\varepsilon_0}\vec{\nabla}\frac{q}{r_v} = -\frac{1}{4\pi\varepsilon_0}\frac{q}{r_v^2}\vec{\nabla}r_v$$

and

$$\vec{\nabla}r_v = \frac{\partial}{\partial r}\left(r - \frac{\vec{r}\cdot\vec{v}}{c}\right)\hat{r} + \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial}{\partial \theta}\left(r - \frac{\vec{r}\cdot\vec{v}}{c}\right) + \frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\frac{\partial}{\partial \phi}\left(r - \frac{\vec{r}\cdot\vec{v}}{c}\right)$$

We can choose our axes with polar axis along the instantaneous direction of $\vec{v}$. Then $\vec{r}\cdot\vec{v} = rv\cos\theta$, and

$$\vec{\nabla}r_v = \left(1 - \frac{v}{c}\cos\theta\right)\hat{r} + \frac{\hat{\boldsymbol{\theta}}}{r}\left(r\frac{v}{c}\sin\theta\right)$$

In this coordinate system

$$\vec{v} = v\hat{z} = v\left(\hat{r}\cos\theta - \hat{\theta}\sin\theta\right)$$

so

$$\vec{\nabla}r_v = \hat{r} - \frac{\vec{v}}{c}$$

In the non-relativistic limit, $v/c \ll 1$, to zeroth order in v/c, this becomes $\vec{\nabla}r_v = \hat{r}$. We are also going to need

$$\frac{\partial r_v}{\partial t} = -\frac{\vec{r}\cdot\vec{a}}{c}$$

Then we have

$$\begin{aligned}
\vec{E} &= -\vec{\nabla}\Phi\Big|_{\text{const } t_{\text{ret}}} + \frac{1}{c}\frac{\partial \Phi}{\partial t}\hat{r} - \frac{\partial \vec{A}}{\partial t} \\
&= -\vec{\nabla}\Phi\Big|_{\text{const } t_{\text{ret}}} + \frac{1}{c}\frac{\partial \Phi}{\partial t}\hat{r} - \frac{1}{c^2}\frac{\partial}{\partial t}(\vec{v}\Phi) \\
&= -\vec{\nabla}\Phi\Big|_{\text{const } t_{\text{ret}}} + \frac{1}{c}\frac{\partial \Phi}{\partial t}\left(\hat{r} - \frac{\vec{v}}{c}\right) - \frac{\Phi}{c^2}\frac{\partial}{\partial t}\vec{v}
\end{aligned}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r_v^2}\left(\hat{r} - \frac{\vec{v}}{c}\right) - \frac{(\hat{r} - \vec{v}/c)}{4\pi\varepsilon_0 c}\frac{q}{r_v^2}\left(-\frac{\vec{r}\cdot\vec{a}}{c}\right) - \frac{q}{4\pi\varepsilon_0 c^2}\frac{\vec{a}}{r_v}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r_v^2}\left(\hat{r}-\frac{\vec{v}}{c}\right)+\frac{1}{4\pi\varepsilon_0 c^2}\frac{q}{r_v}\frac{(\hat{r}-\vec{v}/c)\,(\hat{r}\cdot\vec{a})-\vec{a}\,(1-\hat{r}\cdot\vec{v}/c)}{(1-\hat{r}\cdot\vec{v}/c)} \quad (26)$$

$$= \frac{q}{4\pi\varepsilon_0 r_v^2}\left(\hat{r}-\frac{\vec{v}}{c}\right)+\frac{\mu_0 q}{4\pi r_v\,(1-\hat{r}\cdot\vec{v}/c)}\hat{r}\times\left[\left(\hat{r}-\frac{\vec{v}}{c}\right)\times\vec{a}\right] \quad (27)$$

Taking the non-relativistic limit[2] $v/c \ll 1$, (27) becomes:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{r^2}\hat{r}-\frac{q}{c^2}\frac{[\hat{r}\times(\hat{r}\times\vec{a})]}{r}\right]$$

The first term is the usual Coulomb field which goes as $1/r^2$. The second term depends on $\vec{a}$: this is the radiation field.

$$\vec{E}_{\text{rad}} = \frac{\mu_0}{4\pi}\frac{q}{r}\left[\hat{r}\times(\hat{r}\times\vec{a})\right] \quad (28)$$

This term decreases as $1/r$ and dominates at large r.

Next let's calculate the magnetic field:

$$\vec{B} = \vec{\nabla}\times\vec{A}\Big|_{\text{const } t}$$

$$= \left(\vec{\nabla}\Big|_{\text{const } t_{\text{ret}}}-\frac{1}{c}\hat{r}\frac{\partial}{\partial t}\right)\times\frac{\mu_0}{4\pi}\frac{q\vec{v}}{r_v}$$

$$\vec{B} = \frac{\mu_0}{4\pi}\left[q\vec{\nabla}\left(\frac{1}{r_v}\right)\times\vec{v}-\frac{q}{c}\hat{r}\times\left(\frac{\vec{a}}{r_v}-\frac{\vec{v}}{r_v^2}\frac{\partial r_v}{\partial t}\right)\right]$$

$$= -\frac{\mu_0}{4\pi}\left[\frac{q}{r_v^2}\left(\hat{r}-\frac{\vec{v}}{c}\right)\times\vec{v}+\frac{q}{c}\hat{r}\times\left(\frac{\vec{a}}{r_v}+\frac{\vec{v}}{r_v^2}\frac{\vec{r}\cdot\vec{a}}{c}\right)\right]$$

$$= \frac{\mu_0 q}{4\pi r_v^2}\vec{v}\times\hat{r}-\frac{\mu_0}{4\pi}\frac{q/c}{r_v\,(1-\vec{v}\cdot\hat{r}/c)}\left(\hat{r}\times\vec{a}\,(1-\vec{v}\cdot\hat{r}/c)+\hat{r}\times\vec{v}\frac{\vec{r}\cdot\vec{a}}{c}\right)$$

We can simpify this result using (26) for $\vec{E}_{\text{rad}}$ and the fact that $\hat{r}\times\hat{r}\equiv 0$.

$$\vec{B} = \frac{\mu_0 q}{4\pi r_v^2}\vec{v}\times\hat{r}+\frac{\mu_0}{4\pi}\frac{q}{r_v\,(1-\vec{v}\cdot\hat{r}/c)}\frac{\hat{r}\times[(\hat{r}-\vec{v}/c)\,(\hat{r}\cdot\vec{a})-\vec{a}\,(1-\hat{r}\cdot\vec{v}/c)]}{c} \quad (29)$$

$$= \vec{B}_{\text{B-S}}+\hat{r}\times\vec{E}_{\text{rad}}/c$$

In the limit $v/c \ll 1$, the first term, which goes as $1/r^2$, is the usual Biot-Savart law result . The second term, which goes as $1/r$, is the radiation field. Notice that

$$\vec{B}_{\text{rad}} = \hat{r}\times\vec{E}_{\text{rad}}/c$$

as expected for an EM wave. In the non-relativistic limit,

$$\vec{B}_{\text{rad}} = \frac{\mu_0}{4\pi}\frac{q}{rc}\vec{a}\times\hat{r} \quad (30)$$

The Poynting flux for the radiation field is:

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0}\vec{E}\times\left(\frac{\hat{r}\times\vec{E}}{c}\right)$$

$$= \frac{E^2}{\mu_0 c}\,\hat{r} \quad (31)$$

[2] Eqn (27) is not quite correct if v/c is not small, as we have not done a correct relativistic treatment of time. We are missing some factors of γ and $1-\vec{\beta}\cdot\hat{r}$.

where from equation 28:

$$E = \frac{\mu_0}{4\pi}\frac{q}{r}a\left|\sin\theta\right| = \frac{1}{4\pi\varepsilon_0}\frac{q}{rc^2}a\left|\sin\theta\right|$$

and θ is the angle between $\vec{a}$ and $\hat{r}$. Thus in the non-relativistic limit

$$S = \frac{1}{\mu_0 c}\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{rc^2}a\sin\theta\right)^2 = \frac{q^2}{\left(4\pi\right)^2\varepsilon_0 c^3}\frac{a^2}{r^2}\sin^2\theta$$

Notice that $S \propto 1/r^2$, the usual inverse square law for light. Writing $dA = r^2 d\Omega$, we find the power radiated per unit solid angle is:

$$\frac{dP}{d\Omega} = r^2 S = \frac{q^2 a^2}{\left(4\pi\right)^2\varepsilon_0 c^3}\sin^2\theta \tag{32}$$

Finally the total power radiated is

$$P = \int\frac{dP}{d\Omega}d\Omega = \frac{q^2a^2}{\left(4\pi\right)^2\varepsilon_0 c^3}\int_0^{2\pi}\int_{-1}^{+1}\left(1-\mu^2\right)d\phi d\mu$$

where as usual $\mu = \cos\theta$. Thus

$$\begin{aligned} P &= \frac{q^2a^2}{\left(4\pi\varepsilon_0\right)2c^3}\left(\mu - \frac{\mu^3}{3}\right)\Bigg|_{-1}^{+1} \\ &= \frac{2}{3}\frac{q^2a^2}{4\pi\varepsilon_0 c^3} \end{aligned} \tag{33}$$

This result is called the Larmor formula.

In the relativistic case, we use (27) in (31) to get:

$$\vec{S} = \frac{1}{\mu_0 c}\hat{r}\left|\frac{\mu_0 q}{4\pi r_v\left(1-\hat{r}\cdot\vec{v}/c\right)}\hat{r}\times\left[\left(\hat{r}-\frac{\vec{v}}{c}\right)\times\vec{a}\right]\right|^2$$

and

$$\frac{dP}{d\Omega} = \frac{1}{c}\frac{\mu_0 q^2}{\left(4\pi\right)^2\left(1-\hat{r}\cdot\vec{v}/c\right)^4}\left|\hat{r}\times\left[\left(\hat{r}-\frac{\vec{v}}{c}\right)\times\vec{a}\right]\right|^2$$

The denominator $\left(1-\hat{r}\cdot\vec{v}/c\right)^4$ indicates that the radiation is beamed into a small cone around the velocity vector when $v/c \to 1$. Note here that this derivation is not strictly correct when v/c is not negligible, and it leads to the wrong power of $(1-\hat{r}\cdot\vec{v}/c)$ in the denominator. It does indicate qualitatively how the radiation is beamed. For the correct relativistic derivation see Jackson Ch 14 or http://www.physics.sfsu.edu/~lea/courses/grad/radgen.PDF.

Example

A particle of charge q and mass m is moving in the presence of a uniform magnetic field $\vec{B} = B_0\hat{z}$. Its speed $v \ll c$. Find the power radiated.

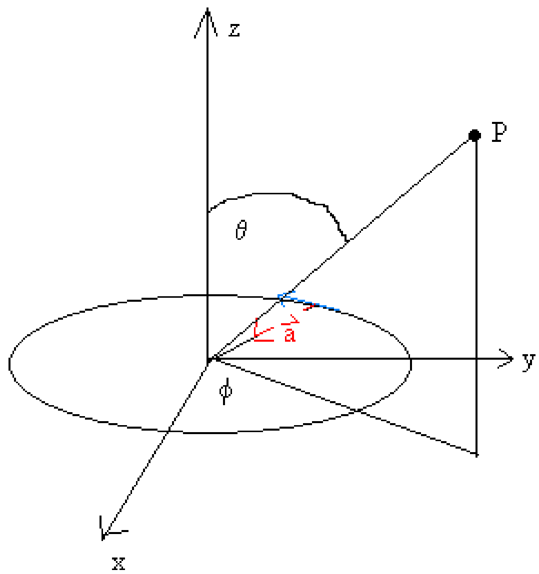
First we compute the acceleration:

$$\vec{F} = m\vec{a} = q\vec{v}\times\vec{B}$$

so

$$\vec{a} = \frac{q}{m}\vec{v}\times\vec{B}$$

Only the component of $\vec{v}$ perpendicular to $\vec{B}$ contributes, and the motion is a circle with $\vec{a}$ pointing toward the center.



The angle χ between $\hat{r}$ and $\vec{a}$ isfound from

$$\hat{r} \cdot \hat{a} = \cos \chi = - \sin \theta \cos (\omega t - \phi)$$

where

$$\omega = \frac{qB}{m}$$

is the cyclotron frequency. Thus from (32),

$$\frac{dP}{d\Omega} = \frac{q^2}{(4\pi)^2 \varepsilon_0 c^3} \left(\frac{q}{m} v_\perp B_0 \right)^2 \left(1 - \sin^2 \theta \cos^2 (\omega t - \phi)\right)$$

Taking the time average, we get

$$< \frac{dP}{d\Omega} > \; = \frac{q^4 (v_\perp B_0)^2}{m^2 (4\pi)^2 \varepsilon_0 c^3} \left(1 - \frac{\sin^2 \theta}{2} \right) = \frac{q^4 B_0^2 K_\perp}{16\pi^2 \varepsilon_0 m^3 c^3} \left(1 + \cos^2 \theta\right)$$

where $K_\perp$ is the particle's kinetic energy $\frac{1}{2} m v_\perp^2$. Radiation is maximum along the direction of $\vec{B}$, and minimum in the plane perpendicular to $\vec{B}$. The total power radiated is (33):

$$P = \frac{2}{3} \frac{q^2}{4\pi\varepsilon_0 c^3} \left(\frac{q}{m} v_\perp B_0 \right)^2 = \frac{2}{3} \frac{q^4 (v_\perp B_0)^2}{4\pi\varepsilon_0 c^3 m^2} = \frac{q^4 B_0^2 K_\perp}{3\pi\varepsilon_0 c^3 m^3}$$

The radiated power is proportional to the square of the magnetic field strength.

Things get much more interesting as $v \to c$. Synchrotron radiation is discussed in J Ch 14 and also in http://www.physics.sfsu.edu/~lea/courses/grad/radiation.PDF.