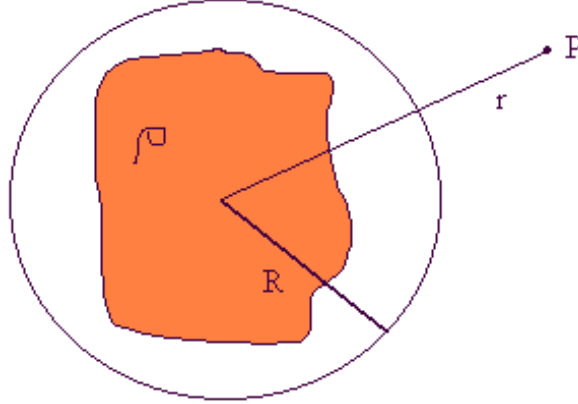


1 Spherical multipole moments



Suppose we have a charge distribution $\rho(\vec{x})$ where all of the charge is contained within a spherical region of radius R . Then for $r > R$ we may write the potential in spherical coordinates as:

$$\Phi(r, \theta, \phi) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \frac{4\pi}{2l+1} \frac{q_{lm}}{r^{l+1}} Y_{lm}(\theta, \phi) \quad (1)$$

where the constants $4\pi/(2l+1)$ have been included for future convenience. We may also write the potential in terms of the charge density as:

$$\Phi(r, \theta, \phi) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x' \quad (2)$$

and then expand the factor $1/|\vec{x} - \vec{x}'|$ in spherical harmonics using J eqn 3.70

$$\begin{aligned} \Phi(r, \theta, \phi) &= \frac{1}{4\pi\epsilon_0} \int \rho(\vec{x}') \sum_{lm} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}(\theta, \phi) Y_{lm}^*(\theta', \phi') d^3x' \\ &= \frac{1}{4\pi\epsilon_0} \sum_{lm} \frac{4\pi}{2l+1} Y_{lm}(\theta, \phi) \int \rho(\vec{x}') \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}^*(\theta', \phi') d^3x' \end{aligned}$$

The integral is over the entire volume where ρ is not zero, which is inside the sphere of radius R . Then since $r' < R$ and $r > R$, we have $r_{<} = r'$ and $r_{>} = r$, so

$$\Phi(r, \theta, \phi) = \frac{1}{4\pi\epsilon_0} \sum_{lm} \frac{4\pi}{2l+1} \frac{Y_{lm}(\theta, \phi)}{r^{l+1}} \int \rho(\vec{x}') (r')^l Y_{lm}^*(\theta', \phi') d^3x' \quad (3)$$

and then comparing equations (1) and (3) we find

$$q_{lm} = \int \rho(\vec{x}') (r')^l Y_{lm}^*(\theta', \phi') d^3x' \quad (4)$$

These are the *spherical multipole moments* of the source. With $l = 1$ we have the dipole, $l = 2$ the quadrupole etc.

2 Cartesian multipole moments

This time we expand $\frac{1}{|\vec{x}-\vec{x}'|}$ in expression (2) in a Taylor series about $\vec{x}' = 0$:

$$\Phi(\vec{x}) = k \int \rho(\vec{x}') \left(\frac{1}{r} + \vec{x}' \cdot \vec{\nabla}' \frac{1}{R} \Big|_{\vec{x}'=0} + \frac{1}{2} \sum_{i,j=1}^3 x'_i x'_j \frac{\partial}{\partial x'_i} \frac{\partial}{\partial x'_j} \frac{1}{R} \Big|_{\vec{x}'=0} + \dots \right) d^3x'$$

Now let's evaluate the derivatives:

$$\vec{x}' \cdot \vec{\nabla}' \frac{1}{R} \Big|_{\vec{x}'=0} = \vec{x}' \cdot \frac{(\vec{x} - \vec{x}')}{R^3} \Big|_{\vec{x}'=0} = \vec{x}' \cdot \frac{\vec{x}}{r^3}$$

and

$$\begin{aligned} \frac{\partial}{\partial x'_i} \frac{\partial}{\partial x'_j} \frac{1}{R} \Big|_{\vec{x}'=0} &= \frac{\partial}{\partial x'_i} \frac{(x_j - x'_j)}{R^3} \Big|_{\vec{x}'=0} = \frac{-\delta_{ij}}{R^3} + 3 \frac{(x_i - x'_i)(x_j - x'_j)}{R^5} \Big|_{\vec{x}'=0} \\ &= \frac{-\delta_{ij}}{r^3} + 3 \frac{x_i x_j}{r^5} \end{aligned}$$

Putting these expressions back into the integrals, we get:

$$\begin{aligned} \Phi(\vec{x}) &= k \int \rho(\vec{x}') \left[\frac{1}{r} + \vec{x}' \cdot \frac{\vec{x}}{r^3} + \frac{1}{2} \sum_{i,j} x'_i x'_j \left(3 \frac{x_i x_j}{r^5} - \frac{\delta_{ij}}{r^3} \right) + \dots \right] d^3x' \\ &= k \left\{ \frac{1}{r} \int \rho(\vec{x}') d^3x' + \frac{\vec{x}}{r^3} \cdot \int \vec{x}' \rho(\vec{x}') d^3x' + \frac{1}{2} \sum_{i,j} \int \left(3 \frac{x_i x_j}{r^5} - \frac{\delta_{ij}}{r^3} \right) \rho(\vec{x}') x'_i x'_j d^3x' + \dots \right\} \end{aligned}$$

The first two terms are easily integrated to give:

$$\Phi(\vec{x}) = k \frac{q}{r} + k \frac{\vec{p} \cdot \vec{x}}{r^3} + \frac{k}{r^5} \sum_{i,j} \int (3x_i x_j - r^2 \delta_{ij}) \rho(\vec{x}') \frac{x'_i x'_j}{2} d^3x' + \dots$$

where q is the total charge in the distribution and

$$\vec{p} = \int \rho(\vec{x}') \vec{x}' d^3x' \quad (5)$$

is the Cartesian dipole moment. The last term needs some work, since we would like to express the result in terms of the quadrupole tensor:

$$Q_{ij} \equiv \int \rho(\vec{x}') \left(3x'_i x'_j - (r')^2 \delta_{ij} \right) d^3 x' \quad (6)$$

but our integral has unprimed variables where we need primes. However, the first term is symmetric in primed and unprimed variables, while the second is non-zero only if $i = j$. The second term is:

$$(x')^2 r^2 + (y')^2 r^2 + (z')^2 r^2 = (r')^2 r^2 = (r')^2 x^2 + (r')^2 y^2 + (r')^2 z^2$$

and so we may interchange prime and unprime and rewrite the integral as:

$$\begin{aligned} \int \rho(\vec{x}') \sum_{i,j} x'_i x'_j (3x_i x_j - r^2 \delta_{ij}) d^3 x' &= \sum_{i,j} x_i x_j \int \rho(\vec{x}') \left(3x'_i x'_j - (r')^2 \delta_{ij} \right) d^3 x' \\ &= \sum_{i,j} x_i x_j Q_{ij} \end{aligned}$$

Thus we have for the potential

$$\Phi(\vec{x}) = k \frac{q}{r} + k \frac{\vec{p} \cdot \vec{x}}{r^3} + \frac{k}{2} \sum_{i,j} \frac{x_i x_j}{r^5} Q_{ij} + \dots \quad (7)$$

The quadrupole tensor Q_{ij} (6) is symmetric, real-valued and traceless. It has three real eigenvalues.

3 Relations between the two sets of multipoles

Comparing the expressions (1) and (7) allows us to relate the two sets of components. Both expressions are series in increasing powers of $1/r$.

The $l = 0$ ($1/r$) term is the monopole term. From equation (4) with $l = m = 0$,

$$q_{00} = \int \rho(\vec{x}') Y_{00}^* (\theta', \phi') d^3 x' = \frac{1}{\sqrt{4\pi}} \int \rho(\vec{x}') d^3 x' = \frac{q}{\sqrt{4\pi}} \quad (8)$$

and the first term in the spherical multipole expansion (1) is

$$\Phi_0 = k4\pi \frac{q_{00}}{r} Y_{00} = k4\pi \frac{q}{\sqrt{4\pi}} \frac{1}{r} \frac{1}{\sqrt{4\pi}} = k \frac{q}{r}$$

which is also the first term in expression (7).

With $l = 1$ ($1/r^2$ term) there are three contributions, with $m = \pm 1$ and zero. First note that

$$\begin{aligned} q_{l,-m} &= \int \rho(\vec{x}') (r')^l Y_{l-m}^* (\theta', \phi') d^3 x' \\ &= (-1)^m \int \rho(\vec{x}') (r')^l Y_{lm} (\theta', \phi') d^3 x' \\ &= (-1)^m q_{lm}^* \end{aligned} \quad (9)$$

So there are actually only two values to calculate.

With $l = 1, m = 0$

$$\begin{aligned}
q_{10} &= \int \rho(\vec{x}') r' Y_{10}^* (\theta', \phi') d^3 x' \\
&= \int \rho(\vec{x}') r' \sqrt{\frac{3}{4\pi}} \cos \theta' d^3 x' \\
q_{10} &= \sqrt{\frac{3}{4\pi}} \int \rho(\vec{x}') z' d^3 x' = \sqrt{\frac{3}{4\pi}} p_z
\end{aligned} \tag{10}$$

and with $l = 1, m = 1$

$$\begin{aligned}
q_{11} &= \int \rho(\vec{x}') r' Y_{11}^* (\theta', \phi') d^3 x' \\
&= - \int \rho(\vec{x}') r' \sqrt{\frac{3}{8\pi}} \sin \theta' e^{-i\phi'} d^3 x' \\
&= - \int \rho(\vec{x}') r' \sqrt{\frac{3}{8\pi}} \sin \theta' (\cos \phi' - i \sin \phi') d^3 x' \\
&= - \sqrt{\frac{3}{8\pi}} \int \rho(\vec{x}') (x' - iy') d^3 x' \\
q_{11} &= \sqrt{\frac{3}{8\pi}} (ip_y - p_x)
\end{aligned} \tag{11}$$

Equivalently,

$$p_x = -\sqrt{\frac{2\pi}{3}} (q_{11} + q_{11}^*), \quad p_y = \frac{1}{i} \sqrt{\frac{2\pi}{3}} (q_{11} - q_{11}^*), \quad p_z = \sqrt{\frac{4\pi}{3}} q_{10} \tag{12}$$

There are three independent values in both cases: the three components of \vec{p} , and q_{10} , $\text{Re}(q_{11})$ and $\text{Im}(q_{11})$ respectively.

The corresponding term in the potential is

$$\begin{aligned}
\Phi_1 &= \frac{1}{4\pi\epsilon_0} \sum_{m=-1}^{+1} \frac{4\pi}{3} \frac{q_{1m}}{r^2} Y_{1m} (\theta, \phi) \\
&= \frac{1}{4\pi\epsilon_0 r^2} \frac{4\pi}{3} (q_{11} Y_{11} (\theta, \phi) + q_{10} Y_{10} + q_{1,-1} Y_{1,-1}) \\
&= \frac{1}{4\pi\epsilon_0 r^2} \frac{4\pi}{3} (q_{11} Y_{11} (\theta, \phi) + q_{10} Y_{10} + (-1) q_{11}^* (-1) Y_{11}^*)
\end{aligned}$$

Inserting our expressions for q_{1m} in terms of the components of \vec{p} , we have

$$\begin{aligned}
\Phi_1 &= \frac{1}{3\epsilon_0 r^2} \left[\sqrt{\frac{3}{8\pi}} (ip_y - p_x) \sqrt{\frac{3}{8\pi}} (-\sin \theta) e^{i\phi} + \sqrt{\frac{3}{4\pi}} p_z \sqrt{\frac{3}{4\pi}} \cos \theta \right. \\
&\quad \left. + \sqrt{\frac{3}{8\pi}} (-ip_y - p_x) \sqrt{\frac{3}{8\pi}} (-\sin \theta) e^{-i\phi} \right] \\
&= \frac{1}{4\pi\epsilon_0 r^2} [p_z \cos \theta + p_x \sin \theta \cos \phi + p_y \sin \theta \sin \phi] = \frac{\vec{p} \cdot \hat{r}}{4\pi\epsilon_0 r^2}
\end{aligned}$$

which is the second term in (7).

With $l = 2$ ($1/r^3$ term) we have $m = 2, 1$, and 0 . The multipole moment q_{20} is

$$\begin{aligned}
q_{20} &= \int \rho(\vec{x}') (r')^2 Y_{20}^*(\theta', \phi') d^3x' \\
&= \int \rho(\vec{x}') (r')^2 \frac{1}{2} \sqrt{\frac{5}{4\pi}} (3 \cos^2 \theta' - 1) d^3x' \\
&= \frac{1}{2} \sqrt{\frac{5}{4\pi}} \int \rho(\vec{x}') [3(z')^2 - (r')^2] d^3x' \\
q_{20} &= \frac{1}{2} \sqrt{\frac{5}{4\pi}} Q_{33}
\end{aligned} \tag{13}$$

Similarly we may show that

$$q_{21} = \frac{1}{3} \sqrt{\frac{15}{8\pi}} (iQ_{23} - Q_{13}) \tag{14}$$

and

$$q_{22} = \frac{1}{12} \sqrt{\frac{15}{2\pi}} (Q_{11} - Q_{22} - 2iQ_{12}) \tag{15}$$

1. Equivalently:

$$Q_{11} - Q_{22} = \operatorname{Re} \left(12 \sqrt{\frac{2\pi}{15}} q_{22} \right)$$

and so on. We have 3 independent values of q_{2m} , and q_{21} and q_{22} each have a real and imaginary part, giving us 5 real numbers, but there are 6 independent values in a general symmetric rank 2 tensor. However, in this case the six values are *not* all independent as we have the additional constraint that Q_{ij} is traceless, leaving us with 5 independent values. For the extension of this discussion to higher multipoles, see Jackson problem 4.3.

Example

A ring of charge of radius a carries linear charge density $\lambda = \lambda_0 \cos \phi$. Let's find the multipole moments and the potential for $r > a$.

The ring is most easily described in spherical coordinates, so let's first find the spherical multipoles. With polar axis along the axis of the ring, the charge density is (Lea Example 6.7)

$$\rho(\vec{x}) = \frac{\lambda_0 \cos \phi \delta(\mu) \delta(r - a)}{r}$$

and thus

$$\begin{aligned} q_{lm} &= \int \rho(\vec{x}) r^l Y_{lm}^*(\theta, \phi) d^3x \\ &= \int \frac{\lambda_0 \cos \phi \delta(\mu) \delta(r - a)}{r} r^l Y_{lm}^*(\theta, \phi) d^3x \\ &= \lambda_0 \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} \int \cos \phi \delta(\mu) \delta(r - a) r^{l+1} P_l^m(\mu) e^{-im\phi} d\mu d\phi dr \end{aligned}$$

Using the sifting property, we have immediately

$$\begin{aligned} q_{lm} &= \lambda_0 \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} a^{l+1} P_l^m(0) \int_0^{2\pi} \cos \phi e^{-im\phi} d\phi \\ &= \lambda_0 \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} a^{l+1} P_l^m(0) \int_0^{2\pi} \left(\frac{e^{i\phi} + e^{-i\phi}}{2} \right) e^{-im\phi} d\phi \\ &= \lambda_0 \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} a^{l+1} P_l^m(0) \pi (\delta_{m1} + \delta_{m,-1}) \end{aligned}$$

Thus

$$q_{l,1} = \lambda_0 \pi \sqrt{\frac{2l+1}{4\pi}} a^{l+1} \sqrt{\frac{(l-1)!}{(l+1)!}} P_l^1(0)$$

We must have $l > 0$, since with $l = 0$ there is no $m = 1$. We also need $l + 1$ to be even so that $P_l^1(0)$ is not zero. Thus we have all multipoles of odd order, and $m = 1$. We can further simplify using

$$q_{l,-1} = -q_{l1}^*$$

so, since q_{l1} is real,

$$q_{l,-1} Y_{l,-1} = (-1)^2 q_{l1}^* Y_{l1}^* = q_{l1} Y_{l1}^*$$

Thus

$$\begin{aligned} q_{l1} Y_{l1} + q_{l,-1} Y_{l,-1} &= q_{l1} (Y_{l1} + Y_{l1}^*) = 2q_{l1} \operatorname{Re}(Y_{l1}) \\ &= \lambda_0 \pi \sqrt{\frac{2l+1}{4\pi}} a^{l+1} \sqrt{\frac{(l-1)!}{(l+1)!}} P_l^1(0) 2 \sqrt{\frac{2l+1}{4\pi} \frac{(l-1)!}{(l+1)!}} P_l^1(\mu) \cos \phi \\ &= \lambda_0 a^{l+1} \frac{2l+1}{2} \frac{(l-1)!}{(l+1)!} P_l^1(0) P_l^1(\mu) \cos \phi \end{aligned}$$

Thus the potential for $r > a$ is (eqn1)

$$\begin{aligned}
\Phi &= \frac{1}{4\pi\epsilon_0} \sum_{l=1, \text{ odd}}^{\infty} \frac{4\pi}{2l+1} \lambda_0 \frac{a^{l+1}}{r^{l+1}} \frac{2l+1}{2} \frac{(l-1)!}{(l+1)!} P_l^1(0) P_l^1(\mu) \cos\phi \\
&= \frac{\lambda_0}{2\epsilon_0} \sum_{l=1, \text{ odd}}^{\infty} \frac{a^{l+1}}{r^{l+1}} \frac{(l-1)!}{(l+1)!} P_l^1(0) P_l^1(\mu) \cos\phi \\
&= \frac{\lambda_0 \cos\phi}{2\epsilon_0} \left[\frac{a^2}{r^2} \frac{1}{2} \sin\theta + \frac{a^4}{r^4} \frac{2}{4!} \frac{3}{2} \frac{3}{2} (1 - 5 \cos^2\theta) \sin\theta + \dots \right] \\
&= \frac{\lambda_0 \sin\theta \cos\phi}{4\epsilon_0} \frac{a^2}{r^2} \left[1 + \frac{3}{8} \frac{a^2}{r^2} (1 - 5 \cos^2\theta) + \dots \right]
\end{aligned}$$

The dominant term is a dipole, because the ring has zero net charge. The dipole moment is (cf eqn 7)

$$\vec{p} = \pi\lambda_0 a^2 \hat{x}$$

as expected from (5).

$$\begin{aligned}
\vec{p} &= \int \frac{\lambda_0 \cos\phi}{r} \frac{\delta(\mu) \delta(r-a)}{r} r \left[\mu \hat{z} + \sqrt{1-\mu^2} (\cos\phi \hat{x} + \sin\phi \hat{y}) \right] r^2 dr d\mu d\phi \\
&= \lambda_0 a^2 \pi \hat{x}
\end{aligned}$$

The next term is the octupole ($l = 3$) term. The potential has the same azimuthal dependence ($\cos\phi$) as the charge density.

Multipole moments are very important in computing the radiation from a time-dependent charge distribution. (See Ch 9.)

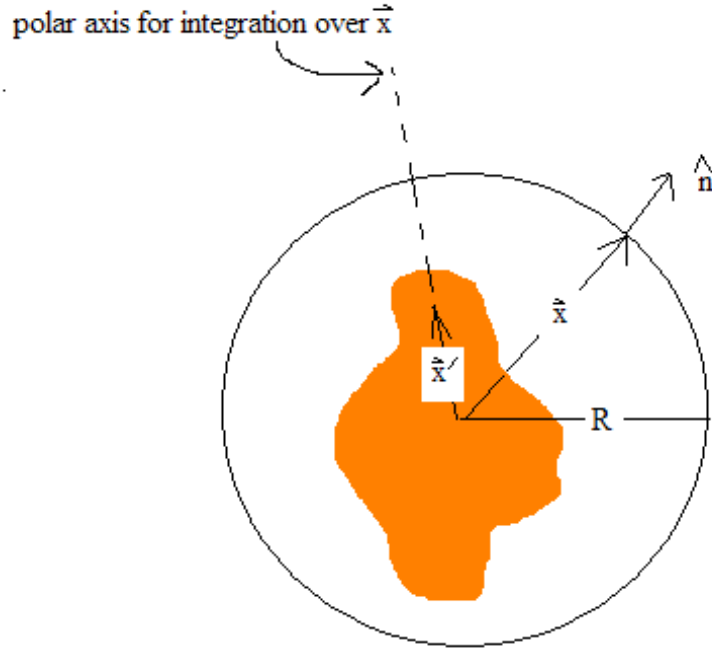
4 A surprising result

Let us evaluate the integral:

$$\int \vec{E}(\vec{x}) d^3x$$

where the integral is taken over a sphere of radius R that contains all the sources of \vec{E} , as shown in the diagram. We express the electric field in terms of the potential, and convert to a surface integral:

$$\begin{aligned}
\int_{\text{sphere}} \vec{E} d^3x &= \int_{\text{sphere}} -\vec{\nabla}\Phi dV = - \int_S \Phi \hat{\mathbf{n}} dA = - \int_S \Phi \hat{\mathbf{n}} R^2 d\Omega \\
&= -k \int_S \int_{\text{all space}} \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} dV' \hat{\mathbf{n}} R^2 d\Omega \\
&= -k \int_S \int_{\text{all space}} \rho(\vec{x}') \sum_l \frac{r_{>}^l}{r_{<}^{l+1}} P_l(\cos\gamma) dV' \hat{\mathbf{n}} R^2 d\Omega
\end{aligned}$$



where we have used the usual expansion of $1/|\vec{x} - \vec{x}'|$, and $k = 1/4\pi\epsilon_0$. Now we interchange the order of integration, and put the polar axis along the vector \vec{x}' . Then we can write $\hat{\mathbf{n}}$ in Cartesian components as:

$$\hat{\mathbf{n}} = \hat{\mathbf{x}} \sin \gamma \cos \phi + \hat{\mathbf{y}} \sin \gamma \sin \phi + \hat{\mathbf{z}} \cos \gamma$$

where γ is the angle between \vec{x} and \vec{x}' , to obtain:

$$\begin{aligned} & \int \vec{E} d^3x \\ = & -k \int_{\text{all space}} \int_0^{2\pi} \int_{-1}^{+1} \rho(\vec{x}') \sum_l \frac{r_{>}^l}{r_{<}^{l+1}} P_l(\mu_\gamma) (\hat{\mathbf{x}} \sin \gamma \cos \phi + \hat{\mathbf{y}} \sin \gamma \sin \phi + \hat{\mathbf{z}} \cos \gamma) R^2 d\mu_\gamma d\phi dV' \end{aligned}$$

The x - and y -components vanish upon integration over ϕ , leaving:

$$\int \vec{E} d^3x = -k \int_{\text{all space}} 2\pi \int_{-1}^{+1} \rho(\vec{x}') \sum_{l=0}^{\infty} \frac{r_{<}^l}{r_{>}^{l+1}} P_l(\cos \gamma) \hat{\mathbf{z}} \cos \gamma R^2 d\mu_\gamma dV'$$

Now since $\cos \gamma = P_1(\cos \gamma)$, orthogonality of the P_l requires that only the $l = 1$ term survive the integration over γ . Then eqn (3.21) with $l = 1$ gives

$$\int \vec{E} d^3x = -k \int_{\text{all space}} 2\pi \frac{2}{3} \rho(\vec{x}') \frac{r_{<}^2}{r_{>}^2} \hat{\mathbf{z}} R^2 dV'$$

Now \vec{x} is on the surface of the sphere, where $|\vec{x}| = r = R$, and $|\vec{x}'| = r' < r$ (because all of the charge is inside the sphere). Also we chose our polar axis along \vec{x}' , so $\hat{z} = \hat{r}'$. Thus:

$$\begin{aligned} \int_{\text{sphere}} \vec{E} d^3x &= -\frac{4\pi}{3}k \int_{\text{all space}} \rho(\vec{x}') \frac{r'}{R^2} \hat{r}' R^2 dV' \\ &= -\frac{1}{3\epsilon_0} \int_{\text{all space}} \rho(\vec{x}') \vec{x}' dV' = -\frac{\vec{p}}{3\epsilon_0} \end{aligned} \quad (16)$$

where \vec{p} is the dipole moment with respect to the center of the sphere.

This is a completely general result: we did not need to say anything about the details of the charge distribution.

Now let's look at the usual expression for the potential due to an ideal "point" dipole:

$$\Phi = k \frac{\vec{p} \cdot \vec{x}}{r^3}$$

The electric field is:

$$\vec{E} = -\vec{\nabla}\Phi = -k \left\{ \frac{\vec{p}}{r^3} - 3 \frac{\vec{x}}{r^5} \vec{p} \cdot \vec{x} \right\} \quad (17)$$

(Here we used the expression for $\vec{\nabla}(\vec{a} \cdot \vec{b})$ from J's front cover, together with the fact that \vec{p} is a constant vector, $\vec{x}/r^3 = -\vec{\nabla}(1/r)$, and the curl of a gradient is zero. See also Jackson 4.13.) With this electric field, we have:

$$\int_{\text{sphere}} \vec{E} d^3x = \int_{\text{sphere}} -k \left\{ \frac{\vec{p}}{r^3} - 3 \frac{\vec{x}}{r^5} (\vec{p} \cdot \vec{x}) \right\} d^3x$$

Put the polar axis along \vec{p} , so that:

$$\int_{\text{sphere}} \vec{E} d^3x = - \int_{\text{sphere}} k \left\{ \frac{\vec{p}}{r^3} - 3 \frac{\hat{r}}{r^3} p \cos \theta \right\} r^2 dr d\Omega$$

Now

$$\hat{r} = \hat{\mathbf{p}} \cos \theta + \hat{\mathbf{x}} \sin \theta \cos \phi + \hat{\mathbf{y}} \sin \theta \sin \phi$$

Again the x - and y -components vanish when we perform the ϕ -integration. Thus:

$$\begin{aligned} \int_{\text{sphere}} \vec{E} d^3x &= -2\pi k \int_0^R \int_{-1}^{+1} \frac{\vec{p}}{r} (1 - 3 \cos^2 \theta) d\mu dr \\ &= -2\pi k \int_0^R \frac{\vec{p}}{r} (\mu - \mu^3) \Big|_{-1}^{+1} dr = 0 \end{aligned}$$

How can this be? We should have obtained $-k \frac{4\pi}{3} \vec{p}$ (equation 16). The problem is that when we took the derivatives to get the field (eqn. 17), those operations are not valid at the origin. There is a missing delta-function! You can also

understand this by looking at the field line diagram for a pair of equal and opposite point charges. As the separation goes to zero, the field lines between the charges (whose direction is opposite \vec{p}) get packed into zero space— the field becomes infinite!

For consistency, the dipole field must be:

$$\vec{E}_{\text{dipole}} = -k \left\{ \frac{\vec{p}}{r^3} - 3 \frac{\vec{x}}{r^5} \vec{p} \cdot \vec{x} \right\} - \frac{4\pi}{3} k \vec{p} \delta(\vec{x}) \quad (18)$$

To see why this result makes sense, let's look again at the derivative of the potential:

$$\Phi = k \frac{\vec{p} \cdot \vec{x}}{r^3} = -k \vec{p} \cdot \vec{\nabla} \left(\frac{1}{r} \right)$$

Thus, with z -axis along \vec{p} ,

$$-\vec{\nabla} \Phi = k \vec{\nabla} \left\{ \vec{p} \cdot \vec{\nabla} \left(\frac{1}{r} \right) \right\} = k p \frac{\partial}{\partial z} \vec{\nabla} \left(\frac{1}{r} \right)$$

whereas

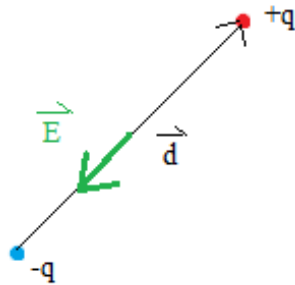
$$\nabla^2 \frac{1}{r} = \vec{\nabla} \cdot \left\{ \vec{\nabla} \left(\frac{1}{r} \right) \right\} = -4\pi \delta(\vec{x}) \quad (19)$$

Now

$$\frac{\partial}{\partial z} \vec{\nabla} \left(\frac{1}{r} \right)$$

is only one of the three second derivatives in the standard delta-function result (19), which helps us to understand the factor of $1/3$ in equation (18). We can understand the direction by looking at the field line diagram.

To understand the magnitude, look at the electric field half way between two point charges separated by a displacement \vec{d} :



$$\begin{aligned}\vec{E} &= -2 \frac{kq}{(d/2)^2} \hat{d} = -\frac{kq}{(d/2)^3} \vec{d} \\ &= -\frac{k\vec{p}}{(d/2)^3} = -\frac{4\pi k}{3} \frac{\vec{p}}{\frac{4}{3}\pi \left(\frac{d}{2}\right)^3}\end{aligned}$$

Now in the limit $d \rightarrow 0$ the dipole moment density $\vec{p}/\left(\frac{4\pi(d/2)^3}{3}\right) \rightarrow \vec{p}\delta(\vec{x})$ and so

$$\vec{E} \rightarrow -\frac{4\pi k}{3} \vec{p}\delta(\vec{x})$$

which is what we obtained in (18).

We can write our result (18) as a purely mathematical statement:

$$\frac{\partial}{\partial z} \vec{\nabla} \left(\frac{1}{r} \right) = \vec{\nabla} \left(\frac{\partial}{\partial z} \frac{1}{r} \right) = - \left\{ \frac{\hat{z}}{r^3} - 3 \frac{\vec{x}}{r^5} \hat{z} \cdot \vec{x} \right\} - \frac{4\pi}{3} \delta(\vec{x}) \hat{z} \quad (20)$$

You'll need this result in problem 6.20.

5 Energy

If our charge distribution is now placed in an *external* field $\vec{E}_{\text{ext}} = -\vec{\nabla}\Phi_{\text{ext}}$, the energy of the system is (Notes 2 eqn 3. Do you understand why there is no factor of 1/2 here?)

$$U = \int_V \rho(\vec{x}) \Phi_{\text{ext}}(\vec{x}) d^3\vec{x}$$

The integral is over all space, but reduces to the volume V where ρ is not zero. To exhibit the result in terms of the multipoles of the charge distribution, we expand the external potential in a Taylor series about the origin;

$$\begin{aligned}U &= \int_V \rho(\vec{x}) \left[\Phi_{\text{ext}}(0) + \vec{x} \cdot \vec{\nabla}\Phi_{\text{ext}} \Big|_0 + \frac{1}{2} x_i x_j \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \Phi_{\text{ext}} \Big|_0 + \dots \right] d^3\vec{x} \\ &= \Phi_{\text{ext}}(0) q - \vec{E}_{\text{ext}}(0) \cdot \vec{p} - \frac{1}{2} \frac{\partial E_{\text{ext},j}}{\partial x_i} \Big|_0 \int_V \rho(\vec{x}) x_i x_j d^3\vec{x} + \dots\end{aligned}$$

where \vec{p} is the dipole moment with respect to the same origin O . We want to express the third term in terms of the Q_{ij} , but we are missing a term. We can add it in because it is zero! Since the sources of the external field \vec{E}_{ext} are not in the volume V that contains the charge ρ , then

$$\vec{\nabla} \cdot \vec{E}_{\text{ext}} = 0 \quad \text{in } V$$

Thus

$$\begin{aligned}\frac{1}{2} \frac{\partial E_{\text{ext},j}}{\partial x_i} \Big|_0 \int_V \rho(\vec{x}) x_i x_j d^3\vec{x} &= \frac{1}{2} \frac{\partial E_{\text{ext},j}}{\partial x_i} \Big|_0 \int_V \rho(\vec{x}) \left(x_i x_j - \frac{r^2}{3} \delta_{ij} \right) d^3\vec{x} \\ &= \frac{1}{6} \frac{\partial E_{\text{ext},j}}{\partial x_i} \Big|_0 Q_{ij}\end{aligned}$$

Thus

$$U = q\Phi_{\text{ext}}(0) - \vec{E}_{\text{ext}}(0) \cdot \vec{p} - \frac{1}{6} \left. \frac{\partial E_{\text{ext},j}}{\partial x_i} \right|_0 Q_{ij} + \dots \quad (21)$$

The second term shows that a dipole is in stable equilibrium in an external field when it is aligned parallel to \vec{E}_{ext} . (This is the minimum energy state.)

Similar techniques may be used to express the force and torque on a charge distribution in terms of its multipoles, as in J problem 4.5.