1 Scattering in the long wavelength limit

Suppose a plane wave with $\vec{E}_{\text{inc}} = \vec{E}_0 e^{ik_0 \cdot \vec{x}}$ is incident on a small scattering object with dimension $d \ll \lambda$. The incoming wave induces electric and magnetic dipole moments in the scattering object, which then radiates.

The scattered fields are

$$\vec{E}_{\text{sc}} = \frac{k^2 e^{ikr}}{r} [(\hat{n} \times \vec{p}) \times \hat{n} - \hat{n} \times \vec{m}]$$

and

$$\vec{B}_{\text{sc}} = \hat{n} \times \vec{E}_{\text{sc}}$$

The power radiated into direction $\vec{n}$ with polarization $\vec{\varepsilon}$ per unit solid angle is

$$\frac{dP}{d\Omega} = r^2 \frac{c}{4\pi} \left| \vec{\varepsilon} \cdot \vec{E}_{\text{sc}} \right|^2$$

and the differential scattering cross section is

$$\frac{d\sigma}{d\Omega} (\hat{n}, \vec{\varepsilon}, \hat{n}_0, \vec{\varepsilon}_0) = \frac{r^2 c}{8\pi} \left| \frac{\varepsilon^* \cdot \vec{E}_{\text{sc}}}{\varepsilon^* \cdot \vec{E}_{\text{inc}}} \right|^2$$

$$= \frac{k^4}{E_0^2} \left| \varepsilon^* \cdot [(\hat{n} \times \vec{p}) \times \hat{n} - \hat{n} \times \vec{m}] \right|^2$$

$$= \frac{k^4}{E_0^2} \left| \varepsilon^* \cdot \vec{p} + (\hat{n} \times \vec{\varepsilon}^*) \cdot \vec{m} \right|^2$$

(1)
where we used the fact that \( \vec{e} \cdot \hat{n} = 0 \), and we rearranged the triple scalar product on the right.

The result shows that the differential scattering cross section is proportional to \( \lambda^4 \). This is Rayleigh’s law. It applies to all scattering in the long-wavelength limit.

**Example:** scattering by a small conducting sphere, radius \( a \).

Since the sphere is small \( (a \ll \lambda) \) it sees the incident field as slowly varying. The sphere can adjust to the electric field in a time \( t \sim a/c \ll \lambda/c = T \), the wave period. Thus we may use the results for the static fields from chapter 3.

With polar axis along \( \vec{e}_0 \), the potential is

\[
\phi = -E_0 \left( r - \frac{a^3}{r} \right) \cos \theta
\]

The first term is the incident field and the second, dipole term is the field due to the charge distribution on the surface of the sphere. The dipole moment is \( \vec{p} = \vec{E}_0 a^3 \). We may also find the scalar magnetic potential. The boundary condition is \( B_r = 0 \) at \( r = a \), giving

\[
\phi_M = -B_0 \left( r + \frac{a^3}{2r} \right) \cos \theta
\]

where here the polar axis is along \( \vec{B}_0 \), and hence

\[
B_r = B_0 \left( 1 - \frac{a^3}{r^3} \right) \cos \theta
\]

which is clearly zero at \( r = a \). Again the first term is the incident field, and the second term is a dipole field. Here the magnetic moment is

\[
\vec{m} = -\frac{\vec{B}_0 a^3}{2} = -\frac{\vec{E}_0 a^3}{2} \left( \hat{n}_0 \times \vec{e}_0 \right)
\]

We may now put these moments into the general result (1):

\[
\frac{d\sigma}{d\Omega} = \frac{k^4}{E_0^2} \frac{a^6}{\varepsilon_0} \left| \vec{e}^* \cdot \vec{e}_0 - \frac{1}{2} (\hat{n} \times \vec{e}^*) \cdot (\hat{n}_0 \times \vec{e}_0) \right|^2
\]

The vectors \( \hat{n}_0 \) and \( \hat{n} \) define the plane of scattering. \( \vec{e} \) is perpendicular to \( \hat{n} \) and \( \vec{e}_0 \) is perpendicular to \( \hat{n}_0 \). We choose polarization vectors \( \vec{e}_{0,1} \) and \( \vec{e}_1 \) in the plane of scattering. Thus \( \vec{e}_1 \) makes angle \( \theta \) with \( \vec{e}_{0,1} \). Similarly \( \vec{e}_{0,2} \) and \( \vec{e}_2 \) are perpendicular to the plane of scattering and parallel to each other. For scattering of unpolarized incident radiation into polarization \( \vec{e}_1 \), in the plane of
scattering, we have
\[
\frac{d\sigma}{d\Omega} = \frac{(ka)^4 a^2}{2} \left\{ \left| \vec{e}_1^* \cdot \vec{e}_{01} - \frac{1}{2} (\hat{n} \times \vec{e}_1^*) \cdot (\hat{n} \times \vec{e}_{01}) \right|^2 + \left| \vec{e}_1^* \cdot \vec{e}_{02} - \frac{1}{2} (\hat{n} \times \vec{e}_1^*) \cdot (\hat{n} \times \vec{e}_{02}) \right|^2 \right\}
\]
\[
= \frac{(ka)^4 a^2}{2} \left\{ \cos \theta - \frac{1}{2} \right\}^2 + 0 \right\}
\]

For scattering into polarization \( \vec{e}_2 \),
\[
\frac{d\sigma}{d\Omega} = \frac{(ka)^4 a^2}{2} \left\{ \left| \vec{e}_2^* \cdot \vec{e}_{01} - \frac{1}{2} (\hat{n} \times \vec{e}_2^*) \cdot (\hat{n} \times \vec{e}_{01}) \right|^2 + \left| \vec{e}_2^* \cdot \vec{e}_{02} - \frac{1}{2} (\hat{n} \times \vec{e}_2^*) \cdot (\hat{n} \times \vec{e}_{02}) \right|^2 \right\}
\]
\[
= \frac{(ka)^4 a^2}{2} \left\{ 0 - \frac{1}{2} \vec{e}_1^* \cdot \vec{e}_{02} \right\}^2 + \left| 1 - \frac{1}{2} (\vec{e}_1^* \cdot \vec{e}_{02}) \right|^2 \right\}
\]
\[
= \frac{(ka)^4 a^2}{2} \right\}\right\}
\]

The sum of the two gives the differential scattering cross section
\[
\frac{d\sigma}{d\Omega} = \frac{(ka)^4 a^2}{2} \left( \cos^2 \theta - \cos \theta + \frac{1}{4} + 1 - \cos \theta + \frac{\cos^2 \theta}{4} \right)
\]
\[
= \frac{(ka)^4 a^2}{2} \left[ \frac{5}{8} (\cos^2 \theta + 1) - \cos \theta \right]
\]

The polarization is
\[
\Pi (\theta) = \frac{\frac{d\sigma}{d\Omega}}{\frac{d\sigma}{d\Omega} + \frac{d\sigma}{d\Omega}} = \frac{\left( 1 - \cos \theta + \frac{\cos^2 \theta}{4} \right) - \left( \cos^2 \theta - \cos \theta + \frac{1}{4} \right)}{\frac{5}{4} (\cos^2 \theta + 1) - 2 \cos \theta}
\]
\[
= \frac{\frac{5}{4} (1 - \cos^2 \theta)}{\frac{5}{4} (\cos^2 \theta + 1) - 2 \cos \theta}
\]
\[
= \frac{3 \sin^2 \theta}{5 (\cos^2 \theta + 1) - 8 \cos \theta}
\]

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The dashed line is the differential scattering cross section, the solid line is the polarization. The polarization peaks at $\theta = \pi/3$. The scattering peaks at $\theta = \pi$ (backward scattering) and is minimum at $\theta = \pi/3$. 