

Generation of EM waves

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1 The Green's function

In Lorentz gauge, we obtained the wave equation:

$$\square^2 A^\beta = \frac{4\pi}{c} J^\beta \quad (1)$$

The corresponding Green's function for the problem satisfies the simpler differential equation

$$\square^2 D \left(\overset{\leftrightarrow}{x}, \overset{\leftrightarrow}{x}' \right) = \delta^{(4)} \left(\overset{\leftrightarrow}{x} - \overset{\leftrightarrow}{x}' \right)$$

and the boundaries are at infinity. Thus D can depend only on the vector $\overset{\leftrightarrow}{z} = \overset{\leftrightarrow}{x} - \overset{\leftrightarrow}{x}'$. We find D by taking the 4-dimensional Fourier transform:

$$D \left(\overset{\leftrightarrow}{z} \right) = \frac{1}{(2\pi)^4} \int \tilde{D}(k) e^{-i\overset{\leftrightarrow}{k} \cdot \overset{\leftrightarrow}{z}} d^4 k$$

(Note that $\overset{\leftrightarrow}{k} \cdot \overset{\leftrightarrow}{z} = \frac{\omega}{c} c(t - t') - \vec{k} \cdot (\vec{x} - \vec{x}') = \omega(t - t') - \vec{k} \cdot (\vec{x} - \vec{x}')$. This explains the usual convention of transforming the time variable with the opposite sign from that for the space variable.)

Also recall the delta-function integral:

$$\delta^{(4)} \left(\overset{\leftrightarrow}{z} \right) = \frac{1}{(2\pi)^4} \int d^4 k e^{-i\overset{\leftrightarrow}{k} \cdot \overset{\leftrightarrow}{z}}$$

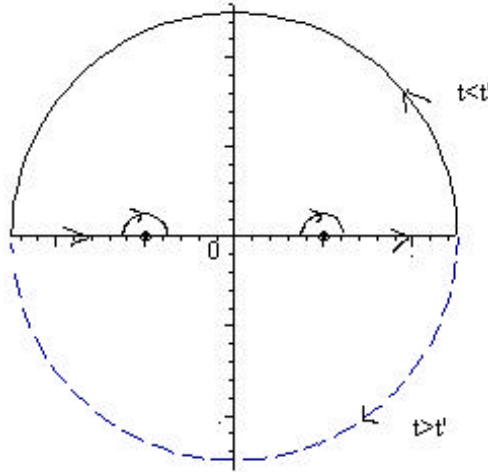
The transformed differential equation reads:

$$-k \cdot k \tilde{D} \left(\overset{\leftrightarrow}{k} \right) = \frac{1}{(2\pi)^2}$$

and transforming back, we get:

$$\begin{aligned} D(\vec{z}) &= -\frac{1}{(2\pi)^4} \int \frac{e^{-i\vec{k}\cdot\vec{z}}}{k \cdot k} d^4k \\ &= -\frac{1}{(2\pi)^4} \int d^3k e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \int_{-\infty}^{\infty} \frac{d\omega}{c} \frac{e^{-i\omega(t-t')}}{\omega^2/c^2 - k^2} \end{aligned}$$

We'll do the frequency integral first. For $t < t'$, we close the contour with a big semi-circle in the upper half plane, while for $t > t'$ we close downward. The integrand has two poles, at $\omega = \pm ck$, both on the real axis. Since the result must be zero for $t < t'$ (event precedes its source) we must evaluate the integral along the real axis by deforming the path to go above the two poles.



Contour for evaluating the inverse transform

For $t > t'$, we use the lower contour and we traverse it *clockwise*. Each of the poles is simple, and the residues are

$$\lim_{\omega \rightarrow -ck} (\omega + ck) \frac{e^{-i\omega(t-t')}}{\omega^2/c^2 - k^2} = c^2 \frac{e^{-i(t-t')(-ck)}}{-2ck} = -c \frac{e^{i(t-t')ck}}{2k}$$

and

$$\lim_{\omega \rightarrow ck} (\omega - ck) \frac{e^{-i\omega(t-t')}}{\omega^2/c^2 - k^2} = c^2 \frac{e^{-i(t-t')(ck)}}{2ck} = c \frac{e^{-i(t-t')ck}}{2k}$$

Then we have:

$$\begin{aligned}
D\left(\frac{\mathfrak{q}_+}{z}\right) &= -\frac{1}{(2\pi)^4}\Theta(t-t')\int d^3k e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}(-2\pi i)\left(c\frac{e^{-i(t-t')ck}}{2k}-c\frac{e^{i(t-t')ck}}{2k}\right) \\
&= \frac{ic}{2(2\pi)^3}\Theta(t-t')\int d^3k e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\frac{1}{k}\left(e^{-i(t-t')ck}-e^{i(t-t')ck}\right)
\end{aligned}$$

The first exponential represents an outgoing wave, while the second represents an incoming wave.

Now do the remaining integrals, we choose our polar axis for \vec{k} along the vector $\vec{z} = \vec{x} - \vec{x}'$. Then:

$$\begin{aligned}
D\left(\frac{\mathfrak{q}_+}{z}\right) &= \frac{ic}{2(2\pi)^3}\Theta(t-t')\int_0^\infty\int_{-1}^{+1}\int_0^{2\pi}k^2dkd\mu d\phi e^{ikz\mu}\frac{1}{k}\left(e^{-i(t-t')ck}-e^{i(t-t')ck}\right) \\
&= \frac{ic}{2(2\pi)^2}\Theta(t-t')\int_0^\infty\int_{-1}^{+1}kdkd\mu e^{ikz\mu}\left(e^{-i(t-t')ck}-e^{i(t-t')ck}\right) \\
&= \frac{ic}{2(2\pi)^2}\Theta(t-t')\int_0^\infty kdk\left.\frac{e^{ikz\mu}}{ikz}\right|_{-1}^{+1}\left(e^{-i(t-t')ck}-e^{i(t-t')ck}\right) \\
&= \frac{c}{2(2\pi)^2z}\Theta(t-t')\int_0^\infty dk\left(e^{ikz}-e^{-ikz}\right)\left(e^{-i(t-t')ck}-e^{i(t-t')ck}\right) \\
&= \frac{c}{2(2\pi)^2z}\Theta(t-t')\int_0^\infty dk\left(\begin{array}{l} \exp(ik(z-c(t-t'))) - \exp ik(z+c(t-t')) \\ -\exp[-ik(z+c(t-t'))] + \exp -ik(z-c(t-t')) \end{array}\right) \\
&= \frac{c}{2(2\pi)^2z}\Theta(t-t')\int_{-\infty}^\infty dk\left(-\exp ik(z+c(t-t')) + \exp(ik(z-c(t-t')))\right) \\
&= \frac{c}{4\pi z}\Theta(t-t')\left(\delta(z-c(t-t')) - \delta(z+c(t-t'))\right)
\end{aligned}$$

The second term can never contribute since both z and $t-t'$ are positive.

Thus we finally have:

$$D(z) = \frac{c}{4\pi z}\Theta(t-t')\delta(z-c(t-t'))$$

and thus the solution to equation (1) is:

$$\begin{aligned}
A^\alpha\left(\frac{\mathfrak{q}_+}{x}\right) &= \frac{4\pi}{c}\int\frac{c}{4\pi z}\Theta(t-t')\delta(z-c(t-t'))J^\alpha\left(\frac{\mathfrak{q}_+}{x}\right)d^4x' \\
&= \int\frac{1}{z}\Theta(t-t')\delta(z-c(t-t'))J^\alpha\left(\frac{\mathfrak{q}_+}{x}\right)d^4x' \quad (2)
\end{aligned}$$

Note: Jackson prefers to write this using the fully covariant expression

$$\begin{aligned}
\delta \left[\left(\frac{\eta}{x} - \frac{\eta'}{x'} \right)^2 \right] &= \delta \left\{ (ct^*)^2 - |\vec{x} - \vec{x}'|^2 \right\} \\
&= \delta \{ (ct^* - R) (ct^* + R) \} \\
&= \frac{1}{2R} \{ \delta(ct^* - R) + \delta(ct^* + R) \}
\end{aligned}$$

where we wrote $t^* = t - t'$, $R = |\vec{x} - \vec{x}'|$, and used the result

$$\delta(f(x)) = \sum_{\text{zeros}} \frac{\delta(x - x_i)}{|f'(x_i)|}$$

Thus

$$D \left(\frac{\eta}{z} \right) = \frac{1}{2\pi} \Theta(t - t') \delta \left[\left(\frac{\eta}{x} - \frac{\eta'}{x'} \right)^2 \right]$$

This is a covariant expression in spite of the explicit appearance of t and t' , since the step function serves to locate the result on the forward light cone from the source event, and this is an invariant statement. The expression for the potential becomes:

$$\begin{aligned}
A^\alpha \left(\frac{\eta}{x} \right) &= \frac{4\pi}{c} \int \frac{1}{2\pi} \Theta(t - t') \delta \left[\left(\frac{\eta}{x} - \frac{\eta'}{x'} \right)^2 \right] J^\alpha \left(\frac{\eta'}{x'} \right) d^4 x' \\
&= \frac{2}{c} \int \Theta(t - t') \delta \left[\left(\frac{\eta}{x} - \frac{\eta'}{x'} \right)^2 \right] J^\alpha \left(\frac{\eta'}{x'} \right) d^4 x' \quad (3)
\end{aligned}$$

2 Charge density and current for a point charge.

The charge and current densities are given by

$$\rho(\vec{x}, t) = q \delta(\vec{x} - \vec{r}(t))$$

and

$$\vec{J}(\vec{x}, t) = q \vec{v} \delta(\vec{x} - \vec{r}(t))$$

and so the 4-vector is

$$J^\alpha = (c\rho, \vec{v}) = q(c, \vec{v}) \delta(\vec{x} - \vec{r}(t)) = \frac{q}{\gamma} u^\alpha \delta(\vec{x} - \vec{r}(t)) \quad (4)$$

We can write this covariantly as

$$J^\alpha(t_0, \vec{x}) = qc \int u^\alpha \delta(\vec{x} - \vec{x}_0(\tau)) d\tau \quad (5)$$

where

$$x_0^\alpha = (ct_0, \vec{r}(t_0))$$

is the charge's position vector at time t_0 . To see why this works, note that

$$\delta(\vec{x} - \vec{x}_0(\tau)) = \delta(ct - ct_0(\tau)) \delta(\vec{x} - \vec{r}(\tau))$$

and

$$\delta(ct - ct_0(\tau)) = \frac{\delta(\tau(t) - \tau(t_0))}{|cdt/d\tau|} = \frac{\delta(\tau(t) - \tau(t_0))}{c\gamma}$$

Thus

$$\begin{aligned} J^\alpha(t_0, \vec{x}) &= qc \int u^\alpha \frac{\delta(\tau(t) - \tau(t_0))}{c\gamma} \delta(\vec{x} - \vec{r}(\tau)) d\tau \\ &= \frac{q}{\gamma} u^\alpha \delta(\vec{x} - \vec{r}(t_0)) \end{aligned}$$

which agrees with equation (4).

3 Radiation from a moving charge- the potential

We insert the current vector (5) into the expression (3) for the potential:

$$\begin{aligned} A^\alpha(\vec{x}) &= \frac{2}{c} \int \Theta(t - t') \delta\left[\left(\vec{x} - \vec{x}'\right)^2\right] qc \int u^\alpha \delta(\vec{x}' - \vec{x}_0(\tau)) d\tau d^4x' \\ &= 2q \int \Theta(t - t') \delta\left[\left(\vec{x} - \vec{x}'\right)^2\right] \int u^\alpha \delta(\vec{x}' - \vec{x}_0(\tau)) d\tau d^4x' \end{aligned}$$

Using the delta function, the integration over \vec{x}' is easy:

$$A^\alpha(\vec{x}) = 2q \int \Theta(t - t_0(\tau)) \delta\left[\left(\vec{x} - \vec{x}_0(\tau)\right)^2\right] u^\alpha d\tau$$

Now again we need to evaluate the delta function of a function- here the function $f(\tau) = \left(\vec{x} - \vec{x}_0(\tau)\right)^2 = (x^\alpha - x_0^\alpha)(x_\alpha - x_{0\alpha})$. The derivative

$$f'(\tau) = -2\frac{dx_0^\alpha}{d\tau}(x_\alpha - x_{0\alpha}) = -2v^\alpha(x_\alpha - x_{0\alpha})$$

and the zeros are those values of τ for which $(x^\alpha - x_0^\alpha)(x_\alpha - x_{0\alpha}) = 0$. Expanding this out, we get

$$\begin{aligned} c(t - t_0)^2 - (\vec{x} - \vec{r}(t_0))^2 &= 0 \\ t - t_0 &= R/c \end{aligned}$$

where

$$R = |\vec{x} - \vec{r}(t_0)|$$

Only the positive value contributes to the integral because of the factor $\Theta(t - t_0(\tau))$. (The event is on the positive light cone from the source.) Thus:

$$t = t_0 + R/c$$

or

$$t_0 = t - R/c = t_{\text{ret}} \tag{6}$$

the *retarded time*.

Then the denominator is

$$2v^\alpha(x_\alpha - x_{0\alpha}) = 2(\gamma c, \gamma \vec{v}) \cdot (c(t - t_0), -\vec{R}) = 2(\gamma c^2(t - t_0) - \gamma \vec{v} \cdot \vec{R})$$

or, using equation (6),

$$2(\gamma c^2 R/c - \gamma \vec{v} \cdot \vec{R}) = 2\gamma R c (1 - \vec{\beta} \cdot \hat{R})$$

Then

$$\begin{aligned} A^\alpha\left(\frac{\vec{x}}{x}\right) &= 2q \int \Theta(t - t_0(\tau)) \delta\left[\left(\vec{x} - \vec{x}_0(\tau)\right)^2\right] u^\alpha(\tau) d\tau \\ &= \frac{q\gamma(c, \vec{v})}{\gamma R c (1 - \vec{\beta} \cdot \hat{R})} \Bigg|_{t_{\text{ret}}} \\ &= \frac{q(1, \vec{\beta})}{R (1 - \vec{\beta} \cdot \hat{R})} \Bigg|_{t_{\text{ret}}} \end{aligned}$$

These are the Lienard-Wiechert potentials. They may also be written covariantly as:

$$A^\alpha = q \frac{u^\alpha}{u^\beta \Delta x_\beta}$$

with

$$\Delta x_\beta = x_\beta - x_{0\beta}$$

all evaluated at the retarded time.

4 Fields due to a moving charge

To compute the fields from the potentials (2), we need to take derivatives. That is, we need to compare the potentials at two nearby events. The potential A^α changes as x^α changes both because of its explicit dependence on x^α but also because of the implicit dependence on $\tau_0(x^\alpha)$ – that is, because of the need to look back along the light cone to find the source event.

$$dA^\mu = \frac{\partial A^\mu}{\partial x^\nu} dx^\nu + \frac{dA^\mu}{d\tau_0} d\tau_0 \quad (7)$$

Differentiating the light cone constraint $(x^\alpha - r^\alpha(\tau_0))(x_\alpha - r_\alpha(\tau_0)) = 0$, we get:

$$2(x^\alpha - r^\alpha(\tau_0))(dx_\alpha - dr_\alpha(\tau_0)) = 0 \quad (8)$$

and from the definition of the particle's velocity

$$dr_\alpha(\tau_0) = u_\alpha d\tau_0$$

So

$$\begin{aligned} (x^\alpha - r^\alpha(\tau_0))(dx_\alpha - u_\alpha d\tau_0) &= 0 \\ d\tau_0 &= \frac{(x^\alpha - r^\alpha(\tau_0)) dx_\alpha}{(x^\beta - r^\beta(\tau_0)) u_\beta} \\ &= \frac{\Delta x^\alpha dx_\alpha}{\Delta x^\beta u_\beta} \end{aligned} \quad (9)$$

where I defined $\Delta x^\alpha \equiv (x^\alpha - r^\alpha(\tau_0))$

Thus from equation (7), we find

$$\begin{aligned} dA^\mu &= \partial_\nu \left(\frac{qu^\mu}{u^\beta \Delta x_\beta} \right) dx^\nu + \frac{d}{d\tau_0} \left(\frac{qu^\mu}{u^\beta \Delta x_\beta} \right) d\tau_0 \\ &= qu^\mu \left\{ \frac{-u^\beta \partial_\nu \Delta x_\beta}{[u^\gamma \Delta x_\gamma]^2} \right\} dx^\nu + q \left\{ \frac{a^\mu}{u^\beta \Delta x_\beta} - u^\mu \frac{a^\beta \Delta x_\beta + u^\beta (-u_\beta)}{[u^\gamma \Delta x_\gamma]^2} \right\} d\tau_0 \end{aligned}$$

Now we insert the expression (9) for $d\tau_0$, note that

$$\partial_\nu \Delta x_\beta = \frac{\partial}{\partial x^\nu} (x_\beta - r_\beta(\tau_0)) = \frac{\partial}{\partial x^\nu} g_{\beta\alpha} (x^\alpha - r^\alpha(\tau_0)) = g_{\beta\alpha} \delta_\nu^\alpha = g_{\beta\nu}$$

and factor a bit:

$$\begin{aligned} dA^\mu &= \frac{qdx^\nu}{[u^\gamma \Delta x_\gamma]^2} \left\{ -u^\mu u^\beta g_{\beta\nu} + \Delta x_\nu \left(a^\mu - u^\mu \frac{a^\beta \Delta x_\beta - u^\beta u_\beta}{u^\varepsilon \Delta x_\varepsilon} \right) \right\} \\ &= \frac{qdx^\nu}{[u^\gamma \Delta x_\gamma]^2} \left\{ -u^\mu u_\nu + \Delta x_\nu \left(a^\mu + u^\mu \frac{c^2 - a^\beta \Delta x_\beta}{u^\varepsilon \Delta x_\varepsilon} \right) \right\} \end{aligned}$$

So

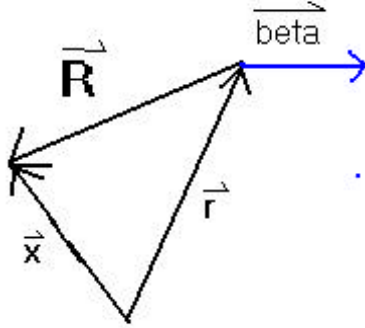
$$\frac{\partial A^\mu}{\partial x^\nu} = \frac{q}{[u^\gamma \Delta x_\gamma]^2} \left\{ -u^\mu u_\nu + \Delta x_\nu \left(a^\mu + u^\mu \frac{c^2 - a^\beta \Delta x_\beta}{u^\varepsilon \Delta x_\varepsilon} \right) \right\}$$

Now we compute the field tensor components

$$F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}$$

The symmetric term $u^\mu u^\nu$ will cancel in the subtraction, leaving

$$F^{\mu\nu} = \frac{q}{[u^\gamma \Delta x_\gamma]^2} \left\{ \Delta x^\mu a^\nu - \Delta x^\nu a^\mu + \frac{c^2 - a^\beta \Delta x_\beta}{u^\varepsilon \Delta x_\varepsilon} (u^\nu \Delta x^\mu - u^\mu \Delta x^\nu) \right\}$$



Now, using the notation in the diagram above,

$$\begin{aligned}\overset{\curvearrowright}{\Delta}x &= R(1, \tilde{\mathbf{n}}) \\ \overset{\curvearrowright}{u} &= c\gamma(1, \tilde{\boldsymbol{\beta}})\end{aligned}$$

and

$$\overset{\curvearrowright}{a} = \frac{d\overset{\curvearrowright}{u}}{d\tau} = \gamma \frac{d\overset{\curvearrowright}{u}}{dt} = c\gamma \left(\frac{d\gamma}{dt}, \tilde{\boldsymbol{\beta}} \frac{d\gamma}{dt} + \gamma \frac{d\tilde{\boldsymbol{\beta}}}{dt} \right)$$

and

$$\frac{d\gamma}{dt} = \gamma^3 \tilde{\boldsymbol{\beta}} \cdot \frac{d\tilde{\boldsymbol{\beta}}}{dt}$$

so

$$\overset{\curvearrowright}{a} = c\gamma^2 \left(\gamma^2 \tilde{\boldsymbol{\beta}} \cdot \frac{d\tilde{\boldsymbol{\beta}}}{dt}, \gamma^2 \left(\tilde{\boldsymbol{\beta}} \cdot \frac{d\tilde{\boldsymbol{\beta}}}{dt} \right) \tilde{\boldsymbol{\beta}} + \frac{d\tilde{\boldsymbol{\beta}}}{dt} \right)$$

Then we can compute the dot products

$$u^\gamma \Delta x_\gamma = c\gamma R(1 - \tilde{\boldsymbol{\beta}} \cdot \hat{\mathbf{n}})$$

and

$$\begin{aligned}
a^\beta \Delta x_\beta &= c\gamma^2 \left(\gamma^2 \vec{\beta} \cdot \frac{d\vec{\beta}}{dt}, \gamma^2 \left(\vec{\beta} \cdot \frac{d\vec{\beta}}{dt} \right) \vec{\beta} + \frac{d\vec{\beta}}{dt} \right) \cdot R(1, \vec{n}) \\
&= c\gamma^2 R \left(\gamma^2 \vec{\beta} \cdot \frac{d\vec{\beta}}{dt} - \gamma^2 \left(\vec{\beta} \cdot \frac{d\vec{\beta}}{dt} \right) \vec{\beta} \cdot \hat{n} - \frac{d\vec{\beta}}{dt} \cdot \hat{n} \right) \\
&= c\gamma^2 R \left(\gamma^2 \vec{\beta} \cdot \frac{d\vec{\beta}}{dt} (1 - \vec{\beta} \cdot \hat{n}) - \frac{d\vec{\beta}}{dt} \cdot \hat{n} \right)
\end{aligned}$$

So, writing

$$K = \frac{q}{\left[c\gamma R (1 - \vec{\beta} \cdot \hat{n}) \right]^2}$$

$$\begin{aligned}
F^{\mu\nu} &= K \left\{ \begin{array}{c} \Delta x^\mu a^\nu - \Delta x^\nu a^\mu + \\ \frac{(u^\nu \Delta x^\mu - u^\mu \Delta x^\nu)}{c\gamma R(1 - \vec{\beta} \cdot \hat{n})} \left(c^2 - c\gamma^2 R \left(\gamma^2 \vec{\beta} \cdot \frac{d\vec{\beta}}{dt} (1 - \vec{\beta} \cdot \hat{n}) - \frac{d\vec{\beta}}{dt} \cdot \hat{n} \right) \right) \end{array} \right\} \\
&= K \left\{ \Delta x^\mu a^\nu - \Delta x^\nu a^\mu + (u^\nu \Delta x^\mu - u^\mu \Delta x^\nu) \left(\frac{c + \gamma^2 R \frac{d\vec{\beta}}{dt} \cdot \hat{n}}{\gamma R (1 - \vec{\beta} \cdot \hat{n})} - \gamma^3 \vec{\beta} \cdot \frac{d\vec{\beta}}{dt} \right) \right\}
\end{aligned}$$

The row

$$\begin{aligned}
F^{0i} &= K \left\{ \Delta x^0 a^i - \Delta x^i a^0 + (u^i \Delta x^0 - u^0 \Delta x^i) \left(\frac{c + \gamma^2 R \frac{d\vec{\beta}}{dt} \cdot \hat{n}}{\gamma R (1 - \vec{\beta} \cdot \hat{n})} - \gamma^3 \vec{\beta} \cdot \frac{d\vec{\beta}}{dt} \right) \right\} \\
&= K \left\{ R \left(a^i - n^i c \gamma^4 \vec{\beta} \cdot \frac{d\vec{\beta}}{dt} \right) + c\gamma R (\beta^i - n^i) \left(\frac{c + \gamma^2 R \frac{d\vec{\beta}}{dt} \cdot \hat{n}}{\gamma R (1 - \vec{\beta} \cdot \hat{n})} - \gamma^3 \vec{\beta} \cdot \frac{d\vec{\beta}}{dt} \right) \right\} \\
&= \frac{q}{cR (1 - \vec{\beta} \cdot \hat{n})^2} \left\{ \begin{array}{c} \gamma^2 \left(\vec{\beta} \cdot \frac{d\vec{\beta}}{dt} \right) \beta^i + \frac{d\beta^i}{dt} - n^i \gamma^2 \vec{\beta} \cdot \frac{d\vec{\beta}}{dt} + \\ \frac{(\beta^i - n^i)}{\gamma} \left(\frac{c + \gamma^2 R \frac{d\vec{\beta}}{dt} \cdot \hat{n}}{\gamma R (1 - \vec{\beta} \cdot \hat{n})} - \gamma^3 \vec{\beta} \cdot \frac{d\vec{\beta}}{dt} \right) \end{array} \right\}
\end{aligned}$$

So

$$\begin{aligned}
E^i &= -F^{0i} \\
&= \gamma^2 K \left\{ (n^i - \beta^i) \left(\gamma^2 \vec{\beta} \cdot \frac{d\vec{\beta}}{dt} + \frac{c + \gamma^2 R \frac{d\vec{\beta}}{dt} \cdot \hat{n}}{\gamma^2 R (1 - \vec{\beta} \cdot \hat{n})} - \gamma^2 \vec{\beta} \cdot \frac{d\vec{\beta}}{dt} \right) - \frac{d\beta^i}{dt} \right\} \\
&= \frac{q}{cR (1 - \vec{\beta} \cdot \hat{n})^2} \left\{ (n^i - \beta^i) \frac{c + \gamma^2 R \frac{d\vec{\beta}}{dt} \cdot \hat{n}}{\gamma^2 R (1 - \vec{\beta} \cdot \hat{n})} - \frac{d\beta^i}{dt} \right\}
\end{aligned}$$

We may divide this result into two parts:

$$E_{\text{coulomb}}^i = \frac{q}{\gamma^2 R^2 (1 - \vec{\beta} \cdot \hat{n})^3} (n^i - \beta^i)$$

This field has the usual $1/R^2$ dependence and is in fact the same as Jackson's equation 11.154. (See Jackson page 664 for the details.) The other term is the radiation field: it depends on the particle's acceleration.

$$E_{\text{rad}}^i = \frac{q}{cR (1 - \vec{\beta} \cdot \hat{n})^2} \left\{ (n^i - \beta^i) \frac{\frac{d\vec{\beta}}{dt} \cdot \hat{n}}{(1 - \vec{\beta} \cdot \hat{n})} - \frac{d\beta^i}{dt} \right\} \quad (10)$$

It decreases as $1/R$ rather than $1/R^2$, and so dominates the Coulomb field at large distances. We can simplify the expression by noting that:

$$\hat{n} \times \left[(\hat{n} - \vec{\beta}) \times \frac{d\vec{\beta}}{dt} \right] = (\hat{n} - \vec{\beta}) \left(\hat{n} \cdot \frac{d\vec{\beta}}{dt} \right) - \frac{d\vec{\beta}}{dt} (1 - \vec{\beta} \cdot \hat{n})$$

Thus

$$\vec{E}_{\text{rad}} = \frac{q}{cR (1 - \vec{\beta} \cdot \hat{n})^3} \hat{n} \times \left[(\hat{n} - \vec{\beta}) \times \frac{d\vec{\beta}}{dt} \right] \quad (11)$$

Remember: everything is evaluated at t_{retarded} . Also notice that \vec{E}_{rad} is perpendicular to \hat{n} , and is proportional to the acceleration $d\vec{\beta}/dt$.

The magnetic field is given by:

$$\vec{B} = (-F^{23}, F^{13}, -F^{12})$$

For example:

$$\begin{aligned}
B_x &= -F^{23} \\
&= -K \left\{ \Delta x^2 a^3 - \Delta x^3 a^2 + (u^3 \Delta x^2 - u^2 \Delta x^3) \left(\frac{c + \gamma^2 R \frac{d\vec{\beta}}{dt} \cdot \hat{n}}{\gamma R (1 - \vec{\beta} \cdot \hat{n})} - \gamma^3 \vec{\beta} \cdot \frac{d\vec{\beta}}{dt} \right) \right\} \\
&= -K \left\{ n^2 a^3 - n^3 a^2 + (u^3 n^2 - u^2 n^3) \left(\frac{c + \gamma^2 R \frac{d\vec{\beta}}{dt} \cdot \hat{n}}{\gamma R (1 - \vec{\beta} \cdot \hat{n})} - \gamma^3 \vec{\beta} \cdot \frac{d\vec{\beta}}{dt} \right) \right\} \\
&= \frac{-q}{c\gamma R (1 - \vec{\beta} \cdot \hat{n})^2} \left\{ \begin{aligned} &\gamma^3 \left(\vec{\beta} \cdot \frac{d\vec{\beta}}{dt} \right) (n^2 \beta^3 - n^3 \beta^2) + \gamma \left(n^2 \frac{d\beta^3}{dt} - n^3 \frac{d\beta^2}{dt} \right) \\ &+ (\beta^3 n^2 - \beta^2 n^3) \left(\frac{c + \gamma^2 R \frac{d\vec{\beta}}{dt} \cdot \hat{n}}{\gamma R (1 - \vec{\beta} \cdot \hat{n})} - \gamma^3 \vec{\beta} \cdot \frac{d\vec{\beta}}{dt} \right) \end{aligned} \right\}
\end{aligned}$$

Thus:

$$\vec{B} = \frac{-q}{c\gamma R (1 - \vec{\beta} \cdot \hat{n})^2} \left\{ \frac{c + \gamma^2 R \frac{d\vec{\beta}}{dt} \cdot \hat{n}}{\gamma R (1 - \vec{\beta} \cdot \hat{n})} \hat{n} \times \vec{\beta} + \gamma \hat{n} \times \frac{d\vec{\beta}}{dt} \right\}$$

Again we may divide the result into two parts:

$$\vec{B}_{B-S} = \frac{q}{\gamma^2 R^2 (1 - \vec{\beta} \cdot \hat{n})^3} \vec{\beta} \times \hat{n}$$

This term, which goes as $1/R^2$, is the usual Biot -Savart law result, with relativistic corrections. (Compare J eqn 11.152 and our expression for \vec{E}_{coulomb})
The radiation term is:

$$\begin{aligned}
\vec{B}_{\text{rad}} &= \frac{-q}{cR (1 - \vec{\beta} \cdot \hat{n})^2} \left\{ \frac{\frac{d\vec{\beta}}{dt} \cdot \hat{n}}{(1 - \vec{\beta} \cdot \hat{n})} \hat{n} \times \vec{\beta} + \hat{n} \times \frac{d\vec{\beta}}{dt} \right\} \\
&= -\hat{n} \times \frac{q}{cR (1 - \vec{\beta} \cdot \hat{n})^2} \left\{ \frac{\frac{d\vec{\beta}}{dt} \cdot \hat{n}}{(1 - \vec{\beta} \cdot \hat{n})} \vec{\beta} + \frac{d\vec{\beta}}{dt} \right\} \\
&= \hat{n} \times \vec{E}_{\text{rad}}
\end{aligned}$$

where we used equation (10) for the electric field.

5 Radiated power

The Poynting vector is

$$\begin{aligned}
 \vec{S} &= \frac{c}{4\pi} \vec{E}_{\text{rad}} \times \vec{B}_{\text{rad}} \\
 &= \frac{c}{4\pi} \vec{E}_{\text{rad}} \times (\hat{n} \times \vec{E}_{\text{rad}}) \\
 &= \frac{c}{4\pi} E_{\text{rad}}^2 \hat{n} \\
 &= \frac{c}{4\pi} \left(\frac{q}{cR(1 - \vec{\beta} \cdot \hat{n})^3} \hat{n} \times \left[(\hat{n} - \vec{\beta}) \times \frac{d\vec{\beta}}{dt} \right] \right)^2 \hat{n} \\
 &= \frac{q^2}{4\pi R^2 c (1 - \vec{\beta} \cdot \hat{n})^6} \left| \hat{n} \times \left[(\hat{n} - \vec{\beta}) \times \frac{d\vec{\beta}}{dt} \right] \right|^2 \hat{n}
 \end{aligned}$$

Thus the power radiated per unit solid angle is:

$$\begin{aligned}
 \frac{dP}{d} &= R^2 \vec{S} \cdot \hat{n} \\
 &= \frac{q^2}{4\pi c (1 - \vec{\beta} \cdot \hat{n})^6} \left| \hat{n} \times \left[(\hat{n} - \vec{\beta}) \times \frac{d\vec{\beta}}{dt} \right] \right|^2
 \end{aligned} \tag{12}$$

For non-relativistic motion, $\beta \ll 1$, we retrieve the Larmor formula:

$$\begin{aligned}
 \frac{dP}{d} &= \frac{q^2}{4\pi c} \left| \hat{n} \times \left[\hat{n} \times \frac{d\vec{\beta}}{dt} \right] \right|^2 \\
 &= \frac{q^2}{4\pi c^3} a^2 \sin^2 \theta
 \end{aligned}$$

where $a = dv/dt$ is the usual 3-acceleration and θ is the angle between \vec{a} and \hat{n} . The total power radiated in the non-relativistic case is:

$$\begin{aligned}
 P &= \int \frac{dP}{d} d = \int_{-1}^{+1} d\mu \int_0^{2\pi} d\phi \frac{q^2}{4\pi c^3} a^2 (1 - \mu^2) \\
 &= \frac{q^2}{2c^3} a^2 \left(\mu - \frac{\mu^3}{3} \right) \Big|_{-1}^{+1} = \frac{2q^2}{3c^3} a^2
 \end{aligned}$$

When β is not small, there are important changes. The denominator of equation (12) gets small when $\vec{\beta} \cdot \hat{n}$ is close to 1, that is for direction of propagation close to the particle's velocity $\vec{\beta}$. Thus the radiation is beamed along the direction of the particle's motion.