EM Waves in vacuum

S.M.Lea

2012

1 Polarization

As usual, Maxwell's equations tell the whole tale. In a source-free region:

$$\vec{\nabla} \cdot \vec{E} = 0$$
$$\vec{\nabla} \cdot \vec{B} = 0$$
$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\vec{\nabla} \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

If we now assume that each field has the form $\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}$, then we find:

$$\vec{k} \cdot \vec{E} = 0$$
$$\vec{k} \cdot \vec{B} = 0$$

i.e. both $\vec{E}$ and $\vec{B}$ are perpendicular to $\vec{k}$, and:

$$\vec{k} \times \vec{E} = \omega \vec{B} \tag{1}$$

so $\vec{B}$ is also perpendicular to $\vec{E}$. Now if we put the $z-$axis along $\vec{k}$, $\vec{k} = k\hat{z}$, then we may express the wave amplitudes as:

$$\vec{E}_0 = E_1\hat{x} + E_2\hat{y}$$

and similarly for $\vec{B}_0$. Finally we may allow for phase shifts by letting $E_j = E_{j0}e^{i\phi_j}$. Since $\vec{E}$ and $\vec{B}$ are perpendicular, and have related magnitudes (by eqn (1), then

$$B_2 = \frac{k}{\omega}E_1 \text{ and } B_1 = -\frac{k}{\omega}E_2$$

so we can focus attention on the components of $\vec{E}_0$.

1.1 Linear polarization

Remember that the real physical field is the real part of the complex number. If $\phi_1 = \phi_2$, then both components of $\vec{E}$ vary in phase, and $E_1/E_2 = E_{10}/E_{20}$ is constant in time and space. Thus the wave vector has a constant direction as its magnitude varies. The wave is *linearly polarized*.

1.2 Elliptical polarization

If ϕ_1 and ϕ_2 differ by $\frac{\pi}{2}$, then we find at $z = 0$:

$$
\begin{aligned}
\vec{E} &= \mathrm{Re}(E_{10}e^{i(\phi_1-\omega t)}\hat{x} + E_{20}e^{i(\phi_2-\omega t)}\hat{y}) \qquad (2)\\
&= \mathrm{Re}(E_{10}e^{i(\phi_1-\omega t)}\hat{x} + E_{20}e^{i\left(\phi_1+\frac{\pi}{2}-\omega t\right)}\hat{y}) \text{ if } \phi_2 = \phi_1 + \pi/2\\
&= E_{10}\cos(\phi_1 - \omega t)\,\hat{\mathbf{x}} - E_{20}\sin(\phi_1 - \omega t)\,\hat{\mathbf{y}} = E_{10}\cos(\omega t - \phi_1)\,\hat{\mathbf{x}} + E_{20}\sin(\omega t - \phi_1)\,\hat{\mathbf{y}}
\end{aligned}
$$

As t increases from ϕ_1/ω, E_x decreases and E_y increases: The electric field vector rotates counter-clockwise. This is an *elliptically polarized* wave (*right-hand circularly polarized* if $E_{10} = E_{20}$). Stick the thumb of your right hand in the direction of propagation (the $z-$direction in this case) and your fingers curl in the direction that $\vec{E}$ rotates. If instead we take $\phi_2 = \phi_1 - \frac{\pi}{2}$, then we get:

$$\vec{E} = E_{10}\cos(\omega t - \phi_1)\ \hat{x} - E_{20}\sin(\omega t - \phi_1)\ \hat{y}$$

and the $y-$component becomes increasingly negative: i.e. the vector rotates clockwise. This is *left-hand circular polarization* when the amplitudes E_{10} and $E_{20}$are equal.

1.3 Circular polarization

In the case of circular polarization, with $E_{10} = E_{20} = E_0/\sqrt{2}$, we have

$$\vec{E} = E_0\left(\frac{\hat{x} \pm i\hat{y}}{\sqrt{2}}\right)e^{i(\phi_1-\omega t)} \qquad (3)$$

so it is convenient to use the complex polarization vectors

$$\hat{e}_{\mathrm{R,L}} = \frac{\hat{x} \pm i\hat{y}}{\sqrt{2}} \qquad (4)$$

which have the properties

$$\hat{e}_{\mathrm{R}} \cdot \hat{e}_{\mathrm{L}}^* = 0 \quad \text{and } \hat{e}_{\mathrm{R}} \cdot \hat{e}_{\mathrm{R}}^* = \hat{e}_{\mathrm{L}} \cdot \hat{e}_{\mathrm{L}}^* = 1$$

1.4 General case

Any wave may be decomposed into a sum of linearly polarized waves or a sum of circularly polarized waves. In the most general case we have elliptical polarization with the axes of the ellipse oriented at an angle θ to the x and y axes, where θ is unknown for the moment. We'd like to find the shape and orientation of the ellipse. It's most convenient to do the analysis using the circular polarization vectors (4). Let's rewrite $\vec{E}$ (eqn 2) in terms of $\hat{e}_{\mathrm{R}}$ and $\hat{e}_{\mathrm{L}}$:

$$
\begin{aligned}
\vec{E} &= E_{10}e^{i(\phi_1-\omega t)}\hat{x} + E_{20}e^{i(\phi_2-\omega t)}\hat{y}\\
&= E_{10}e^{i(\phi_1-\omega t)}\left(\frac{\hat{e}_{\mathrm{R}} + \hat{e}_{\mathrm{L}}}{\sqrt{2}}\right) + E_{20}e^{i(\phi_2-\omega t)}\left(\frac{\hat{e}_{\mathrm{R}} - \hat{e}_{\mathrm{L}}}{\sqrt{2}i}\right)\\
&= \left[\left(E_{10}e^{i\phi_1} - iE_{20}e^{i\phi_2}\right)\left(\frac{\hat{e}_{\mathrm{R}}}{\sqrt{2}}\right) + \left(E_{10}e^{i\phi_1} + iE_{20}e^{i\phi_2}\right)\left(\frac{\hat{e}_{\mathrm{L}}}{\sqrt{2}}\right)\right]e^{-i\omega t}\\
&= \left(E_{\mathrm{R}}\hat{e}_{\mathrm{R}} + E_{\mathrm{L}}\hat{e}_{\mathrm{L}}\right)e^{-i\omega t} \qquad (5)
\end{aligned}
$$

where

$$E_{\mathrm{R}} = \frac{1}{\sqrt{2}} \left[E_{10} \cos\phi_1 + E_{20} \sin\phi_2 + i \left(E_{10} \sin\phi_1 - E_{20} \cos\phi_2 \right) \right] = \left| E_{\mathrm{R}} \right| e^{i\chi_{\mathrm{R}}}$$

The magnitude is given by

$$\begin{aligned} \left| E_{\mathrm{R}} \right| &= \frac{1}{\sqrt{2}} \sqrt{\left(E_{10} \cos\phi_1 + E_{20} \sin\phi_2 \right)^2 + \left(E_{10} \sin\phi_1 - E_{20} \cos\phi_2 \right)^2} \\ &= \frac{1}{\sqrt{2}} \sqrt{E_{10}^2 + E_{20}^2 + 2E_{10}E_{20} \sin\left(\phi_2 - \phi_1 \right)} \end{aligned}$$

and the phase by

$$\tan\chi_{\mathrm{R}} = \frac{E_{10} \sin\phi_1 - E_{20} \cos\phi_2}{E_{10} \cos\phi_1 + E_{20} \sin\phi_2}$$

For the left-hand component, we have

$$E_{\mathrm{L}} = \frac{1}{\sqrt{2}} \left[E_{10} \cos\phi_1 - E_{20} \sin\phi_2 + i \left(E_{10} \sin\phi_1 + E_{20} \cos\phi_2 \right) \right] = \left| E_{\mathrm{L}} \right| e^{i\chi_{\mathrm{L}}}$$

with

$$\left| E_{\mathrm{L}} \right| = \frac{1}{\sqrt{2}} \sqrt{E_{10}^2 + E_{20}^2 - 2E_{10}E_{20} \sin\left(\phi_2 - \phi_1 \right)}$$

and

$$\tan\chi_{\mathrm{L}} = \frac{E_{10} \sin\phi_1 + E_{20} \cos\phi_2}{E_{10} \cos\phi_1 - E_{20} \sin\phi_2}$$

If $\phi_2 - \phi_1 = \pi/2$, then $|E_{\mathrm{R}}| = E_{10} + E_{20}$ and $|E_{\mathrm{L}}| = |E_{10} - E_{20}|$. Additionally, if $E_{10} = E_{20}$, $|E_{\mathrm{L}}| = 0$ as expected for right circular polarization.

We may factor (5) to get:

$$\begin{aligned} \vec{E} &= \left| E_{\mathrm{R}} \right| \left[\hat{e}_{\mathrm{R}} + \hat{e}_{\mathrm{L}} \frac{|E_{\mathrm{L}}|}{|E_{\mathrm{R}}|} \exp i \left(\chi_{\mathrm{L}} - \chi_{\mathrm{R}} \right) \right] e^{i(\chi_{\mathrm{R}} - \omega t)} \\ &= \left| E_{\mathrm{R}} \right| \left[\hat{e}_{\mathrm{R}} + \varepsilon \hat{e}_{\mathrm{L}} \exp i\alpha \right] e^{i(\chi_{\mathrm{R}} - \omega t)} \end{aligned} \tag{6}$$

where the last line defines ε and α. In particular

$$\begin{aligned} \alpha &= \chi_{\mathrm{L}} - \chi_{\mathrm{R}} = \tan^{-1} \frac{E_{10} \sin\phi_1 + E_{20} \cos\phi_2}{E_{10} \cos\phi_1 - E_{20} \sin\phi_2} - \tan^{-1} \frac{E_{10} \sin\phi_1 - E_{20} \cos\phi_2}{E_{10} \cos\phi_1 + E_{20} \sin\phi_2} \\ &= \tan^{-1} \frac{\frac{E_{10} \sin\phi_1 + E_{20} \cos\phi_2}{E_{10} \cos\phi_1 - E_{20} \sin\phi_2} - \frac{E_{10} \sin\phi_1 - E_{20} \cos\phi_2}{E_{10} \cos\phi_1 + E_{20} \sin\phi_2}}{1 + \frac{E_{10} \sin\phi_1 + E_{20} \cos\phi_2}{E_{10} \cos\phi_1 - E_{20} \sin\phi_2} \frac{E_{10} \sin\phi_1 - E_{20} \cos\phi_2}{E_{10} \cos\phi_1 + E_{20} \sin\phi_2}} \\ &= \tan^{-1} \frac{2E_{10}E_{20} \cos\left(\phi_2 - \phi_1 \right)}{E_{10}^2 \cos^2\phi_1 - E_{20}^2 \sin^2\phi_2 + E_{10}^2 \sin^2\phi_1 - E_{20}^2 \cos^2\phi_2} \\ &= \tan^{-1} \frac{2E_{10}E_{20} \cos\left(\phi_2 - \phi_1 \right)}{E_{10}^2 - E_{20}^2} \end{aligned} \tag{7}$$

Expanding the circular polarization vectors in (6), we get

$$\vec{E} = \frac{E_{\mathrm{R}}}{\sqrt{2}} \left[\left(1 + \varepsilon e^{i\alpha} \right) \hat{x} + i \left(1 - \varepsilon e^{i\alpha} \right) \hat{y} \right] e^{i(\chi_{\mathrm{R}} - \omega t)}$$

The physical field is the real part:

$$\vec{E}_{\text{phys}} = \frac{E_{\mathrm{R}}}{\sqrt{2}} \left\{ \left[\cos\left(\chi_{\mathrm{R}} - \omega t \right) + \varepsilon \cos\left(\alpha + \chi_{\mathrm{R}} - \omega t \right) \right] \hat{x} - \left[\sin\left(\chi_{\mathrm{R}} - \omega t \right) - \varepsilon \sin\left(\alpha + \chi_{\mathrm{R}} - \omega t \right) \right] \hat{y} \right\}$$

with magnitude

$$\begin{aligned}\left|\vec{E}_{\text{phys}}\right| &= \frac{E_{\text{R}}}{\sqrt{2}}\sqrt{\left[\cos\left(\chi_{\text{R}}-\omega t\right)+\varepsilon\cos\left(\alpha+\chi_{\text{R}}-\omega t\right)\right]^2+\left[\sin\left(\chi_{\text{R}}-\omega t\right)-\varepsilon\sin\left(\alpha+\chi_{\text{R}}-\omega t\right)\right]^2} \\ &= \frac{E_{\text{R}}}{\sqrt{2}}\sqrt{\begin{array}{c}\cos^2\left(\chi_{\text{R}}-\omega t\right)+\varepsilon^2\cos^2\left(\alpha+\chi_{\text{R}}-\omega t\right)+2\varepsilon\cos\left(\chi_{\text{R}}-\omega t\right)\cos\left(\alpha+\chi_{\text{R}}-\omega t\right) \\ +\sin^2\left(\chi_{\text{R}}-\omega t\right)+\varepsilon^2\sin^2\left(\alpha+\chi_{\text{R}}-\omega t\right)-2\varepsilon\sin\left(\alpha+\chi_{\text{R}}-\omega t\right)\sin\left(\chi_{\text{R}}-\omega t\right)\end{array}} \\ &= \frac{E_{\text{R}}}{\sqrt{2}}\sqrt{1+\varepsilon^2+2\varepsilon\cos\left[\alpha+2\left(\chi_{\text{R}}-\omega t\right)\right]}\end{aligned}$$

Thus $\left|\vec{E}\right|$ is maximum when $\omega t=\alpha/2+\chi_{\text{R}}$ and minimum when $\omega t=\alpha/2+\pi/2+\chi_{\text{R}}$. At the maximum, $\vec{E}$ makes angle θ with the $x-$axis, where

$$\tan\theta=\frac{E_y}{E_x}=\frac{\sin\left(\alpha/2\right)+\varepsilon\sin\left(\alpha/2\right)}{\cos\left(\alpha/2\right)+\varepsilon\cos\left(\alpha/2\right)}=\tan\frac{\alpha}{2} \tag{8}$$

and thus we have elliptical polarization with major axis rotated through angle $\alpha/2$ from the x-axis. (See eqn (7). θ is zero if $|\phi_1-\phi_2|=\pi/2$, as expected.) The ratio of major to minor axes is

$$\frac{E_{\text{max}}}{E_{\text{min}}}=\frac{E_{\text{R}}+E_{\text{L}}}{|E_{\text{R}}-E_{\text{L}}|}=\sqrt{1-e^2} \tag{9}$$

where e is the eccentricity.

1.5 Results in terms of linear polarizations

The math is tougher with the linear polarizations. The analysis is at the end of these notes. The results are as follows. Writing $e_1=E_{10}/E_0$ and similarly for e_2, we have:

$$\tan 2\theta=2\frac{e_1e_2}{e_1^2-e_2^2}\cos\left(\phi_1-\phi_2\right) \tag{10}$$

with major axis

$$a=\sqrt{e_1^2\cos^2\theta+e_1e_2\sin 2\theta\cos\left(\phi_1-\phi_2\right)+e_2^2\sin^2\theta} \tag{11}$$

and minor axis

$$b=\sqrt{e_1^2\sin^2\theta-e_1e_2\sin 2\theta\cos\left(\phi_1-\phi_2\right)+e_2^2\cos^2\theta} \tag{12}$$

1.6 Stokes Parameters

What do we actually observe? We can design our detectors to measure either linear or circular polarizations. From those observations, we would like to determine the polarization characteristics of the observed radiation. That is, we will observe

$$\hat{x}\cdot\vec{E}=E_x=a_1e^{i\delta_1},\ \ \hat{y}\cdot\vec{E}=E_y=a_2e^{i\delta_2}$$

or

$$\hat{e}_{\text{R}}\cdot\vec{E}=E_{\text{R}}=a_Re^{i\delta_R},\ \ \ \hat{e}_{\text{L}}\cdot\vec{E}=E_{\text{L}}=a_Le^{i\delta_L}$$

(In our previous notation ($E_{10}=a_1$ and $\phi_1=\delta_1,\ E_{20}=a_2$ and $\phi_2=\delta_2$.)

The Stokes parameters are defined as:

$$\begin{aligned}
s_0 &= \left|E_x^2\right| + \left|E_y\right|^2 = a_1^2 + a_2^2 = \text{ total intensity} \\
s_1 &= \left|E_x^2\right| - \left|E_y\right|^2 = a_1^2 - a_2^2 \\
s_2 &= 2\,\mathrm{Re}\left(E_x^* E_y\right) = 2a_1 a_2 \cos\left(\delta_2 - \delta_1\right) \\
s_3 &= 2\,\mathrm{Im}\left(E_x^* E_y\right) = 2a_1 a_2 \sin\left(\delta_2 - \delta_1\right)
\end{aligned}$$

with a similar set for the circular polarizations. The four parameters are not independent:

$$s_0^2 = s_1^2 + s_2^2 + s_3^2$$

They describe physical parameters as follows:

$$\frac{s_1}{s_0} = \text{percent linear polarization}$$

From equation (10) we get

$$\frac{s_2}{s_1} = \tan(2 \times \text{angle of polarization ellipse})$$

The eccentricity of the polarization ellipse is given by (equations 11 and 12)

$$\begin{aligned}
e^2 &= \frac{a^2 - b^2}{a^2} = \frac{\left(a_1^2 - a_2^2\right)\cos 2\theta + 2a_1 a_2 \sin 2\theta \cos\left(\delta_2 - \delta_1\right)}{a_1^2 \cos^2\theta + a_1 a_2 \sin 2\theta \cos\left(\delta_2 - \delta_1\right) + a_2^2 \sin^2\theta} \\
&= \frac{\left(a_1^2 - a_2^2\right)\cos 2\theta + 2a_1 a_2 \sin 2\theta \cos\left(\delta_2 - \delta_1\right)}{\left(a_1^2 - a_2^2\right)\cos^2\theta + a_1 a_2 \sin 2\theta \cos\left(\delta_2 - \delta_1\right) + a_2^2} \\
&= \frac{\left(a_1^2 - a_2^2\right) + 2a_1 a_2 \tan 2\theta \cos\left(\delta_2 - \delta_1\right)}{\left(a_1^2 - a_2^2\right)\left(1 + \sec 2\theta\right)/2 + a_1 a_2 \tan 2\theta \cos\left(\delta_2 - \delta_1\right) + a_2^2 \sec 2\theta} \\
&= \frac{s_1 + s_2^2/s_1}{s_1\left(1 + \sqrt{s_1^2 + s_2^2}/s_1\right)/2 + s_2^2/2s_1 + \left(s_0 - s_1\right)\sqrt{s_1^2 + s_2^2}/2s_1} \\
e^2 &= 2\frac{s_1^2 + s_2^2}{s_1^2 + s_2^2 + s_0\sqrt{s_1^2 + s_2^2}} = 2\frac{\sqrt{s_1^2 + s_2^2}}{\sqrt{s_1^2 + s_2^2} + s_0}
\end{aligned}$$

and so on. To get this result, we used

$$\begin{aligned}
\tan 2\theta &= 2\frac{a_1 a_2}{a_1^2 - a_2^2}\cos\left(\delta_1 - \delta_2\right) = \frac{s_2}{s_1} \\
\sec^2 2\theta &= 1 + \left(\frac{s_2}{s_1}\right)^2 \\
\sec 2\theta &= \frac{\sqrt{s_1^2 + s_2^2}}{s_1}
\end{aligned}$$

Check: If $\delta_1 = \delta_2$ then $s_2 = 2a_1 a_2$ and

$$\sqrt{s_1^2 + s_2^2} = \sqrt{\left(a_1^2 - a_2^2\right)^2 + 4a_1^2 a_2^2} = s_0$$

So we get $e = 1$, as expected. The polarization is linear with no circularly polarized component.

See also Pacholczyk, Radio Astrophysics, Appendix I

2 Analysis in terms of linear polarizations

We define new coordinates x' and y' aligned along the axes of the ellipse. These are obtained from the old coordinates by a rotation through angle θ:

$$\hat{x} = \hat{x}' \cos\theta - \hat{y}' \sin\theta$$

and

$$\hat{y} = \hat{x}' \sin\theta + \hat{y}' \cos\theta$$

Then the electric field can be expressed as:

$$\begin{aligned}
\vec{E} &= E_0 \left(a \cos\left(\omega t - \eta\right) \ \hat{x}' + b \sin\left(\omega t - \eta\right) \ \hat{y}'\right) \\
&= E_0 \operatorname{Re}\left(a \exp i\left(\eta - \omega t\right) \ \hat{x}' + ib \exp i\left(\eta - \omega t\right) \ \hat{y}'\right)
\end{aligned}$$

where a and b are the major and minor axes of the ellipse. Let's start with the general expression (where Re is implied)

$$\begin{aligned}
\vec{E} &= E_{10} e^{i(\phi_1 - \omega t)} \hat{x} + E_{20} e^{i(\phi_2 - \omega t)} \hat{y} \\
&= E_{10} e^{i(\phi_1 - \omega t)} \left(\hat{\mathbf{x}}' \cos\theta - \hat{\mathbf{y}}' \sin\theta\right) + E_{20} e^{i(\phi_2 - \omega t)} \left(\hat{\mathbf{x}}' \sin\theta + \hat{\mathbf{y}}' \cos\theta\right) \\
&= \hat{\mathbf{x}}' \left(E_{10} e^{i(\phi_1 - \omega t)} \cos\theta + E_{20} e^{i(\phi_2 - \omega t)} \sin\theta\right) \\
&\quad + \hat{\mathbf{y}}' \left(-E_{10} e^{i(\phi_1 - \omega t)} \sin\theta + E_{20} e^{i(\phi_2 - \omega t)} \cos\theta\right)
\end{aligned}$$

Then setting the two expressions for $\vec{E}$ equal, and writing $e_1 = E_{10}/E_0$ and similarly for e_2, we have:

$$a e^{i\eta} = e_1 e^{i\phi_1} \cos\theta + e_2 e^{i\phi_2} \sin\theta$$

and

$$i b e^{i\eta} = -e_1 e^{i\phi_1} \sin\theta + e_2 e^{i\phi_2} \cos\theta$$

We have four unknowns: a, b, θ and η. So let's write the real and imaginary parts of these two relations, so we'll have a total of 4 equations.

$$a \cos\eta = e_1 \cos\phi_1 \cos\theta + e_2 \cos\phi_2 \sin\theta$$

$$\begin{aligned}
a \sin\eta &= e_1 \sin\phi_1 \cos\theta + e_2 \sin\phi_2 \sin\theta \\
-b \sin\eta &= -e_1 \cos\phi_1 \sin\theta + e_2 \cos\phi_2 \cos\theta \\
b \cos\eta &= -e_1 \sin\phi_1 \sin\theta + e_2 \sin\phi_2 \cos\theta
\end{aligned}$$

First eliminate a and b by dividing each pair of equations:

$$\tan\eta = \frac{e_1 \sin\phi_1 \cos\theta + e_2 \sin\phi_2 \sin\theta}{e_1 \cos\phi_1 \cos\theta + e_2 \cos\phi_2 \sin\theta} = -\frac{-e_1 \cos\phi_1 \sin\theta + e_2 \cos\phi_2 \cos\theta}{-e_1 \sin\phi_1 \sin\theta + e_2 \sin\phi_2 \cos\theta}$$

Now cross multiply and solve for θ :

$$\begin{aligned}
& \left(e_1 \sin\phi_1 \cos\theta + e_2 \sin\phi_2 \sin\theta\right)\left(-e_1 \sin\phi_1 \sin\theta + e_2 \sin\phi_2 \cos\theta\right) \\
= & -\left(-e_1 \cos\phi_1 \sin\theta + e_2 \cos\phi_2 \cos\theta\right)\left(e_1 \cos\phi_1 \cos\theta + e_2 \cos\phi_2 \sin\theta\right)
\end{aligned}$$

$$\begin{aligned}
& -e_1^2 \sin^2\phi_1 \cos\theta \sin\theta + e_1 e_2 \sin\phi_1 \sin\phi_2 \left(\cos^2\theta - \sin^2\theta\right) + e_2^2 \sin^2\phi_2 \sin\theta \cos\theta \\
= & \ e_1^2 \cos^2\phi_1 \sin\theta \cos\theta + e_1 e_2 \cos\phi_2 \cos\phi_1 \left(\sin^2\theta - \cos^2\theta\right) - e_2^2 \cos^2\phi_2 \cos\theta \sin\theta
\end{aligned}$$

$$
\begin{aligned}
\left(e_1^2 - e_2^2\right)\frac{\sin 2\theta}{2} &= e_1 e_2 \cos 2\theta \left(\sin\phi_1 \sin\phi_2 + \cos\phi_2 \cos\phi_1\right) \\
\tan 2\theta &= 2\frac{e_1 e_2}{e_1^2 - e_2^2}\cos\left(\phi_1 - \phi_2\right) \qquad (13)
\end{aligned}
$$

Compare with eqn (7). Let's check this answer.

If $\phi_1 = \phi_2$,we have linear polarization. Equation 13 gives $\theta = 0$ if either of e_1 or $e_2 = 0$,as expected. If $e_1 = e_2$, $\theta = \pi/4$, and the polarization is at 45° to the original axes, also as expected. The general case is expected to be

$$
\tan\theta = \frac{e_2}{e_1}
$$

But

$$
\tan 2\theta = \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta} = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2e_2}{e_1\left(1 - e_2^2/e_1^2\right)} = \frac{2e_2 e_1}{e_1^2 - e_2^2}
$$

in agreement with equation 13.

If $|\phi_1 - \phi_2| = \pi/2$, equation 13 gives $\theta = 0$, which means that the original axes are the axes of the ellipse, also as expected.

Once we have θ, we can find a and b from:

$$
\begin{aligned}
a^2 &= \left(e_1\cos\phi_1\cos\theta + e_2\cos\phi_2\sin\theta\right)^2 + \left(e_1\sin\phi_1\cos\theta + e_2\sin\phi_2\sin\theta\right)^2 \\
&= e_1^2\cos^2\phi_1\cos^2\theta + 2e_1e_2\cos\phi_1\cos\theta\cos\phi_2\sin\theta + e_2^2\cos^2\phi_2\sin^2\theta \\
&\quad + e_1^2\sin^2\phi_1\cos^2\theta + 2e_1e_2\sin\phi_1\cos\theta\sin\phi_2\sin\theta + e_2^2\sin^2\phi_2\sin^2\theta \\
&= e_1^2\cos^2\theta + e_1e_2\sin 2\theta\cos\left(\phi_1 - \phi_2\right) + e_2^2\sin^2\theta \qquad (14)
\end{aligned}
$$

and

$$
\begin{aligned}
b^2 &= \left(-e_1\cos\phi_1\sin\theta + e_2\cos\phi_2\cos\theta\right)^2 + \left(-e_1\sin\phi_1\sin\theta + e_2\sin\phi_2\cos\theta\right)^2 \\
&= e_1^2\cos^2\phi_1\sin^2\theta - 2e_1e_2\cos\phi_1\cos\phi_2\sin\theta\cos\theta + e_2^2\cos^2\phi_2\cos^2\theta \\
&\quad + e_1^2\sin^2\phi_1\sin^2\theta - 2e_1e_2\sin\phi_1\sin\phi_2\sin\theta\cos\theta + e_2^2\sin^2\phi_2\cos^2\theta \\
&= e_1^2\sin^2\theta - e_1e_2\cos\left(\phi_1 - \phi_2\right)\sin 2\theta + e_2^2\cos^2\theta \qquad (15)
\end{aligned}
$$

Finally we get η from any of the prior relations. Notice that if $e_1 = e_2 = c$, then $\sin\theta = \cos\theta = \sqrt{2}/2$, and

$$
\begin{aligned}
a &= c\sqrt{1 + \cos\left(\phi_1 - \phi_2\right)} \\
b &= c\sqrt{1 - \cos\left(\phi_1 - \phi_2\right)}
\end{aligned}
$$

and again we get back the expected results for $|\phi_1 - \phi_2| = \pi/2$ (circular polarization: $a = b = c$) and $\phi_1 = \phi_2$ (linear polarization: $b = 0$).