Waves in plasmas

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1 Plasma as an example of a dispersive medium

We shall now discuss the propagation of electromagnetic waves through a hydrogen plasma–an electrically neutral fluid of protons and electrons. This will allow us to develop a specific expression for the dielectric constant as a function of frequency. Maxwell's equations include the charge density $\rho = e(n_i - n_e)$ and the current $\vec{j} = -e(n_e\vec{v}_e - n_i\vec{v}_i)$. We will be looking for normal modes of the complete system of particles plus fields. If we assume that each field has the form $\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}$, as usual, (or equivalently, take the Fourier transform of the equations) then we find:

$$i\vec{k}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \tag{1}$$

$$\vec{k}\cdot\vec{B} = 0 \tag{2}$$

$$\vec{k}\times\vec{E} = \omega\vec{B} \tag{3}$$

and

$$i\vec{k}\times\vec{B} = \mu_0\vec{j} - i\frac{\omega}{c^2}\vec{E} \tag{4}$$

We also need the equation of motion for the electrons:

$$m\frac{d\vec{v}_e}{dt} = -e\left(\vec{E} + \vec{v}\times\vec{B}\right)$$

which becomes:

$$-i\omega m\vec{v}_e = -e\left(\vec{E} + \vec{v}\times\vec{B}\right) \tag{5}$$

We consider first high frequency waves. Because of their greater mass, the ions accelerate much more slowly than the electrons, and do not have time to respond to the wave fields before they reverse again. Thus for high-frequency waves, we may assume that the ions remain at rest:

$$\vec{v}_i \simeq 0;\ \ n_i = n_0 = \text{ constant}$$

Then the current is due to the electrons alone:

$$\vec{j} = -n_e e\vec{v}_e$$

and equation 4 becomes:

$$i\vec{k}\times\vec{B} = \mu_0(-n_e e\vec{v}_e) - i\frac{\omega}{c^2}\vec{E}$$

The electron density does not remain constant:

$$n_e\left(\vec{x},t\right) = n_0 + n_1\left(\vec{x},t\right)$$

where n_1 is the perturbation to the electron density. We now assume that the waves have small amplitude, which means that $n_1 \ll n_0$, $v \ll c$, and so on. Thus we will ignore all products of wave amplitudes in what follows. (We *linearize* the equations.)

The equation for charge conservation is:

$$\frac{\partial n_e}{\partial t} + \vec{\nabla}\cdot\left(n_e\vec{v}_e\right) = 0$$

which becomes (ignoring the term in $n_1\vec{v}$):

$$-i\omega n_1 + i\vec{k}\cdot\vec{v}_e n_0 = 0 \tag{6}$$

Now let's look for transverse waves $\left(\vec{k}\cdot\vec{v}_e = 0\right)$. For such waves equation 6 shows that the electron density perturbation n_1 is zero. Thus the right hand side of equation 1 is zero and $\vec{E}$ is perpendicular to $\vec{k}$ just as in a vacuum, and the waves are transverse in that sense too. The equation of motion (5) relates the electron velocity to the electric field. For small amplitude waves, the term in $\vec{v}\times\vec{B}$ is second order, and we neglect it. Then:

$$-i\omega m\vec{v}_e = -e\vec{E}$$

and

$$\vec{j} = -n_0 e\frac{e\vec{E}}{i\omega m} = i\frac{n_0e^2}{\omega m}\vec{E} = \sigma\vec{E} \tag{7}$$

The conductivity is imaginary, indicating a 90° phase shift between $\vec{E}$ and $\vec{j}$. Putting this result into Ampere's law, we get:

$$\begin{aligned} i\vec{k}\times\vec{B} &= \frac{1}{c^2\varepsilon_0}i\frac{n_0e^2}{\omega m}\vec{E} - i\frac{\omega}{c^2}\vec{E} \\ \vec{k}\times\vec{B} &= -\frac{\omega}{c^2}\left(1-\frac{n_0e^2}{\omega^2 m\varepsilon_0}\right)\vec{E} \end{aligned} \tag{8}$$

The quantity

$$\frac{n_0e^2}{\varepsilon_0 m} \equiv \omega_p^2$$

where ω_p is the *plasma frequency* , the frequency of natural electrostatic oscillations in the ionized plasma. Now we may interpret equation 8 in terms of the dielectric constant ε for the plasma (*cf* "Brewster" notes eqn 2):

$$\vec{k}\times\vec{B} = -\frac{\omega}{c^2}\left(1-\frac{\omega_p^2}{\omega^2}\right)\vec{E} = -\frac{\omega}{c^2}\frac{\varepsilon}{\varepsilon_0}\vec{E}$$

with

$$\varepsilon = \left(1-\frac{\omega_p^2}{\omega^2}\right)\varepsilon_0 \tag{9}$$

Recall that we may express the wave speed in terms of the dielectric constant ("Brewster" notes eqn 4):

$$v_\phi = \frac{\omega}{k} = c\sqrt{\frac{\varepsilon_0}{\varepsilon}} = \frac{c}{\sqrt{1-\frac{\omega_p^2}{\omega^2}}} \tag{10}$$

Since $\varepsilon/\varepsilon_0 < 1$, the wave phase speed is greater than c! This is physically possible, because the speed at which information travels, the group speed $d\omega/dk$, is less than c :

$$\begin{aligned} \frac{\omega^2}{k^2}\left(1-\frac{\omega_p^2}{\omega^2}\right) &= c^2 \\ \omega^2-\omega_p^2 &= c^2k^2 \end{aligned} \tag{11}$$

Thus

$$\begin{aligned} 2\omega\frac{d\omega}{dk} &= 2c^2k \\ v_g &= \frac{d\omega}{dk}=\frac{c^2}{v_\phi} \end{aligned} \tag{12}$$

If $\omega < \omega_p$ the wave number k becomes imaginary and the wave ceases to propagate. Wave energy is dissipated by the fields as the electrons gain kinetic energy. The rate of energy dissipation per unit volume is

$$P=\vec{j}\cdot\vec{E}$$

where we have to take the real part before multiplying. Since σ is imaginary (eqn 7), there is a phase shift of $\pi/2$ and so

$$\begin{aligned} P(t) &= \operatorname{Re}\left(\sigma\vec{E}\right)\cdot\operatorname{Re}\left(\vec{E}\right)=\operatorname{Re}\left(i\left|\sigma\right|\vec{E}\right)\cdot\operatorname{Re}\left(\vec{E}\right) \\ &= -\left|\sigma\right|E_0\sin\left(\vec{k}\cdot\vec{x}-\omega t\right)E_0\cos\left(\vec{k}\cdot\vec{x}-\omega t\right) \\ &= -\frac{\omega_p^2}{\omega}\frac{\varepsilon_0E_0^2}{2}\sin 2\left(\vec{k}\cdot\vec{x}-\omega t\right) \end{aligned}$$

Looking at $\vec{x}=0$ for simplicity:

$$P(t)=\omega u_{E,0}\frac{\omega_p^2}{\omega^2}\sin 2\omega t$$

If k is imaginary, we get

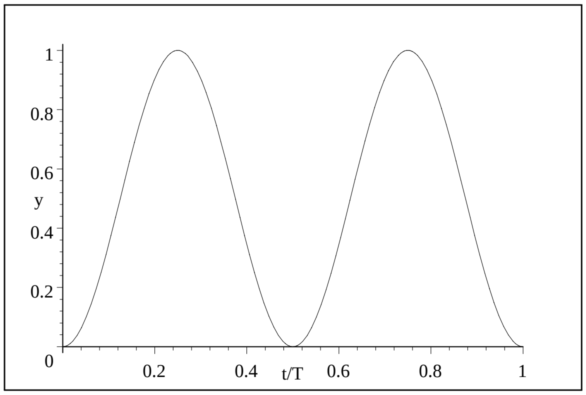
$$P(t)=-\left|\sigma\right|E_0e^{-2\hat{k}\cdot\vec{x}|k|}\sin(-\omega t)E_0\cos(-\omega t)=\omega u_{E,0}\frac{\omega_p^2}{\omega^2}\sin 2\omega t \text{ at } x=0$$

In both cases, the energy converted in time t is

$$\mathcal{E}(t)=\int_0^t P(t)\,dt=u_{E,0}\frac{\omega_p^2}{2\omega^2}(1-\cos 2\omega t)$$

which shows that electric field energy is converted to electron kinetic energy fast enough to dissipate all the wave energy in less than a quarter of a wave period if $\omega < \omega_p$. For $\omega > \omega_p$ the time-averaged energy transfer is zero.

The plot shows $(\mathcal{E}/u_{E,0})\left(\omega^2/\omega_p^2\right)$ versus t/T.



2 Waves in magnetized plasmas

If the plasma is magnetized with a uniform magnetic field $\vec{B}_0$, there is an additional term in the equation of motion:

$$-i\omega m\vec{v}_e = -e\left(\vec{E} + \vec{v}_e \times \vec{B}_0\right)$$

To simplify the solution of this equation for $\vec{v}_e$, choose the $z-$axis along $\vec{B}_0$. Then we have:

$$-i\omega m v_x = -eE_x - ev_y B_0$$
$$-i\omega m v_y = -eE_y + ev_x B_0$$
$$-i\omega m v_z = -eE_z$$

and thus solving for the components of $\vec{v}$, we have:

$$v_z = -i\frac{e}{\omega m}E_z \tag{13}$$

$$\begin{aligned} -i\omega m v_x &= -eE_x - eB_0\left(\frac{-eE_y + ev_x B_0}{-i\omega m}\right) \\ &= -eE_x + i\frac{\Omega}{\omega}eE_y - i\frac{\Omega^2}{\omega}mv_x \\ -i\omega m v_x\left(1 - \frac{\Omega^2}{\omega^2}\right) &= -e\left(E_x - i\frac{\Omega}{\omega}E_y\right) \\ v_x &= -i\frac{e}{\omega m}\frac{\left(E_x - i\frac{\Omega}{\omega}E_y\right)}{1 - \Omega^2/\omega^2} \end{aligned} \tag{14}$$

where $\Omega = eB_0/m$ is the cyclotron frequency, and

$$\begin{aligned} v_y &= \frac{-eE_y + \Omega m v_x}{-i\omega m} \\ &= -i\frac{e}{\omega m}\left[E_y + \frac{\Omega m}{-e}\left(-i\frac{e}{\omega m}\frac{\left(E_x - i\frac{\Omega}{\omega}E_y\right)}{1-\Omega^2/\omega^2}\right)\right] \\ &= -i\frac{e}{\omega m}\frac{\left(E_y + i\frac{\Omega}{\omega}E_x\right)}{1-\Omega^2/\omega^2} \qquad (15) \end{aligned}$$

With $\vec{j} = -n_0 e\vec{v}$, we see that j_x is related to both E_x and E_y components. Then $\vec{j}$ is related to $\vec{E}$ by the tensor relation:

$$j_i = -n_0 e v_i = \sigma_{ij} E_j$$

with the conductivity tensor

$$\begin{aligned} \widetilde{\sigma} &= \frac{-n_0 e\left(-i\frac{e}{\omega m}\right)}{1-\Omega^2/\omega^2}\begin{pmatrix} 1 & -i\Omega/\omega & 0 \\ i\Omega/\omega & 1 & 0 \\ 0 & 0 & 1-\Omega^2/\omega^2 \end{pmatrix} \\ &= \frac{i\varepsilon_0\omega_p^2/\omega}{1-\Omega^2/\omega^2}\begin{pmatrix} 1 & -i\Omega/\omega & 0 \\ i\Omega/\omega & 1 & 0 \\ 0 & 0 & 1-\Omega^2/\omega^2 \end{pmatrix} \qquad (16) \end{aligned}$$

Stuffing back into Ampere's law gives the dielectric "constant", which is also now a rank 2 tensor:

$$i\vec{k}\times\vec{B} = \mu_0\widetilde{\sigma}\vec{E} - i\frac{\omega}{c^2}\vec{E} = -i\frac{\omega}{c^2}\frac{\widetilde{\varepsilon}}{\varepsilon_0}\vec{E}$$

The dielectric tensor has components

$$\begin{aligned} \varepsilon_{ij} &= \left(\delta_{ij} + i\frac{\mu_0}{\omega}c^2\sigma_{ij}\right)\varepsilon_0 \\ \frac{\varepsilon_{ij}}{\varepsilon_0} &= \delta_{ij} + \frac{i}{\varepsilon_0\omega}\frac{i\varepsilon_0\omega_p^2/\omega}{1-\Omega^2/\omega^2}\begin{pmatrix} 1 & -i\Omega/\omega & 0 \\ i\Omega/\omega & 1 & 0 \\ 0 & 0 & 1-\Omega^2/\omega^2 \end{pmatrix} \\ &= \delta_{ij} - \frac{\omega_p^2/\omega^2}{1-\Omega^2/\omega^2}\begin{pmatrix} 1 & -i\Omega/\omega & 0 \\ i\Omega/\omega & 1 & 0 \\ 0 & 0 & 1-\Omega^2/\omega^2 \end{pmatrix} \qquad (17) \end{aligned}$$

We can tidy this up by rewriting the matrix as

$$\begin{pmatrix} 1 & -i\frac{\Omega}{\omega} & 0 \\ i\frac{\Omega}{\omega} & 1 & 0 \\ 0 & 0 & 1-\frac{\Omega^2}{\omega^2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{\omega^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Omega^2 \end{pmatrix} - i\frac{\Omega}{\omega}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Now define a vector cyclotron frequency:

$$\vec{\Omega} = \frac{e}{mc}\vec{B}_0$$

With our coordinate choice, $\vec{\Omega}$ has only one component: Ω_3. Then we can write the dielectric tensor in coordinate-free form as:

$$\frac{\varepsilon_{ij}}{\varepsilon_0} = \delta_{ij} - \frac{\omega_p^2}{(\omega^2 - \Omega^2)} \left(\delta_{ij} - \frac{\Omega_i \Omega_j}{\omega^2} - i\epsilon_{ijk} \frac{\Omega_k}{\omega} \right) \tag{18}$$

2.1 Normal modes

We have now reduced Ampere's law to the form:

$$\epsilon_{ijn} k_j B_n = -\frac{\omega}{c^2} \frac{\varepsilon_{ij}}{\varepsilon_0} E_j$$

Now take $\vec{k} \times$ (equation 3) to get:

$$\begin{aligned} k_i k_j E_j - k^2 E_i &= \omega \epsilon_{ijn} k_j B_n \\ &= -\frac{\omega^2}{c^2} \frac{\varepsilon_{ij}}{\varepsilon_0} E_j \end{aligned}$$

or

$$E_j \left(\frac{\varepsilon_{ij}}{\varepsilon_0} - \frac{c^2 k^2}{\omega^2} \left[\delta_{ij} - \hat{k}_i \hat{k}_j \right] \right) = 0 \tag{19}$$

This equation has a non-trivial solution for $\vec{E}$ only if

$$\det \left(\frac{\varepsilon_{ij}}{\varepsilon_0} - \frac{c^2 k^2}{\omega^2} \left[\delta_{ij} - \hat{k}_i \hat{k}_j \right] \right) = 0 \tag{20}$$

Again, with the $z-$axis along $\vec{B}_0$, and defining the dimensionless quantities:

$$\kappa_0 \equiv 1 - \frac{\omega_p^2}{\omega^2} \tag{21}$$

$$\kappa_1 \equiv 1 - \frac{\frac{\omega_p^2}{\omega^2}}{1 - \frac{\Omega^2}{\omega^2}} \tag{22}$$

and

$$\kappa_2 \equiv \frac{\omega_p^2 \Omega / \omega^3}{1 - \frac{\Omega^2}{\omega^2}} \tag{23}$$

we can write:

$$\frac{\varepsilon_{ij}}{\varepsilon_0} = \begin{pmatrix} \kappa_1 & -i\kappa_2 & 0 \\ i\kappa_2 & \kappa_1 & 0 \\ 0 & 0 & \kappa_0 \end{pmatrix}$$

Choose coordinate axes so that $\hat{k}$ lies in the $y - z$ plane:

$$\hat{k} = \begin{pmatrix} 0, & \sin\theta, & \cos\theta \end{pmatrix}$$

Then equation (20) becomes:

$$\begin{vmatrix} \kappa_1 - \frac{c^2 k^2}{\omega^2} & -i\kappa_2 & 0 \\ i\kappa_2 & \kappa_1 - \frac{c^2 k^2}{\omega^2} \left(1 - \sin^2\theta\right) & \frac{c^2 k^2}{\omega^2} \sin\theta \cos\theta \\ 0 & \frac{c^2 k^2}{\omega^2} \sin\theta \cos\theta & \kappa_0 - \frac{c^2 k^2}{\omega^2} \left(1 - \cos^2\theta\right) \end{vmatrix} = 0 \tag{24}$$

This is the dispersion relation for plasma waves in a magnetized plasma. Rather than solve this equation in the most general case, we'll look at the specific case of propagation along $\vec{B}_0$.

2.2 Propagation along $\vec{B}_0$ ($\theta = 0$)

With $\theta = 0$, equation (24) becomes:

$$\begin{vmatrix} \kappa_1 - \frac{c^2k^2}{\omega^2} & -i\kappa_2 & 0 \\ i\kappa_2 & \kappa_1 - \frac{c^2k^2}{\omega^2} & 0 \\ 0 & 0 & \kappa_0 \end{vmatrix} = 0$$

$$\left[\left(\kappa_1 - \frac{c^2k^2}{\omega^2} \right)^2 - \kappa_2^2 \right] \kappa_0 = 0$$

Thus either $\kappa_0 = 0$ (which corresponds to electrostatic (longitudinal) waves at frequency ω_p- see below), or

$$\kappa_1 - \frac{c^2k^2}{\omega^2} = \pm\kappa_2 \tag{25}$$

Substituting in for the κ from (21, 22, and 23), we get the dispersion relation for transverse waves:

$$\begin{aligned} \frac{c^2k^2}{\omega^2} &= 1 - \frac{\omega_p^2}{\omega^2 - \Omega^2} \mp \frac{\omega_p^2\Omega/\omega}{\omega^2 - \Omega^2} \\ &= 1 - \frac{\omega_p^2}{\omega} \frac{(\omega \pm \Omega)}{\omega^2 - \Omega^2} \\ &= 1 - \frac{\omega_p^2}{\omega(\omega \mp \Omega)} \end{aligned} \tag{26}$$

To show that these are the transverse (electromagnetic waves), let's solve for the corresponding electric field vectors. Equation 19 is:

$$\begin{pmatrix} E_1, & E_2, & E_3 \end{pmatrix} \begin{pmatrix} \kappa_1 - \frac{c^2k^2}{\omega^2} & -i\kappa_2 & 0 \\ i\kappa_2 & \kappa_1 - \frac{c^2k^2}{\omega^2} & 0 \\ 0 & 0 & \kappa_0 \end{pmatrix} = 0$$

Now we use equation 25 to simplify:

$$\begin{aligned} \begin{pmatrix} E_1, & E_2, & E_3 \end{pmatrix} \begin{pmatrix} \pm\kappa_2 & -i\kappa_2 & 0 \\ i\kappa_2 & \pm\kappa_2 & 0 \\ 0 & 0 & \kappa_0 \end{pmatrix} &= 0 \\ \begin{pmatrix} \pm E_1\kappa_2 + iE_2\kappa_2, & -iE_1\kappa_2 \pm E_2\kappa_2, & E_3\kappa_0 \end{pmatrix} &= 0 \end{aligned}$$

The $z-$component shows that either $\kappa_0 = 0$ and $E_3 \neq 0$, (longitudinal waves) or if $\kappa_0 \neq 0$, then $E_3 = 0$. This is the transverse wave. The $x-$component gives either $\kappa_2 = 0$ (which is not possible unless $\omega \to \infty$) or

$$E_2 = \pm iE_1$$

We get the same result from the $y-$component. These solutions corrrespond to right hand (upper sign)and left hand (lower sign) circular polarization. (Polarization notes eqn 3.) Note that the RHC mode (top sign of the pair) corresponds to the resonance in eqn (26). In this mode the electron gyrates in the same sense as the electric field vector, and at $\omega = \Omega$, energy may be transferred continuously from the fields to the electrons.

2.3 Propagation of a plane polarized wave along $\vec{B}_0$

Since the normal modes are circularly polarized, to understand the propagation of a linearly polarized wave we must split it up into two circular polarizations. Let's put the $x-$axis along the direction of the linear polarization. Then (polarization notes eqn 4):

$$\begin{aligned} \vec{E}_0 &= E_0\hat{x} \\ &= \frac{1}{2}E_0\left(\hat{\mathbf{x}}+i\hat{\mathbf{y}}\right)+\frac{1}{2}E_0\left(\hat{\mathbf{x}}-i\hat{\mathbf{y}}\right) \end{aligned}$$

The two circular polarizations have different phase velocities. Thus, after travelling a distance z, the wave is described by:

$$\vec{E} = \frac{1}{2}E_0\left(\hat{\mathbf{x}}+i\hat{\mathbf{y}}\right)e^{ik_Rz-i\omega t}+\frac{1}{2}E_0\left(\hat{\mathbf{x}}-i\hat{\mathbf{y}}\right)e^{ik_Lz-i\omega t}$$

where (equation 26):

$$\frac{c^2k_R^2}{\omega^2} = 1-\frac{\omega_p^2}{\omega\left(\omega-\Omega\right)} \tag{27}$$

and

$$\frac{c^2k_L^2}{\omega^2} = 1-\frac{\omega_p^2}{\omega\left(\omega+\Omega\right)} \tag{28}$$

At time t, the electric field vector makes an angle ϕ with the $x-$axis, where

$$\begin{aligned} \tan\phi &= \frac{E_y}{E_x} = \frac{\mathrm{Re}\left(i\left[e^{ik_Rz-i\omega t}-e^{ik_Lz-i\omega t}\right]\right)}{\mathrm{Re}\left(e^{ik_Rz-i\omega t}+e^{ik_Lz-i\omega t}\right)} \\ &= \frac{-\sin\left(k_Rz-\omega t\right)+\sin\left(k_Lz-\omega t\right)}{\cos\left(k_Rz-\omega t\right)+\cos\left(k_Lz-\omega t\right)} \\ &= \frac{2\cos\left(\frac{k_L+k_R}{2}z-\omega t\right)\sin\left(\frac{k_L-k_R}{2}z\right)}{2\cos\left(\frac{k_L+k_R}{2}z-\omega t\right)\cos\left(\frac{k_L-k_R}{2}z\right)} \\ &= \tan\left(\frac{k_L-k_R}{2}z\right) \end{aligned}$$

Thus:

$$\phi = \frac{k_L-k_R}{2}z\pm 2n\pi \tag{29}$$

For high frequency waves, $\omega \gg \omega_p$, Ω, we may expand the denominator in equations 27 and 28 to get:

$$\begin{aligned} k_{R,L} &= \frac{\omega}{c}\sqrt{1-\frac{\omega_p^2}{\omega^2}\left(1\mp\frac{\Omega}{\omega}\right)^{-1}} \simeq \frac{\omega}{c}\sqrt{1-\frac{\omega_p^2}{\omega^2}\left(1\pm\frac{\Omega}{\omega}\right)} \\ &\simeq \frac{\omega}{c}\left[1-\frac{1}{2}\frac{\omega_p^2}{\omega^2}\left(1\pm\frac{\Omega}{\omega}\right)\right] \end{aligned}$$

and thus

$$k_L - k_R = \frac{\omega}{c}\frac{\omega_p^2}{\omega^2}\frac{\Omega}{\omega}$$

and

$$\begin{aligned} \phi &= \frac{\omega_p^2 \Omega}{2\omega^2}\frac{z}{c} \\ &= \frac{z}{2c\omega^2}\frac{n_0 e^2}{\varepsilon_0 m}\frac{eB_0}{m} \\ &= \frac{z}{2\omega^2}\frac{e^3}{\varepsilon_0 m^2 c} n_0 B_0 \\ &= \frac{e^3}{2\,(2\pi)^2\,\varepsilon_0 m^2 c}\frac{n_0 B_0 z}{\nu^2} \end{aligned} \quad (30)$$

Thus the direction of polarization rotates as the wave travels. The rotation angle ϕ is proportional to the electron density n_0,the magnetic field strength B_0, the distance travelled z, and inversely proportional to the square of the frequency. This effect is known as *Faraday Rotation.*

2.4 Low frequency waves

When the wave frequency $\omega \ll \Omega, \omega_p$ the dispersion relation 26 simplifies:

$$\frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega\,(\mp\Omega)} = 1 \pm \frac{\omega_p^2}{\omega\Omega} \approx \frac{\omega_p^2}{\omega\Omega}$$

(Note: only the positive sign (RHC) makes sense in this frequency range. With the minus sign, $k^2 < 0$ and this mode does not propagate.). For RHC waves, we get:

$$\begin{aligned} \frac{c^2k^2}{\omega^2} &\approx \frac{\omega_p^2}{\omega\Omega} \\ k &\approx \sqrt{\frac{\omega}{\Omega}}\frac{\omega_p}{c} \end{aligned} \quad (31)$$

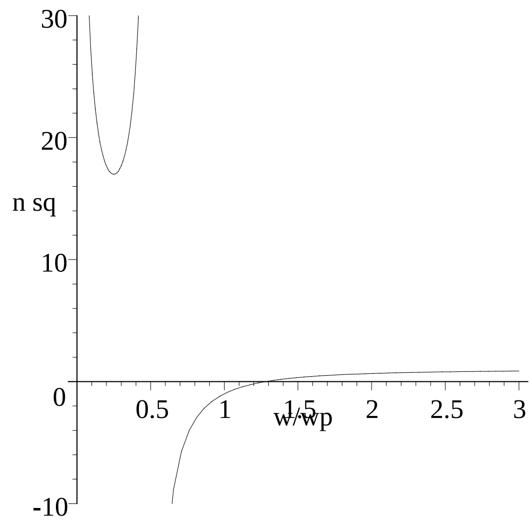
For these waves, the phase speed

$$v_\phi = \frac{\omega}{k} = \frac{\sqrt{\omega\Omega}}{\omega_p} c \quad (32)$$

increases with the frequency ω. Because a signal's high frequencies will arrive before the low frequencies, resulting in a declining pitch “whistle”, these waves are called *whistlers.*

For more on wave propagation in plasmas see P 7.17, 18, 25, http://www.physics.sfsu.edu/~lea/courses/grad/plaswav.PDF, and Chen Ch 4.

The graphs show the square of the refractive index versus ω/ω_p for the RHC wave in the two cases $\Omega = \omega_p/2$. and $\Omega = 2\omega_p$. The Whistler branch is the left (low frequency) side of the upper curve. Negative values of n^2 correspond to "stop bands" where the wave does not propagate. Note the resonance at $\omega = \Omega$. The resonance at $\omega \to 0$ is spurious due to our neglect of ion motion.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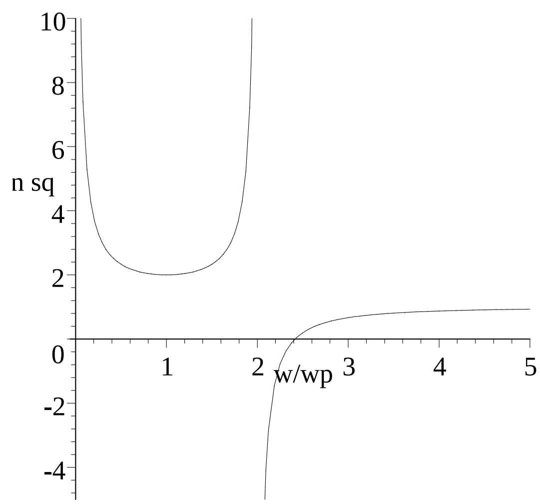
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