

Multipole fields

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1 Introduction

The big idea is to time- transform all physical quantities, such as the current vector and the fields. We have

$$J^\mu(\vec{x}, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} J^\mu(\vec{x}, \omega) \exp(-i\omega t) dt$$

The wave equation for \mathbf{A} is:

$$\square^2 \mathbf{A} = \frac{4\pi}{c} \mathbf{J}$$

and we already have the solution

$$\mathbf{A} = \frac{4\pi}{c} \int \mathbf{J}(\mathbf{x}') \frac{\delta(|\vec{x} - \vec{x}'| - c(t - t'))}{4\pi |\vec{x} - \vec{x}'|} d^4 \mathbf{x}'$$

which we may write in terms of the time transform of \mathbf{J}

$$A^\mu(\mathbf{x}) = \frac{1}{c} \int \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} J^\mu(\vec{x}', \omega) e^{-i\omega t'} d\omega \frac{\delta(|\vec{x} - \vec{x}'| - c(t - t'))}{|\vec{x} - \vec{x}'|} d^4 \mathbf{x}'$$

Now we do the integration over t' :

$$\begin{aligned} A^\mu(\mathbf{x}) &= \frac{1}{c} \int \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} J^\mu(\vec{x}', \omega) e^{-i\omega t'} d\omega \frac{\delta(ct' - c(t - |\vec{x} - \vec{x}'|/c))}{|\vec{x} - \vec{x}'|} dt' d^3 \vec{x}' \\ &= \frac{1}{\sqrt{2\pi}} \int \int_{-\infty}^{+\infty} \frac{J^\mu(\vec{x}', \omega) \exp(-i\omega(t - |\vec{x} - \vec{x}'|/c))}{c |\vec{x} - \vec{x}'|} d\omega d^3 \vec{x}' \end{aligned} \quad (1)$$

The 0th component is:

$$\Phi(\mathbf{x}) = \frac{1}{\sqrt{2\pi}} \int \int_{-\infty}^{+\infty} d\omega e^{-i\omega t} \frac{\rho(\vec{x}', \omega) \exp(i\omega |\vec{x} - \vec{x}'|/c)}{|\vec{x} - \vec{x}'|} d^3 \vec{x}' \quad (2)$$

while the 1,2,3 components are:

$$\vec{A} = \frac{1}{\sqrt{2\pi}} \int \int_{-\infty}^{+\infty} d\omega e^{-i\omega t} \frac{\vec{J}(\vec{x}', \omega) \exp(i\omega |\vec{x} - \vec{x}'|/c)}{c |\vec{x} - \vec{x}'|} d^3 \vec{x}' \quad (3)$$

Transforming the relations between the fields and the potentials, we get:

$$\vec{B}(\vec{x}, \omega) = \vec{\nabla} \times \vec{A}(\vec{x}, \omega)$$

and

$$\frac{1}{c} \frac{\partial \vec{E}}{\partial t} = \vec{\nabla} \times \vec{B} \rightarrow -\frac{i\omega}{c} \vec{E}(\vec{x}, \omega) = \vec{\nabla} \times \vec{B}(\vec{x}, \omega)$$

and thus

$$\vec{E}(\vec{x}, \omega) = \frac{ic}{\omega} \vec{\nabla} \times \vec{B}(\vec{x}, \omega)$$

We have 3 relevant length scales: d , the dimension of the source, λ , the wavelength, and r , the distance from source to observer. The ordering of these lengths determines how we proceed.

- $d \ll r \ll \lambda$. This is the near, or static region. With $r \ll \lambda$, the exponential in equations(3) and (2) is $\exp(2\pi ir/\lambda) \approx 1$ and we get the static results from Chapter 3 for the time transform $\vec{A}(\vec{x}, \omega)$. Thus we get the static fields, but oscillating in time.
- $d \ll r \sim \lambda$. The induction zone. This is tricky.
- $d \ll \lambda \ll r$. The radiation zone. In this zone the source appears almost point-like. We may expand the quantity $|\vec{x} - \vec{x}'|$:

$$\begin{aligned} |\vec{x} - \vec{x}'|^2 &= r^2 + x'^2 - 2\vec{r} \cdot \vec{x}' \\ |\vec{x} - \vec{x}'| &= r \left(1 - \frac{\hat{\mathbf{r}} \cdot \vec{x}'}{r} \right) = r - \hat{\mathbf{r}} \cdot \vec{x}' \end{aligned}$$

and similarly

$$\frac{1}{|\vec{x} - \vec{x}'|} = \frac{1}{r} + \hat{\mathbf{r}} \cdot \vec{x}' + \dots \approx \frac{1}{r}$$

Remember that we need more accuracy in the exponential than in the quantity outside the exponential. From here on we shall assume we are in the radiation zone.

2 The dipole fields

We put the approximations for $|\vec{x} - \vec{x}'|$ into the expression for \vec{A} (equation 3):

$$\vec{A} = \frac{1}{\sqrt{2\pi}} \int \int_{-\infty}^{+\infty} d\omega e^{-i\omega t} \frac{\vec{J}(\vec{x}', \omega) \exp(ikr - ik\hat{\mathbf{r}} \cdot \vec{x}')}{cr} d^3\vec{x}'$$

Thus the time transform of \vec{A} is:

$$\vec{A}(\vec{x}, \omega) = \frac{e^{ikr}}{cr} \int \vec{J}(\vec{x}', \omega) \exp(-ik\hat{\mathbf{r}} \cdot \vec{x}') d^3\vec{x}'$$

where $k = \omega/c$. Now we expand the exponential in the integrand:

$$\begin{aligned} \exp(-ik\hat{\mathbf{r}} \cdot \vec{x}') &= 1 - ik\hat{\mathbf{r}} \cdot \vec{x}' + \frac{1}{2} (-ik\hat{\mathbf{r}} \cdot \vec{x}')^2 + \dots \\ &= 1 - ik\hat{\mathbf{r}} \cdot \vec{x}' - \frac{k^2}{2} (\hat{\mathbf{r}} \cdot \vec{x}')^2 + \dots \end{aligned}$$

The first term in \vec{A} is:

$$\vec{A}_d(\vec{x}, \omega) = \frac{e^{ikr}}{cr} 2 \int \vec{J}(\vec{x}', \omega) d^3\vec{x}'$$

We can simplify this expression by using the equation of charge conservation:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

Taking the time transform, we have:

$$-i\omega\rho + \vec{\nabla} \cdot \vec{J}(\vec{x}, \omega) = 0$$

Now

$$\begin{aligned} \vec{\nabla} (\vec{x} \cdot \vec{J}) &= (\vec{x} \cdot \vec{\nabla}) \vec{J} + (\vec{J} \cdot \vec{\nabla}) \vec{x} + \vec{x} \times (\vec{\nabla} \times \vec{J}) + \vec{J} \times (\vec{\nabla} \times \vec{x}) \\ &= (\vec{x} \cdot \vec{\nabla}) \vec{J} + \vec{J} + \vec{x} \times (\vec{\nabla} \times \vec{J}) \\ &= x_m \partial_m J_i + J_i + \varepsilon_{ijk} x_j \varepsilon_{klm} \partial_l J_m \\ &= x_m \partial_m J_i + J_i + (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) x_j \partial_l J_m \\ &= x_m \partial_m J_i + J_i + x_j \partial_i J_j - x_l \partial_l J_i \\ &= J_i + x_j \partial_i J_j + x_i \partial_m J_m - x_i \partial_m J_m \\ &= \vec{J} + \vec{x} (\vec{\nabla} \cdot \vec{J}) + x_j \partial_i J_j - x_i \partial_m J_m \\ &= \vec{J} + \vec{x} (\vec{\nabla} \cdot \vec{J}) + (\vec{x} \times \vec{\nabla}) \times \vec{J} \end{aligned} \quad (4)$$

and

$$\begin{aligned} \int_{\text{all space}} \vec{\nabla} (\vec{x} \cdot \vec{J}) dV &= \int_{s_\infty} (\vec{x} \cdot \vec{J}) \hat{n} dA = 0 \\ &= \int_{\text{all space}} (\vec{J} + \vec{x} (\vec{\nabla} \cdot \vec{J}) + (\vec{x} \times \vec{\nabla}) \times \vec{J}) dV \end{aligned}$$

since \vec{J} is zero outside the source. Thus:

$$\begin{aligned} \int \vec{J}(\vec{x}', \omega) d^3 \vec{x}' &= - \int \vec{x}' (\vec{\nabla}' \cdot \vec{J}) d^3 \vec{x}' \\ &= -i \int \vec{x}' \omega \rho(\vec{x}', \omega) d^3 \vec{x}' = -i\omega \vec{p}(\omega) \end{aligned}$$

where \vec{p} is the dipole moment of the source.

Then:

$$\begin{aligned} \vec{A}_d(\vec{x}, \omega) &= -i\omega \frac{e^{ikr}}{cr} \vec{p}(\omega) = -ik \frac{e^{ikr}}{r} \vec{p}(\omega) \\ \vec{B} = \vec{\nabla} \times \vec{A} &= -\frac{ik}{r} \left(ik - \frac{1}{r} \right) e^{ikr} \hat{r} \times \vec{p} \end{aligned}$$

and in the radiation zone $k \gg 1/r$, so:

$$\vec{B} = k^2 \frac{e^{ikr}}{r} \hat{r} \times \vec{p}$$

and then

$$\begin{aligned}
\vec{E}(\vec{x}, \omega) &= \frac{i}{k} \vec{\nabla} \times \vec{B}(\vec{x}, \omega) = \frac{i}{k} \vec{\nabla} \times \left(k^2 \frac{e^{ikr}}{r} \hat{\mathbf{r}} \times \vec{p} \right) \\
&= ik \left(ik \frac{e^{ikr}}{r} \right) \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \vec{p}) \\
&= -k^2 \frac{e^{ikr}}{r} \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \vec{p})
\end{aligned}$$

3 Power radiated

The power radiated per unit solid angle is given by the Poynting vector:

$$\frac{dP}{d\Omega} = r^2 |\vec{S}| = r^2 \frac{c}{4\pi} |\vec{E} \times \vec{B}|$$

The the total energy radiated is

$$\frac{dW}{d\Omega} = \frac{c}{4\pi} \int_{-\infty}^{+\infty} r^2 |\vec{E} \times \vec{B}| dt$$

and using Parseval's theorem we may convert to an integral of the transforms over frequency:

$$\frac{dW}{d\Omega} = \frac{c}{4\pi} \int_{-\infty}^{+\infty} r^2 |\vec{E} \times \vec{B}^*| d\omega$$

where

$$\begin{aligned}
r^2 |\vec{E} \times \vec{B}^*| &= r^2 \frac{c}{4\pi} \left| \left(-k^2 \frac{e^{ikr}}{r} \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \vec{p}) \right) \times \left(k^2 \frac{e^{-ikr}}{r} \hat{\mathbf{r}} \times \vec{p}^* \right) \right| \\
&= \frac{c}{4\pi} k^4 |\hat{\mathbf{r}} \times \vec{p}|^2
\end{aligned}$$

and so

$$\frac{d^2W}{d\Omega d\omega} = \frac{c}{4\pi} k^4 |\hat{\mathbf{r}} \times \vec{p}|^2 \quad (5)$$

is the energy radiated per unit solid angle and per unit frequency.

4 Periodic source

If the source is periodic, then the current must be expanded in a Fourier series rather than a Fourier transform. We have

$$\mathbf{J}(\vec{x}, t) = \sum_{n=-\infty}^{+\infty} \mathbf{J}_n(\vec{x}) \exp(in\omega_0 t)$$

where ω_0 is the fundamental frequency $= 2\pi/T$ and T is the period of the source. Then the

expression for \vec{A} becomes:

$$\begin{aligned}\mathbf{A}(\mathbf{x}) &= \frac{1}{c} \int \sum_{n=-\infty}^{+\infty} \mathbf{J}_n(\vec{x}') \exp(in\omega_0 t') \frac{\delta(|\vec{x} - \vec{x}'| - c(t - t'))}{|\vec{x} - \vec{x}'|} d^4 \mathbf{x}' \\ &= \sum_{n=-\infty}^{+\infty} \int \frac{\mathbf{J}_n(\vec{x}) \exp(in\omega_0 t)}{c|\vec{x} - \vec{x}'|} \exp(-in\omega_0 |\vec{x} - \vec{x}'|/c) d^3 \vec{x}'\end{aligned}$$

which is a Fourier series for \mathbf{A} with coefficients:

$$\mathbf{A}_n = \sum_{n=-\infty}^{+\infty} \int \frac{\mathbf{J}_n(\vec{x})}{c|\vec{x} - \vec{x}'|} \exp(-in\omega_0 |\vec{x} - \vec{x}'|/c) d^3 \vec{x}'$$

Following the analysis above, we find the dipole term to be:

$$\vec{A}_{n,d}(\vec{x}) = -in\omega_0 \frac{e^{-ik_n r}}{cr} \vec{p}_n = -ink_0 \frac{e^{-ik_n r}}{r} \vec{p}_n$$

where $k_n = n\omega_0/c$ and

$$\vec{p}_n = \int \vec{x}' \rho_n(\vec{x}') d^3 \vec{x}'$$

with ρ_n being the n th coefficient in the Fourier series for ρ . Then we find the power radiated is:

$$\begin{aligned}\frac{dP}{d} &= r^2 \frac{c}{4\pi} \left| \vec{E} \times \vec{B} \right| \\ &= r^2 \frac{c}{4\pi} \left| \left(\sum_{n=-\infty}^{+\infty} -k_n^2 \frac{e^{-ik_n r}}{r} \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \vec{p}_n) e^{in\omega_0 t} \right) \times \left(\sum_{m=-\infty}^{+\infty} k_m^2 \frac{e^{-ik_m r}}{r} \hat{\mathbf{r}} \times \vec{p}_m e^{im\omega_0 t} \right) \right| \\ &= \frac{c}{4\pi} \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} k_n^2 k_m^2 e^{-ik_n r} e^{-ik_m r} |-(\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \vec{p}_n)) \times (\hat{\mathbf{r}} \times \vec{p}_m)| \exp(i\omega_0 t(n+m))\end{aligned}$$

Now we time average by integrating over one period T and dividing by T . Then only those terms with $m = -n$ survive the integration. We have:

$$\left\langle \frac{dP}{d} \right\rangle = \frac{c}{4\pi} \sum_{n=-\infty}^{+\infty} k_n^4 |\hat{\mathbf{r}} \times \vec{p}_n|^2 \quad (6)$$

This equation is very similar to equation (5), but the meaning is somewhat different.

Equation (5) is the energy radiated per unit frequency – the power spectrum. On the other hand equation (6) gives the time averaged power in the n th harmonic for a periodic source. Note however, that the real physical frequency ω_n is described by both the negative and the positive frequency $\pm\omega_n$, and for a real dipole $\vec{p}_n^* = \vec{p}_{-n}$, so

$$\left\langle \frac{dP_n}{d} \right\rangle = \frac{c}{2\pi} k_n^4 |\hat{\mathbf{r}} \times \vec{p}_n|^2 \quad (7)$$

In this expression \vec{p}_n is the coefficient of the exponential Fourier series. Careful use of these expressions obviates any of the factor-of-2 problems which otherwise arise.

Example: Suppose we have a pure frequency dipole: $\vec{p} = \vec{p}_0 \cos(\omega_0 t) =$

$\frac{\vec{p}_0}{2} (e^{i\omega_0 t} + e^{-i\omega_0 t})$. Thus $\vec{p}_1 = \vec{p}_{-1} = \vec{p}_0/2$. Then

$$\vec{A} = -ink_0 \frac{1}{2r} \vec{p}_0 (e^{ik_0 r} e^{i\omega_0 t} - e^{-ik_0 r} e^{-i\omega_0 t})$$

and

$$\begin{aligned} \frac{dP}{d} &= r^2 \frac{c}{4\pi} \left| \vec{E} \times \vec{B} \right| \\ &= r^2 \frac{c}{4\pi} \left| \left(-k_0^2 \frac{1}{2r} \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \vec{p}_0) (e^{ik_0 r} e^{i\omega_0 t} - e^{-ik_0 r} e^{-i\omega_0 t}) \right) \times \left(k_0^2 \frac{1}{2r} \hat{\mathbf{r}} \times \vec{p}_0 (e^{ik_0 r} e^{i\omega_0 t} - e^{-ik_0 r} e^{-i\omega_0 t}) \right) \right| \\ &= \frac{k_0^4}{4} \frac{c}{4\pi} |\hat{\mathbf{r}} \times \vec{p}_0|^2 \left| 2 - e^{2(ik_0 r)} e^{i2\omega_0 t} - e^{-2(ik_0 r)} e^{-i2\omega_0 t} \right| \end{aligned}$$

Now we time average to get:

$$\left\langle \frac{dP}{d} \right\rangle = 2 \frac{c}{16\pi} k_0^4 |\hat{\mathbf{r}} \times \vec{p}_0|^2 = \frac{c}{8\pi} k_0^4 |\hat{\mathbf{r}} \times \vec{p}_0|^2$$

which agrees with Jackson's equation 9.22.

Alternatively, we can use equation (7) directly. We get:

$$\left\langle \frac{dP_n}{d} \right\rangle = \frac{c}{2\pi} k_n^4 |\hat{\mathbf{r}} \times \vec{p}_n|^2 = \frac{c}{2\pi} \left(\frac{\omega_0}{c} \right)^4 \left| \hat{\mathbf{r}} \times \frac{\vec{p}_0}{2} \right|^2 = \frac{c}{8\pi} \left(\frac{\omega_0}{c} \right)^4 |\hat{\mathbf{r}} \times \vec{p}_0|^2$$

as expected.

5 Quadrupole fields

The second term in the expansion of \vec{A} is:

$$\vec{A}_2 = \frac{e^{ikr}}{cr} \int \vec{J}(\vec{x}', \omega) (-ik\hat{\mathbf{r}} \cdot \vec{x}') d^3\vec{x}'$$

To evaluate this term, first we work on the integrand. Note that

$$(\vec{x}' \times \vec{J}) \times \hat{\mathbf{r}} = \vec{J}(\hat{\mathbf{r}} \cdot \vec{x}') - \vec{x}'(\hat{\mathbf{r}} \cdot \vec{J})$$

Thus we can write

$$\frac{1}{c} \vec{J}(\hat{\mathbf{r}} \cdot \vec{x}') = \frac{1}{2c} \left\{ \vec{J}(\hat{\mathbf{r}} \cdot \vec{x}') + \vec{x}'(\hat{\mathbf{r}} \cdot \vec{J}) + (\vec{x}' \times \vec{J}) \times \hat{\mathbf{r}} \right\}$$

The antisymmetric part of this expression is

$$\frac{1}{2c} (\vec{x}' \times \vec{J}) \times \hat{\mathbf{r}} = \vec{M} \times \hat{\mathbf{r}}$$

where \vec{M} is the magnetization (cf Jackson equation 5.53). Thus:

$$\begin{aligned} \vec{A}_2 &= -ik \frac{e^{ikr}}{r} \left(\int \frac{1}{2c} \left\{ \vec{J}(\hat{\mathbf{r}} \cdot \vec{x}') + \vec{x}'(\hat{\mathbf{r}} \cdot \vec{J}) \right\} d^3\vec{x}' + \vec{m} \times \hat{\mathbf{r}} \right) \\ &= \vec{A}_q + \vec{A}_{md} \end{aligned}$$

where

$$\vec{A}_{md} = -ik \frac{e^{ikr}}{r} \vec{m} \times \hat{\mathbf{r}}$$

is the magnetic dipole term and

$$\vec{A}_q = -ik \frac{e^{ikr}}{r} \int \frac{1}{2c} \left\{ \vec{J}(\hat{\mathbf{r}} \cdot \vec{x}') + \vec{x}' (\hat{\mathbf{r}} \cdot \vec{J}) \right\} d^3 \vec{x}'$$

is the quadrupole term.

Analysis of the magnetic dipole term proceeds exactly as for the electric dipole term, with \vec{p} replaced by $\vec{m} \times \hat{\mathbf{r}}$. The extra cross product with $\hat{\mathbf{r}}$ changes the directions of \vec{E} and \vec{B} , i.e. the polarization of these fields differs from that of the electric dipole terms.

$$\vec{B}_{md} = k^2 \frac{e^{ikr}}{r} \hat{\mathbf{r}} \times (\vec{m} \times \hat{\mathbf{r}}) = k^2 \frac{e^{ikr}}{r} (\vec{m} - \hat{\mathbf{r}} (\vec{m} \cdot \hat{\mathbf{r}}))$$

while

$$\vec{E}_{md} = -k^2 \frac{e^{ikr}}{r} (\hat{\mathbf{r}} \times \vec{m})$$

and the power is given by equation (4) or (6) with \vec{p} replaced by \vec{m} .

Now we proceed with the quadrupole term. We are supposed to do an integration by parts, so let's run through the usual steps. First note that:

$$\begin{aligned} \partial_i (x_k J_i(\hat{\mathbf{r}} \cdot \vec{x})) &= \delta_{ik} J_i(\hat{\mathbf{r}} \cdot \vec{x}) + x_k (\hat{\mathbf{r}} \cdot \vec{x}) \vec{\nabla} \cdot \vec{J} + x_k J_i(\hat{\mathbf{r}} \cdot \vec{\nabla}) \vec{x} \\ &= J_k(\hat{\mathbf{r}} \cdot \vec{x}) + x_k (\hat{\mathbf{r}} \cdot \vec{x}) \vec{\nabla} \cdot \vec{J} + x_k \vec{J} \cdot \hat{\mathbf{r}} \end{aligned}$$

So

$$\begin{aligned} \int x_k J_i(\hat{\mathbf{r}} \cdot \vec{x}) dV &= \int_S \partial_i (x_k J_i(\hat{\mathbf{r}} \cdot \vec{x})) dA = 0 \\ &= \int \left(J_k(\hat{\mathbf{r}} \cdot \vec{x}) + x_k (\hat{\mathbf{r}} \cdot \vec{x}) \vec{\nabla} \cdot \vec{J} + x_k \vec{J} \cdot \hat{\mathbf{r}} \right) dV \end{aligned}$$

Then:

$$\begin{aligned} \vec{A}_q &= -ik \frac{e^{ikr}}{r} \int \frac{1}{2c} \left\{ \vec{J}(\hat{\mathbf{r}} \cdot \vec{x}') + \vec{x}' (\hat{\mathbf{r}} \cdot \vec{J}) \right\} d^3 \vec{x}' \\ &= ik \frac{e^{ikr}}{2cr} \int \vec{x} (\hat{\mathbf{r}} \cdot \vec{x}) (\vec{\nabla} \cdot \vec{J}) dV \\ &= ik \frac{e^{ikr}}{2cr} \int \vec{x} (\hat{\mathbf{r}} \cdot \vec{x}) i\omega \rho dV \\ &= -k^2 \frac{e^{ikr}}{2r} \int \vec{x} (\hat{\mathbf{r}} \cdot \vec{x}) \rho dV \end{aligned}$$

Then

$$\vec{B} = \nabla \times \vec{A} = -\frac{ik^3}{2r} e^{ikr} \int (\hat{\mathbf{r}} \times \vec{x}) (\hat{\mathbf{r}} \cdot \vec{x}) \rho dV$$

Now define the vector $\vec{Q}(\hat{\mathbf{r}})$ by

$$Q_i = \sum_j Q_{ij} \hat{r}_j \tag{8}$$

where Q_{ij} is the quadrupole tensor

$$Q_{ij} = \int (3x_i x_j - |\vec{x}|^2 \delta_{ij}) \rho(\vec{x}) d^3\vec{x}$$

Then

$$Q_i = \sum_j \int (3x_i x_j \hat{r}_j - |\vec{x}|^2 \hat{r}_i) \rho(\vec{x}) d^3\vec{x}$$

$$\vec{Q} = \int 3\vec{x} (\hat{\mathbf{r}} \cdot \vec{x}) \rho dV - \hat{\mathbf{r}} \int |\vec{x}|^2 \rho dV$$

and thus

$$\vec{B} = -\frac{ik^3}{2r} e^{ikr} \frac{\hat{\mathbf{r}} \times \vec{Q}}{3}$$

and then

$$\begin{aligned} \vec{E} &= \frac{i}{k} \left(\frac{k^4}{2r} e^{ikr} \frac{\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \vec{Q})}{3} \right) \\ &= \frac{ik^3}{6r} e^{ikr} \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \vec{Q}) \end{aligned}$$

and thus

$$\begin{aligned} \frac{d^2W}{d \, d\omega} &= \frac{c}{4\pi} \frac{k^6}{36} |\hat{\mathbf{r}} \times \vec{Q}|^2 \\ &= \frac{c}{144\pi} k^6 |\hat{\mathbf{r}} \times \vec{Q}|^2 \end{aligned}$$

or, for a periodic source, the time averaged power is

$$\left\langle \frac{dP_n}{d} \right\rangle = \frac{c}{72\pi} k^6 |\hat{\mathbf{r}} \times \vec{Q}_n|^2$$

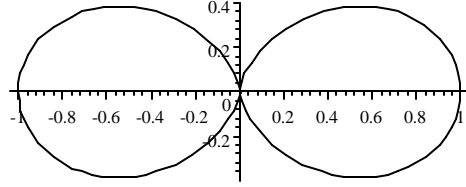
where again Q_n is the coefficient in the exponential Fourier series.

6 Angular distribution

For the dipole:

$$\frac{dP}{d} \propto |\hat{\mathbf{r}} \times \vec{p}|^2 = p^2 \sin^2 \theta$$

where θ is the angle between \vec{p} and the direction of propagation $\hat{\mathbf{r}}$.



Dipole radiation pattern

The total power radiated is:

$$\begin{aligned}
 P_n &= \int \frac{dP_n}{d} d = \frac{c}{2\pi} k_n^4 p_n^2 (2\pi) \int_{-1}^{+1} (1 - \mu^2) d\mu \\
 &= \frac{4}{3} c k_n^4 p_n^2
 \end{aligned}$$

For the quadrupole,

$$\frac{dP}{d} \propto \left| \hat{\mathbf{r}} \times \vec{Q}_n(\hat{\mathbf{r}}) \right|^2$$

which can be quite complicated.

Suppose a charge q oscillates along the z -axis from $z = 0$ to $z = a$, with period T . The dipole moment is:

$$p_z = zq = q \frac{a}{2} (1 + \cos \omega t) = q \frac{a}{4} (2 + e^{i\omega t} + e^{-i\omega t})$$

and the quadrupole moments are

$$\begin{aligned}
 Q_{11} &= Q_{22} = -z^2 q = -q \frac{a^2}{4} (1 + \cos \omega t)^2 \\
 &= -q \frac{a^2}{4} (1 + 2 \cos \omega t + \cos^2 \omega t) \\
 &= -q \frac{a^2}{4} \left(1 + 2 \cos \omega t + \frac{\cos 2\omega t + 1}{2} \right) \\
 &= -q \frac{a^2}{4} \left(\frac{3}{2} + e^{i\omega t} + e^{-i\omega t} + \frac{e^{i2\omega t} + e^{-i2\omega t}}{4} \right)
 \end{aligned}$$

$$Q_{33} = 2z^2 q = q \frac{a^2}{2} \left(\frac{3}{2} + e^{i\omega t} + e^{-i\omega t} + \frac{e^{i2\omega t} + e^{-i2\omega t}}{4} \right)$$

The dipole power radiated is

$$\frac{dP_n}{d} = \frac{c}{2\pi} k_n^4 p_n^2$$

and is all at the fundamental $\omega = \frac{2\pi}{T}$. We have $p_1 = qa/4$, so

$$\frac{dP_1}{d} = \frac{c}{2\pi} \left(\frac{\omega}{c}\right)^4 \left(\frac{qa}{4}\right)^2 \sin^2 \theta = \frac{q^2 a^2 \omega^4}{32\pi c^3} \sin^2 \theta$$

For the quadrupole, we first evaluate the vector $\vec{Q}(\hat{\mathbf{r}})$

$$Q_i = \sum_j Q_{ij} r_j$$

Thus

$$Q_1 = Q_{11} \sin \theta \cos \phi = -q \frac{a^2}{4} \left(\frac{3}{2} + e^{i\omega t} + e^{-i\omega t} + \frac{e^{i2\omega t} + e^{-i2\omega t}}{4} \right) \sin \theta \cos \phi$$

$$Q_2 = Q_{22} \sin \theta \sin \phi = -q \frac{a^2}{4} \left(\frac{3}{2} + e^{i\omega t} + e^{-i\omega t} + \frac{e^{i2\omega t} + e^{-i2\omega t}}{4} \right) \sin \theta \sin \phi$$

and

$$Q_3 = Q_{33} \cos \theta = q \frac{a^2}{2} \left(\frac{3}{2} + e^{i\omega t} + e^{-i\omega t} + \frac{e^{i2\omega t} + e^{-i2\omega t}}{4} \right) \cos \theta$$

Then

$$\begin{aligned} \hat{\mathbf{r}} \times \vec{Q} &= (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \times (\sin \theta \cos \phi, \sin \theta \sin \phi, -2 \cos \theta) Q_{11} \\ &= (3 \sin \theta \cos \theta \sin \phi, -3 \cos \theta \sin \theta \cos \phi, 0) q \frac{a^2}{4} \left(\frac{3}{2} + e^{i\omega t} + e^{-i\omega t} + \frac{e^{i2\omega t} + e^{-i2\omega t}}{4} \right) \\ &= \left(\frac{3}{2} \sin 2\theta \sin \phi, -\frac{3}{2} \sin 2\theta \cos \phi, 0 \right) q \frac{a^2}{4} \left(\frac{3}{2} + e^{i\omega t} + e^{-i\omega t} + \frac{e^{i2\omega t} + e^{-i2\omega t}}{4} \right) \end{aligned}$$

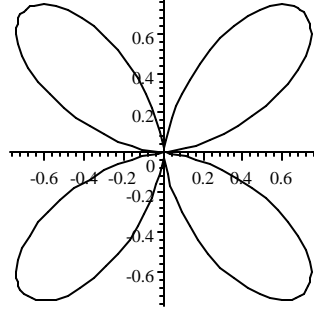
This vector has components at both ω and 2ω .

$$\begin{aligned} \frac{dP_1}{d} &= \frac{c}{72\pi} k^6 \left| \hat{\mathbf{r}} \times \vec{Q}_1(\hat{\mathbf{r}}) \right|^2 \\ &= \frac{c}{72\pi} k^6 \left(q \frac{a^2}{4} \right)^2 \left(\frac{9}{4} \sin^2 2\theta \right) \\ &= \frac{\omega^6 q^2 a^4}{512 c^5 \pi} \sin^2 2\theta \end{aligned}$$

while at the second harmonic:

$$\begin{aligned} \frac{dP_2}{d} &= \frac{c}{72\pi} k^6 \left(q \frac{a^2}{16} \right)^2 \left(\frac{9}{4} \sin^2 2\theta \right) \\ &= \frac{c}{72\pi} \left(\frac{2\omega}{c} \right)^6 \left(q \frac{a^2}{16} \right)^2 \left(\frac{9}{4} \sin^2 2\theta \right) \\ &= \frac{\omega^6 q^2 a^4}{128 c^5 \pi} \sin^2 2\theta \end{aligned} \tag{9}$$

The distribution given in equation (9) is shown below:



The total power radiated at ω is the sum of the dipole and quadrupole terms:

$$\begin{aligned} \frac{dP_1}{d} &= \frac{q^2 a^2 \omega^4}{32\pi c^3} \sin^2 \theta + \frac{\omega^6 q^2 a^4}{2048\pi c^5} \sin^2 2\theta \\ &= \frac{q^2 a^2 \omega^4}{2048c^3\pi} \left(64 \sin^2 \theta + \frac{\omega^2 a^2}{c^2} \sin^2 2\theta \right) \end{aligned}$$

The total power radiated in the quadrupole is:

$$P = \int \frac{dP}{d} d = \frac{c}{72\pi} k^6 \int \left| \hat{\mathbf{r}} \times \vec{Q}_n(\hat{\mathbf{r}}) \right|^2 d$$

where

$$\left(\hat{\mathbf{r}} \times \vec{Q}_n(\hat{\mathbf{r}}) \right)_i = \varepsilon_{ijk} r_j Q_{kl} r_l$$

and thus

$$\begin{aligned} \left| \hat{\mathbf{r}} \times \vec{Q}_n(\hat{\mathbf{r}}) \right|^2 &= \varepsilon_{ijk} r_j Q_{kl} r_l \varepsilon_{imn} r_m Q_{np}^* r_p \\ &= (\delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}) r_j Q_{kl} r_l r_m Q_{np}^* r_p \\ &= r_m r_m Q_{kl} Q_{kp}^* r_l r_p - r_j r_p Q_{ml} Q_{jp}^* r_l r_m \\ &= r_l r_p (Q_{kl} Q_{kp}^* - r_j r_m Q_{ml} Q_{jp}^*) \\ &= \vec{Q} \cdot \vec{Q}^* - \left| \hat{\mathbf{r}} \cdot \vec{Q} \right|^2 \end{aligned}$$

The angle integrals give:

$$\int r_l r_p \delta = \begin{cases} 0 & \text{if } l \text{ or } p = 1, 2 \text{ and } l \neq p \\ \frac{4}{3}\pi & \text{if } l \text{ or } p = 1, 2 \text{ and } l = p \\ 2\pi \frac{2}{3} & \text{if } l = p = 3 \end{cases} = \frac{4\pi}{3} \delta_{lp}$$

while for

$$\int r_l r_p r_j r_m d$$

we note that if any of the indices $l, p, j, m = 1$ or 2 , then one more of them must also equal

that same index to survive the integration over ϕ . We need the results

$$\begin{aligned}\int_0^{2\pi} \cos \phi d\phi &= \int_0^{2\pi} \sin \phi d\phi = 0 \\ \int_0^{2\pi} \cos^2 \phi d\phi &= \int_0^{2\pi} \sin^2 \phi d\phi = \pi \\ \int_0^{2\pi} \cos^3 \phi d\phi &= \int_0^{2\pi} \sin^3 \phi d\phi = 0 \\ \int_0^{2\pi} \cos^4 \phi d\phi &= \int_0^{2\pi} \sin^4 \phi d\phi = \frac{3}{4}\pi \\ \int_0^{2\pi} \sin^2 \phi \cos^2 \phi d\phi &= \frac{\pi}{4}\end{aligned}$$

Each of the components of \hat{r} also contains either $\sin \theta$ or $\cos \theta$, and since we need an even number of $\sin \phi$ and $\cos \phi$ terms, we never have an odd power of $\sin \theta$, so we need:

$$\begin{aligned}\int_{-1}^{+1} \mu^4 d\mu &= \frac{2}{5} \\ \int_{-1}^{+1} \mu^2 (1 - \mu^2) d\mu &= \left. \frac{\mu^3}{3} - \frac{\mu^5}{5} \right|_{-1}^{+1} = \frac{4}{15} \\ \int_{-1}^{+1} (1 - \mu^2)^2 d\mu &= \frac{16}{15}\end{aligned}$$

These couple as follows:

$$\int r_l r_p r_j r_m d = \begin{cases} \frac{\pi}{15} \frac{4}{3} & \text{if } 2 \text{ of indices } = 1, 2 \text{ and } 2 \text{ indices } = 3 \\ 2\pi \frac{2}{5} & \text{if all indices } = 3 \\ \frac{\pi}{4} \frac{16}{15} & \text{if } 2 \text{ indices } = 1 \text{ and } 2 \text{ indices } = 2 \\ \frac{3\pi}{4} \frac{16}{15} & \text{if all indices } = 1 \text{ or } 2. \end{cases}$$

Thus we may write:

$$\int r_l r_p r_j r_m d = \frac{4\pi}{15} (\delta_{lp} \delta_{jm} + \delta_{lj} \delta_{pm} + \delta_{lm} \delta_{jp})$$

So we have

$$\begin{aligned}P &= \frac{c}{72\pi} k^6 \int r_l r_p (Q_{kl} Q_{kp}^* - r_j r_m Q_{ml} Q_{jp}^*) d \\ &= \frac{c}{72\pi} k^6 \left[Q_{kl} Q_{kp}^* \frac{4\pi}{3} \delta_{lp} - Q_{ml} Q_{jp}^* \frac{4\pi}{15} (\delta_{lp} \delta_{jm} + \delta_{lj} \delta_{pm} + \delta_{lm} \delta_{jp}) \right] \\ &= \frac{c}{72\pi} k^6 \frac{4\pi}{3} \left[Q_{kp} Q_{kp}^* - \frac{Q_{mp} Q_{mp}^*}{5} - \frac{Q_{mj} Q_{jm}^*}{5} - \frac{Q_{mm} Q_{pp}^*}{5} \right]\end{aligned}$$

But Q_{ij} is traceless, so the last term is zero.

$$\begin{aligned} \langle P_n \rangle &= \frac{\omega_n^6}{54c^5} \frac{3}{5} Q_{kp} Q_{kp}^* \\ &= \frac{\omega_n^6}{90c^5} \sum_{k,p} |Q_{kp}|^2 \end{aligned} \quad (10)$$

Let's recalculate the power radiated by our oscillating charge. The quadrupole term at 2ω gives

$$\begin{aligned} \langle P_2 \rangle &= \frac{(2\omega)^6}{90c^5} \left(q \frac{a^2}{16} \right)^2 [1 + 1 + 4] \\ &= \frac{1}{60} \frac{\omega^6}{c^5} q^2 a^4 \end{aligned}$$

Now compare with the integral of our previous term (9)

$$\begin{aligned} \langle P_2 \rangle &= \int \frac{\omega^6 q^2 a^4}{128c^5 \pi} \sin^2 2\theta d \\ &= \frac{\omega^6 q^2 a^4}{128c^5 \pi} (2\pi) \int_{-1}^{+1} 4(1 - \mu^2) \mu^2 d\mu \\ &= \frac{\omega^6 q^2 a^4}{128c^5 \pi} (2\pi) (4) \frac{4}{15} \\ &= \frac{1}{60} \frac{\omega^6}{c^5} q^2 a^4 \end{aligned}$$

while at the fundamental

$$\langle P_1 \rangle = \int \frac{\omega^6 q^2 a^4}{512c^5 \pi} \sin^2 2\theta d = \frac{\omega^6 q^2 a^4}{512c^5 \pi} \frac{32\pi}{15} = \frac{1}{240} \frac{\omega^6}{c^5} q^2 a^4$$

and the total power formula gives:

$$\langle P_1 \rangle = \frac{\omega^6}{90c^5} \left(q \frac{a^2}{4} \right)^2 (1 + 1 + 4) = \frac{1}{240} \frac{\omega^6}{c^5} q^2 a^4$$

These expressions agree.