1 Mixed Boundary Conditions

The allowed set of boundary conditions for Laplace's equation (or the Helmholtz equation) include Dirichlet or Neumann conditions, or a mixture in which we have Dirichlet on part of the boundary and Neumann on part. These latter kinds of problems with "mixed" boundary conditions are more tricky. Let's look at an example.

An infinite, thin, conducting plane has a circular hole of radius a in it. We may choose coordinates so that the conducting plane is the $x-y$ plane, with origin at the center of the hole. There is a uniform field $\vec{E}=\vec{E}_0=E_0\hat{z}$ at $z\to-\infty$. We want to solve for the fields everywhere.

The boundary conditions on the plane at $z=0$ are as follows.

Since the plane is a conductor,

$$\Phi = \text{constant for } \ \rho > a$$

and since there is no charge density in the hole

$$E_z \ \text{ is continuous for } \ 0 \le \rho < a$$

So we have Dirichlet conditions on the plane and Neumann conditions in the hole.

The problem has azimuthal symmetry. Far from the hole, $(\rho\to\infty)$ the conducting plane shields the region $z>0$ from the field at $z<0$. Thus we may write the potential as

$$\begin{aligned}\Phi\left(\rho,z<0\right) &= -E_0 z+\Phi^{(1)}\\ \Phi\left(\rho,z>0\right) &= \Phi^{(1)}\end{aligned}$$

where $\Phi^{(1)}$ is the potential produced by the charge distribution on the conducting plane, and $\Phi^{(1)}(\rho,z)\to 0$ as $\rho\to\infty$ and $z\to\pm\infty$. We may express $\Phi^{(1)}$ in terms of the (unknown) surface charge density on the plane.

$$\Phi^{(1)}(\rho,z)=\frac{1}{4\pi\varepsilon_0}\int_0^\infty \frac{\sigma(\rho')}{\sqrt{\rho^2+(\rho')^2-2\rho\rho'\cos\phi'+z^2}}2\pi\rho' d\rho'$$

From this expression, we may conclude that the potential $\Phi^{(1)}(\rho,z)$ is an even function of $z:\Phi^{(1)}(\rho,z)=\Phi^{(1)}(\rho,-z)$. Then the tangential component of $\vec{E}^{(1)}$, $E_\rho^{(1)}=-\partial\Phi^{(1)}/\partial\rho$ is also even in z, but the $z-$component $E_z^{(1)}=-\partial\Phi^{(1)}/\partial z$ is an odd function of z. The boundary condition in the hole is

$$E_0+E_z^{(1)}(z=0-)=E_z^{(1)}(z=0+)$$

Because $E_z^{(1)}$ is odd, this becomes

$$\begin{aligned} E_0 + E_z^{(1)}(z = 0-) &= -E_z^{(1)}(z = 0-) \\ E_z^{(1)}(z = 0-) &= -\frac{E_0}{2} \end{aligned} \tag{1}$$

or

$$\left.\frac{\partial \Phi^{(1)}}{\partial z}\right|_{z=0-} = \frac{E_0}{2} \qquad \rho < a$$

We may choose the reference point for potential on the plane, so that outside the hole

$$\Phi^{(1)}(\rho, 0) = 0 \qquad \rho > a$$

We may express the potential in the general form of the solution to Laplace's equation in cylindrical coordinates. Since $\Phi^{(1)} \to 0$ as $z \to \pm\infty$, and the potential is finite at $\rho = 0$ and as $\rho \to \infty$, the appropriate form is

$$\Phi^{(1)}(\rho, z) = \int_0^\infty A(k) e^{-k|z|} J_0(k\rho)\, dk \tag{2}$$

So the problem is reduced to finding the coefficients $A(k)$ in eqn (2).

Following the usual plan, we start with the known boundary conditions at $z = 0$.

$$\begin{aligned} \Phi^{(1)}(\rho, 0) &= \int_0^\infty A(k) J_0(k\rho)\, dk = 0 \qquad \rho > a \\ \left.\frac{\partial \Phi^{(1)}}{\partial z}\right|_{z=0-} &= \int_0^\infty kA(k) J_0(k\rho)\, dk = \frac{E_0}{2} \qquad \rho < a \end{aligned}$$

This is a special case of the integral equation problem:

$$\begin{aligned} \int_0^\infty f(y) J_0(yx)\, dy &= 0 \qquad x > 1 \\ \int_0^\infty y f(y) J_0(yx)\, dy &= x^n \qquad x < 1 \end{aligned}$$

with $n = 0$, $x = \rho/a$ and $y = ka$. According to Jackson pg 132, the solution is

$$f(y) = \frac{\Gamma(n+1)}{\sqrt{\pi}\,\Gamma(n+3/2)} j_{n+1}(y) = \frac{\Gamma(n+1)}{\sqrt{2y}\,\Gamma(n+3/2)} J_{n+3/2}(y)$$

where $j_n(y)$ is the spherical Bessel function of order n (Lea §8.5). Here we have $n = 0$, $y = ka$ and $x = \rho/a$, so

$$f(ka) = \frac{1}{\sqrt{\pi}\,\Gamma(3/2)} j_1(ka) = \frac{1}{\sqrt{\pi}\frac{1}{2}\Gamma(1/2)} \left[\frac{\sin ka}{(ka)^2} - \frac{\cos ka}{ka}\right]$$

and thus

$$A(k) = \frac{E_0 a^2}{2} \frac{2}{\pi} \left[\frac{\sin ka}{(ka)^2} - \frac{\cos ka}{ka}\right]$$

So the potential is

$$\Phi^{(1)}(\rho, z) = \frac{E_0 a^2}{\pi} \int_0^\infty \left[\frac{\sin ka}{(ka)^2} - \frac{\cos ka}{ka} \right] e^{-k|z|} J_0(k\rho)\, dk \tag{3}$$

which is dimensionally correct.

We can gain more insight by expanding the function $A(k)$ in a series:

$$\begin{aligned} \frac{\sin ka}{(ka)^2} - \frac{\cos ka}{ka} &= \frac{ka - (ka)^3/6}{(ka)^2} - \frac{1 - (ka)^2/2}{ka} + \cdots \\ &= \frac{1}{3} ka + \cdots \end{aligned}$$

At large z or ρ, both functions e^{-kz} and $J_0(k\rho) \to 0$ rapidly unless k is very small, $(k < 1/z,\ k < 1/\rho)$, so the solution is dominated by the small k limit of $A(k)$. Thus at a large distance from the hole, the leading term is

$$\begin{aligned} \Phi^{(1)}(\rho, z) &= \frac{E_0 a^3}{3\pi} \int_0^\infty k e^{-k|z|} J_0(k\rho)\, dk \\ &= -\frac{E_0 a^3}{3\pi} \frac{d}{dz} \int_0^\infty e^{-kz} J_0(k\rho)\, dk \qquad z > 0 \\ &= -\frac{E_0 a^3}{3\pi} \frac{d}{dz} \frac{1}{\sqrt{\rho^2 + z^2}} \qquad \text{JP3.16, Lea p8.34} \\ &= -\frac{E_0 a^3}{3\pi} \hat{z} \cdot \vec{\nabla} \frac{1}{r} = \frac{E_0 a^3}{3\pi} \frac{z}{r^3} \end{aligned}$$

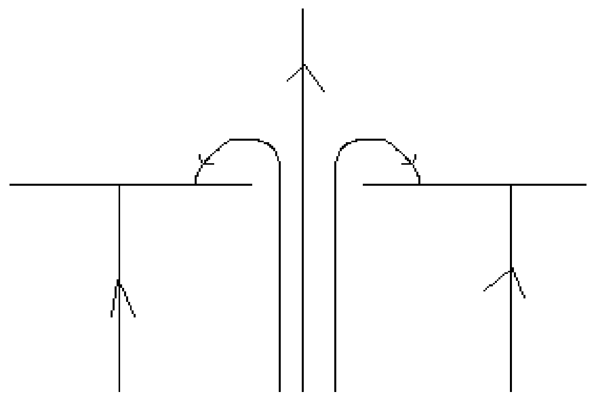
Compare with the dipole potential (Notes 1 eqn 25).

$$\Phi_{\text{dipole}} = -\frac{1}{4\pi\varepsilon_0} \vec{p} \cdot \vec{\nabla} \frac{1}{r} = \frac{1}{4\pi\varepsilon_0} \frac{\vec{p} \cdot \hat{r}}{r^2}$$

so the potential for $z > 0$ is dominantly a dipole potential with dipole moment

$$\vec{p} = \frac{4}{3} \varepsilon_0 \vec{E}_0 a^3$$

Look at a field line diagram to see why it makes sense that $\vec{p}$ is parallel to $\vec{E}_0$. Note also that.the physical dimensions are correct. For $z < 0$ the potential is that due to a dipole $-\vec{p}$.



Solution close in. First note that

$$j_1(x) = -\frac{d}{dx}\frac{\sin x}{x}$$

so we can integrate by parts:

$$\begin{aligned}
\Phi^{(1)}(\rho, z>0) &= \frac{E_0 a^2}{\pi}\int_0^\infty \left[-\frac{d}{adk}\frac{\sin ka}{ka}\right] e^{-kz} J_0(k\rho)\, dk \\
&= -\frac{E_0}{\pi}\left[e^{-k|z|} J_0(k\rho)\frac{\sin ka}{k}\Big|_0^\infty - \int_0^\infty \frac{\sin ka}{k}\left(-ze^{-kz}J_0(k\rho) + \rho e^{-kz} J_0'(k\rho)\right) dk\right] \\
&= -\frac{E_0}{\pi}\left[-a - \int_0^\infty \frac{\sin ka}{k}\left(-ze^{-kz}J_0(k\rho) - \rho e^{-kz} J_1(k\rho)\right) dk\right] \\
&= \frac{E_0}{\pi}\left[a - \operatorname{Im}\int_0^\infty \left(ze^{-k(z-ia)}\frac{J_0(k\rho)}{k} + \rho e^{-k(z-ia)}\frac{J_1(k\rho)}{k}\right) dk\right] \\
&= \frac{E_0}{\pi}\left[a - \operatorname{Im}\left(z\times * + \rho\frac{\sqrt{\rho^2 + (z-ia)^2} - (z-ia)}{\rho}\right)\right] \qquad (\text{GR6.623\#3 with } \nu = 1)
\end{aligned}$$

We do the integral using GR6.623#3 with $\alpha = z - ia$ and $\beta = \rho$. The second term has $\nu = 1$,and is straightforward. Note that we need $\operatorname{Re}(\alpha) > \operatorname{Im}(\beta)$ which is satisfied here. To get the first term we take the limit $\nu \to 0$ using l'Hospital's rule:

$$\begin{aligned}
\lim_{\nu\to 0}\frac{\left(\sqrt{\alpha^2+\beta^2}-\alpha\right)^\nu}{\nu\beta^\nu} &= \lim_{\nu\to 0}\frac{\exp\left[v\ln\left(\sqrt{\alpha^2+\beta^2}-\alpha\right)\right]}{\nu\beta^\nu} \\
&= \lim_{\nu\to 0}\frac{\ln\left(\sqrt{\alpha^2+\beta^2}-\alpha\right)\exp\left[v\ln\left(\sqrt{\alpha^2+\beta^2}-\alpha\right)\right]}{\beta^\nu + \nu\ln\beta\exp(\nu\ln\beta)} \\
&= \ln\left(\sqrt{\alpha^2+\beta^2}-\alpha\right) \\
&= \ln\left(\sqrt{\rho^2+(z-ia)^2}-(z-ia)\right)
\end{aligned}$$

Thus

$$\Phi^{(1)}(\rho,z)=\frac{E_0}{\pi}\left[a-\mathrm{Im}\left\{z\ln\left(\sqrt{\rho^2+(z-ia)^2}-(z-ia)\right)+\sqrt{\rho^2+(z-ia)^2}-z+ia\right\}\right]$$

Now we need the imaginary part, so we write the complex number in polar form to take the square root and the log most easily.

$$\begin{aligned}\sqrt{\rho^2+(z-ia)^2} &= \sqrt{\rho^2+z^2-a^2-2iza}=\sqrt{\sqrt{\lambda^2a^4+4z^2a^2}e^{i\theta}}\\ &= a\sqrt{R}e^{i\theta/2}\end{aligned}$$

where λ and R are dimensionless variables:

$$\lambda a^2\equiv\rho^2+z^2-a^2 \text{ and } R^2\equiv\lambda^2+4z^2/a^2$$

and

$$\tan\theta=\frac{-2za}{\rho^2+z^2-a^2}=-\frac{2z/a}{\lambda}$$

So

$$\begin{aligned}\sec^2\theta &= 1+\frac{4z^2/a^2}{\lambda^2}\\ \cos^2\theta &= \frac{\lambda^2}{\lambda^2+4z^2/a^2}=\frac{\lambda^2}{R^2}\end{aligned}\tag{4}$$

Then

$$\mathrm{Im}\left(\sqrt{\rho^2+(z-ia)^2}-(z-ia)\right)=a+a\sqrt{R}\sin\frac{\theta}{2}$$

Further

$$\cos\theta=1-2\sin^2\frac{\theta}{2}=2\cos^2\frac{\theta}{2}-1$$

so from (4),

$$\sin\frac{\theta}{2}=\sqrt{\frac{1-\cos\theta}{2}}=\sqrt{\frac{1-\lambda/R}{2}};\quad \cos\frac{\theta}{2}=\sqrt{\frac{1+\lambda/R}{2}}$$

So now we have

$$\Phi^{(1)}(\rho,z)=\frac{E_0}{\pi}a\left[\sqrt{\frac{R-\lambda}{2}}-\frac{z}{a}\mathrm{Im}\ln\left(\sqrt{\rho^2+(z-ia)^2}-(z-ia)\right)\right]$$

Next we work on the imaginary part of the log.

$$\mathrm{Im}\ln\left(\sqrt{\rho^2+(z-ia)^2}-(z-ia)\right)=\mathrm{Im}\ln\left[\left(a\sqrt{R}e^{i\theta/2}-z+ia\right)\right]$$

$$
\begin{aligned}
\operatorname{Im}\ln\left[\left(a\sqrt{R}e^{i\theta/2}-z+ia\right)\right] &= \tan^{-1}\frac{a\left(1+\sqrt{R}\sin\frac{\theta}{2}\right)}{-z+a\sqrt{R}\cos\frac{\theta}{2}} \\
&= \tan^{-1}\frac{1-\sqrt{\frac{R-\lambda}{2}}}{-z/a+\sqrt{\frac{R+\lambda}{2}}} \\
&= \tan^{-1}\frac{1-\sqrt{\frac{R-\lambda}{2}}}{-\sqrt{\frac{R^2-\lambda^2}{4}}+\sqrt{\frac{R+\lambda}{2}}} \\
&= \tan^{-1}\frac{1-\sqrt{\frac{R-\lambda}{2}}}{\left(1-\sqrt{\frac{R-\lambda}{2}}\right)\sqrt{\frac{R+\lambda}{2}}}=\tan^{-1}\sqrt{\frac{2}{R+\lambda}}
\end{aligned}
$$

So finally the potential is:

$$
\Phi^{(1)}(\rho, z>0)=\frac{E_0}{\pi}a\left[\sqrt{\frac{R-\lambda}{2}}-\frac{z}{a}\tan^{-1}\sqrt{\frac{2}{R+\lambda}}\right] \tag{5}
$$

In the plane $z=0$

$$
R^2=\lambda^2
$$

so for $\rho>a$, $Ra^2=\sqrt{\left(\rho^2-a^2\right)^2}=\rho^2-a^2=+\lambda a^2$ and

$$
\Phi^{(1)}(\rho,0)=0 \quad \text{for } \rho>a
$$

If $\rho<a$, $Ra^2=\sqrt{\left(\rho^2-a^2\right)^2}=a^2-\rho^2=-\lambda a^2$ and

$$
\Phi^{(1)}(\rho,0)=\frac{E_0}{\pi}\left[\sqrt{\frac{a^2-\rho^2-\left(\rho^2-a^2\right)}{2}}\right]=\frac{E_0}{\pi}\sqrt{a^2-\rho^2}
$$

and

$$
\begin{aligned}
\vec{E}^{(1)}(\rho<a, z)\Big|_{z=0+} &= -\vec{\nabla}\Phi^{(1)}(\rho, z>0)\Big|_{z=0} \\
&= -\hat{\rho}\frac{\partial}{\partial\rho}\frac{E_0}{\pi}\sqrt{a^2-\rho^2}+\hat{z}\frac{E_0}{\pi}\lim_{R\to-\lambda}\tan^{-1}\sqrt{\frac{2}{R+\lambda}} \\
&= \frac{E_0}{\pi}\left(\frac{\rho}{\sqrt{a^2-\rho^2}}\hat{\rho}+\frac{\pi}{2}\hat{z}\right) \\
&= \frac{\vec{E}_0}{2}+\frac{E_0}{\pi}\frac{\vec{\rho}}{\sqrt{a^2-\rho^2}}
\end{aligned}
$$

Compare this result with (1). Note that $\vec{E}$ diverges at the edge of the opening, as expected.