Magnetic moments

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1 The magnetic moment tensor

Our goal here is to develop a set of moments to describe the magnetic field as we did for the electric field in Ch 4. Because there are no magnetic monopoles, the dominant contribution to $\vec{B}$ at a great distance from a current distribution is a dipole, so we start by looking at the dipole moment.

The magnetic moment tensor is defined by:

$$M_{ij} \equiv I \oint x_i dx_j \tag{1}$$

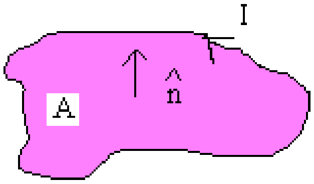
for a current loop, or equivalently

$$M_{ij} \equiv \int x_i J_j dV \tag{2}$$

if the current is not confined to wire loops. Each component of M in equation 1 represents the area of the projection of the loop onto the $i-j$ plane. Thus it should be related to the magnetic moment vector. For a planar loop, $\vec{m}$ is defined to be (see Lea and Burke Ch 29):

$$\vec{m} = \frac{I}{c} A\hat{n} = \frac{I}{2} \oint \vec{x} \times d\vec{\ell} \tag{3}$$

where the loop is traversed counter-clockwise around $\hat{n}$ according to the right hand rule. (See figure)



(Note: the cross product gives the area of the parallelogram formed by $\vec{x}$ and $d\vec{\ell}$, but we need the area of the triangle, or half the area of the parallelogram. The cross product also conveniently gives a direction normal to the area element.)

Now consider the vector *dual* to M_{ij}, defined by (see Lea Optional topic A eqn A.7):

$$m_p = \frac{1}{2}\varepsilon_{prs}M_{rs} \tag{4}$$

Then

$$m_p = \frac{1}{2}\varepsilon_{prs}I\oint x_r dx_s = \frac{1}{2}I\oint \varepsilon_{prs}x_r dx_s = \frac{1}{2}I\oint\left(\vec{x}\times d\vec{l}\right)_p$$

in agreement with equation (3), showing that the magnetic moment vector is dual to the tensor M_{ij} .

The dual relation has an inverse (Lea eqn A.8):

$$\begin{aligned}\varepsilon_{jkp}m_p &= \varepsilon_{jkp}\frac{1}{2}\varepsilon_{prs}M_{rs} = \frac{1}{2}\left(\delta_{jr}\delta_{ks}-\delta_{js}\delta_{kr}\right)M_{rs} \\ &= \frac{1}{2}\left(M_{jk}-M_{kj}\right)\end{aligned}$$

But M_{ij} is antisymmetric:

$$M_{ij} = I\oint x_i dx_j = I\left\{ x_i x_j|_P^P - \oint x_j dx_i\right\} = 0 - M_{ji}$$

Thus:

$$\varepsilon_{jkp}m_p = M_{jk} \tag{5}$$

Alternative proof of antisymmetry for general, localized, current density J: First note that

$$\begin{aligned}\partial_k\left(x_i x_j J_k\right) &= \delta_{ik}x_j J_k + \delta_{jk}x_i J_k + x_i x_j \partial_k J_k \\ &= x_j J_i + x_i J_j - x_i x_j\frac{\partial\rho}{\partial t}\end{aligned}$$

where we used the charge conservation relation in the last step. In a steady state the last term is zero. Then

$$\begin{aligned}\int_V \partial_k\left(x_i x_j J_k\right)\,dV &= \int_{S_\infty} x_i x_j J_k n_k\,dA = 0 \\ \int_V \left(x_j J_i + x_i J_j\right)\,dV &= 0\end{aligned}$$

since $\vec{J}=0$ on the surface at infinity, and so

$$\begin{aligned}\int_V x_j J_i\,dV &= -\int_V x_i J_j\,dV \\ M_{ji} &= -M_{ij}\end{aligned} \tag{6}$$

2 Magnetic field due to a current loop

Here we will find the multipole expansion of the magnetic field due to a current loop.

We start with the vector potential. As we did in the electric case, we use a Taylor expansion of $1/R$ to get

$$\vec{A} = \frac{\mu_0}{4\pi}I\int\frac{d\vec{x}'}{|\vec{x}-\vec{x}'|} = \frac{\mu_0}{4\pi}I\oint d\vec{x}'_2\left(\frac{1}{|\vec{x}|}+\frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|^3}+\frac{\vec{x}'\cdot q\cdot\vec{x}'}{2}+\cdots\right)$$

where the tensor q has components

$$\begin{aligned}
q_{ij} &= \left.\frac{\partial}{\partial x_i'}\frac{\partial}{\partial x_j'}\;\frac{1}{|\vec{x}-\vec{x}'|}\right|_{\vec{x}'=0} \\
&= \left.\frac{\partial}{\partial x_i'}\left(\frac{x_j-x_j'}{|\vec{x}-\vec{x}'|^3}\right)\right|_{\vec{x}'=0} \\
&= \left.\frac{-\delta_{ij}}{|\vec{x}-\vec{x}'|^3}+3\frac{\left(x_i-x_i'\right)\left(x_j-x_j'\right)}{|\vec{x}-\vec{x}'|^5}\right|_{\vec{\mathbf{x}}'=0} \\
&= \frac{-\delta_{ij}}{|\vec{x}|^3}+3\frac{x_ix_j}{|\vec{x}|^5} \qquad (7)
\end{aligned}$$

Then, moving the functions of unprimed coordinates out of the integrals, we get

$$\begin{aligned}
A_i &= \frac{\mu_0}{4\pi}I\left(0+\frac{x_j}{|\vec{x}|^3}\oint x_j'dx_i'+\cdots\right)=\frac{\mu_0}{4\pi}\frac{x_j}{|\vec{x}|^3}M_{ji}+\cdots \\
&= \frac{\mu_0}{4\pi}\frac{x_j}{|\vec{x}|^3}\varepsilon_{jip}m_p+\cdots \qquad (8)
\end{aligned}$$

The leading term is the dipole:

$$\vec{A}_{\text{dipole}}=\frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{x}}{|\vec{x}|^3} \qquad (9)$$

The next term in the expansion is

$$A_i=\frac{\mu_0}{4\pi}I\oint dx_i'x_j'x_k'\frac{q_{jk}}{2}=\frac{\mu_0}{4\pi}\frac{q_{jk}}{2}M_{jki}$$

where

$$M_{jki}=I\oint x_j'x_k'\,dx_i' \qquad (10)$$

is a rank three tensor. (More on this below.)

Then we can get $\vec{B}$ from $\vec{A}$. The leading term is:

$$\begin{aligned}
\vec{B} &= \vec{\nabla}\times\vec{A}=\vec{\nabla}\times\left(\frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{x}}{|\vec{x}|^3}\right)=-\frac{\mu_0}{4\pi}\vec{\nabla}\times\left(\vec{m}\times\vec{\nabla}\frac{1}{|\vec{x}|}\right) \\
&= -\frac{\mu_0}{4\pi}\left[\vec{m}(\vec{\nabla}\cdot\vec{\nabla}\frac{1}{|\vec{x}|})+\left(\vec{\nabla}\frac{1}{|\vec{x}|}\cdot\vec{\nabla}\right)\vec{m}-\vec{\nabla}\frac{1}{|\vec{x}|}\left(\vec{\nabla}\cdot\vec{m}\right)-\left(\vec{m}\cdot\vec{\nabla}\right)\vec{\nabla}\frac{1}{|\vec{x}|}\right]
\end{aligned}$$

But $\vec{m}$ is a constant vector, so all its derivatives are zero, and temporarily putting $z-$axis along $\vec{m}$, we get

$$\vec{B}=-\frac{\mu_0}{4\pi}\left[\vec{m}(\nabla^2\frac{1}{|\vec{x}|})-m\frac{\partial}{\partial z}\vec{\nabla}\frac{1}{|\vec{x}|}\right]$$

Now we get the last term from eqn 18 of the multipole moments notes:

$$\begin{aligned}\vec{B} &= -\frac{\mu_0}{4\pi}\left[\vec{m}\left[\left(-4\pi\delta\left(\vec{x}\right)\right]+m\left\{\frac{\hat{z}}{r^3}-3\frac{\vec{x}}{r^5}\hat{z}\cdot\vec{x}+\frac{4\pi}{3}\delta\left(\vec{x}\right)\ \hat{z}\right\}\right]\\ &= \frac{\mu_0}{4\pi}\left[3\frac{\vec{x}}{r^5}\vec{m}\cdot\vec{x}-\frac{\vec{m}}{r^3}+\frac{8\pi}{3}\delta\left(\vec{x}\right)\ \vec{m}\right] \qquad (11)\end{aligned}$$

which is J 5.64. As with the electric dipole, there is a delta-function at the origin, but in the magnetic case it is parallel to (not opposite) $\vec{m}$.

The direction and magnitude of the delta-function term may be understood by looking at a tiny current loop model for the magnetic dipole. On axis, the magnetic field due to a loop of radius a is (magloop notes page 4)

$$\begin{aligned}\vec{B} &= \mu_0\frac{I}{2a}\hat{n}\\ &= \mu_0\frac{I\pi a^2}{2\pi a^3}\hat{n}=\frac{2}{3}\mu_0\frac{\vec{m}}{\left(4\pi a^3/3\right)}\end{aligned}$$

As $a\rightarrow 0$, the magnetic dipole density $\rightarrow \vec{m}\delta\left(\vec{x}\right)$ and we get

$$\vec{B}\rightarrow\frac{2}{3}\mu_0\vec{m}\delta\left(\vec{x}\right)$$

as in (11). Note that the electric dipole field delta function term (multipole notes eqn 18) differs from this result by a factor of 2 as well as the sign. See J pg 190 for applications of this to the energy of the hyperfine states of atomic systems.

Alternatively, we can find the field by starting from the Biot-Savart law:

$$\begin{aligned}\vec{B}\left(\vec{x}\right) &= \frac{\mu_0}{4\pi}I\oint\frac{d\vec{\ell}\times\left(\vec{x}-\vec{x}'\right)}{\left|\vec{x}-\vec{x}'\right|^3}\\ &= \frac{\mu_0}{4\pi}I\oint d\vec{\ell}\times\vec{\nabla}'\frac{1}{\left|\vec{x}-\vec{x}'\right|}\end{aligned}$$

Inserting the Taylor expansion of $1/R$, we get

$$\vec{B}\left(\vec{x}\right)=\frac{\mu_0}{4\pi}I\oint d\vec{\ell}\times\vec{\nabla}'\left(\frac{1}{\left|\vec{x}\right|}+\frac{\vec{x}\cdot\vec{x}'}{\left|\vec{x}\right|^3}+\frac{\vec{x}'\cdot q\cdot\vec{x}'}{2}+\cdots\right) \qquad (12)$$

Now $\vec{\nabla}'\frac{1}{|\vec{x}|}=0$, so the first term in equation 12 is zero. In the second term, $\vec{\nabla}'\left(\vec{x}\cdot\vec{x}'\right)=\vec{x}$, and $\oint d\vec{\ell}=0$, so

$$\text{term 2} = \frac{\mu_0}{4\pi}I\oint d\vec{\ell}\times\frac{\vec{x}}{\left|\vec{x}\right|^3}=0\times\frac{\vec{x}}{\left|\vec{x}\right|^3}=0$$

Thus the first non-zero term is the third:

$$B_i=\frac{\mu_0}{8\pi}I\varepsilon_{ijk}\oint dx'_j\nabla'_k\left(x'_l q_{lm}x'_m\right)$$

and

$$\begin{aligned}\nabla'_k\left(x'_l q_{lm}x'_m\right) &= \delta_{kl}q_{lm}x'_m+x'_l q_{lm}\delta_{km}=q_{km}x'_m+x'_l q_{lk}\\ &= 2q_{km}x'_m\end{aligned}$$

since q_{lm} (7) is symmetric. Thus the dominant term in $\vec{B}$ is:

$$\begin{aligned} B_i &= \frac{\mu_0}{4\pi} I \varepsilon_{ijk} q_{km} \oint dx'_j x'_m \\ &= \frac{\mu_0}{4\pi} \varepsilon_{ijk} q_{km} M_{mj} \\ &= \frac{\mu_0}{4\pi} \varepsilon_{ijk} \left(\frac{-\delta_{km}}{|\vec{x}|^3} + 3\frac{x_k x_m}{|\vec{x}|^5} \right) M_{mj} \end{aligned}$$

Then using equation (4),

$$\varepsilon_{ijk}\delta_{km}M_{mj} = \varepsilon_{ijk}M_{kj} = -\varepsilon_{ijk}M_{jk} = -2m_i$$

and using the inverse relation (5):

$$\begin{aligned} \varepsilon_{ijk}M_{mj} &= \varepsilon_{ijk}\varepsilon_{mjp}m_p = \varepsilon_{jki}\varepsilon_{jpm}m_p \\ &= (\delta_{kp}\delta_{im} - \delta_{km}\delta_{ip})\, m_p \\ &= \delta_{im}m_k - \delta_{km}m_i \end{aligned}$$

And thus:

$$B_i = \frac{\mu_0}{4\pi}\left(\frac{2m_i}{|\vec{x}|^3} + 3\frac{x_k x_i m_k - x_k x_k m_i}{|\vec{x}|^5} \right)$$

or:

$$\begin{aligned} \vec{B} &= \frac{\mu_0}{4\pi}\frac{1}{|\vec{x}|^3}\left(-\vec{m} + 3\frac{\vec{x}\,(\vec{m}\cdot\vec{x})}{|\vec{x}|^2} \right) \\ &= \frac{\mu_0}{4\pi}\frac{1}{|\vec{x}|^3}\left(3\hat{r}\,(\vec{m}\cdot\hat{r}) - \vec{m}\right) \end{aligned}$$

which is Jackson equation 5.56. This is a dipole field, as expected, but we do not get the delta function this way. Thus the result is valid for $r > 0$.

3 Force and torque

3.1 Force

The force exerted on a current distribution $\vec{J}$ by an external magnetic field $\vec{B}_{\text{ext}}$ is:

$$\vec{F} = \int \vec{J} \times \vec{B}_{\text{ext}}\, dV$$

As we did in the electric case, we expand the external field in a Taylor series:

$$\begin{aligned} F_i &= \int \varepsilon_{ijk} J_j B_{\text{ext},k}\, d^3x \\ &= \int \varepsilon_{ijk} J_j \left(B_{\text{ext},k}(0) + x_m \left.\frac{\partial B_{\text{ext},k}}{\partial x_m}\right|_0 + \frac{1}{2} x_m x_n \left.\frac{\partial^2 B_{\text{ext},k}}{\partial x_m \partial x_n}\right|_0 \cdots \right) d^3x \end{aligned}$$

We may use the time-independent Maxwell's equations to rewrite the first term:

$$\begin{aligned}\varepsilon_{ijk}B_{\text{ext, }k}(0)\int J_j d^3x &= \frac{\varepsilon_{ijk}B_{\text{ext, }k}(0)}{\mu_0}\int\left(\vec{\nabla}\times\vec{B}\right)_j d^3x \\ &= \frac{\varepsilon_{ijk}B_{\text{ext, }k}(0)}{\mu_0}\int_{S_\infty}\left(\hat{n}\times\vec{B}\right)_j d^2x = 0\end{aligned}$$

since $\vec{B}$ due to $\vec{J}$ is proportional to $1/r^3$, as proved above. (For a current loop we have the more obvious result $\oint I\, dx_j = 0$.) Notice here that $\vec{B}$ (due to $\vec{J}$) and $\vec{B}_{\text{ext}}$ are distinct fields.

Thus the first non-zero term in the force is the second term:

$$\begin{aligned}F_i &= \int \varepsilon_{ijk}J_j x_m \left.\frac{\partial B_{\text{ext, }k}}{\partial x_m}\right|_0 d^3x \\ &= \varepsilon_{ijk}M_{mj}\left.\frac{\partial B_{\text{ext, }k}}{\partial x_m}\right|_0 \\ &= \varepsilon_{ijk}\varepsilon_{mjp}m_p\left.\frac{\partial B_{\text{ext, }k}}{\partial x_m}\right|_0 \\ &= \left(\delta_{im}\delta_{kp}-\delta_{ip}\delta_{km}\right)m_p\left.\frac{\partial B_{\text{ext, }k}}{\partial x_m}\right|_0 \\ &= m_k\left.\frac{\partial B_{\text{ext, }k}}{\partial x_i}\right|_0 - m_i\left.\frac{\partial B_{\text{ext, }k}}{\partial x_k}\right|_0\end{aligned}$$

But $\vec{\nabla}\cdot\vec{B}_{\text{ext}} = 0$, and we can bring the components m_k through the differential operator because they are constants, so

$$\vec{F} = \vec{\nabla}\left.\left(\vec{m}\cdot\vec{B}_{\text{ext}}\right)\right|_0$$

This is the first non-zero term in a Taylor expansion of $\vec{F}$. Note the relation between this expression for $\vec{F}$ and the energy $U = -\vec{m}\cdot\vec{B}_{\text{ext}}$ of a dipole in an external field.

$$\vec{F} = -\vec{\nabla}U$$

as expected.

The next term is

$$F_i = \varepsilon_{ijk}\frac{1}{2}\left.\frac{\partial^2 B_{\text{ext},k}}{\partial x_m \partial x_n}\right|_0 \int J_j x_m x_n d^3x = \varepsilon_{ijk}\frac{1}{2}\left.\frac{\partial^2 B_{\text{ext},k}}{\partial x_m \partial x_n}\right|_0 M_{mnj}$$

3.2 Torque

The torque exerted on a current distribution by the external field is

$$\begin{aligned}\vec{\tau} &= \int \vec{x}\times d\vec{F} = \int \vec{x}\times\left(\vec{J}\times\vec{B}_{\text{ext}}\right) d^3x \\ &= \int\left[\vec{J}\left(\vec{x}\cdot\vec{B}_{\text{ext}}\right) - \vec{B}_{\text{ext}}\left(\vec{x}\cdot\vec{J}\right)\right] d^3x\end{aligned}$$

and using the same expansion as before, and dropping the subscript "ext" on $\vec{B} = \vec{B}_{\text{ext}}$ for clarity, we have:

$$\begin{aligned}\tau_i &= \int \left[J_i x_k B_k(0) + J_i x_k x_m \left.\frac{\partial B_k}{\partial x_m}\right|_0 + \cdots \right. \\ &\quad \left. - B_i(0)\, x_k J_k - x_m \left.\frac{\partial B_i}{\partial x_m}\right|_0 x_k J_k + \cdots \right] d^3x\end{aligned}$$

Let's look at the terms one at a time. The first term is

$$\begin{aligned}\int J_i x_k B_k(0)\ d^3x &= B_k(0)\, M_{ki} = B_k(0)\, \varepsilon_{kip} m_p \\ \tau_{i,1} &= \left(\vec{m} \times \vec{B}(0)\right)_i \end{aligned} \tag{13}$$

The third term is

$$\int B_i(0)\, x_k J_k\ d^3x = B_i(0)\, M_{kk} = 0$$

since M_{ij} is antisymmetric, and thus its trace is zero. Thus (13) is the total torque if $\vec{B}$ is uniform. The other two terms are higher order corrections to the basic result (13)

$$\tau_i\,(\text{correction terms}) = \left.\frac{\partial B_k}{\partial x_m}\right|_0 \int J_i x_k x_m\ d^3x - \left.\frac{\partial B_i}{\partial x_m}\right|_0 \int x_m x_k J_k\ d^3x$$

They involve the third rank tensor

$$M_{ijk} = \int x_i x_j J_k\ d^3x$$

which also appeared in the expansion of $\vec{A}$ (10). This tensor is not easily expressed in terms of the vector $\vec{m}$. It does have some symmetry:

$$M_{ijk} = M_{jik}$$

The correction terms are:

$$\tau_i\,(\text{correction terms}) = \left.\frac{\partial B_k}{\partial x_m}\right|_0 M_{kmi} - \left.\frac{\partial B_i}{\partial x_m}\right|_0 M_{mkk} \tag{14}$$

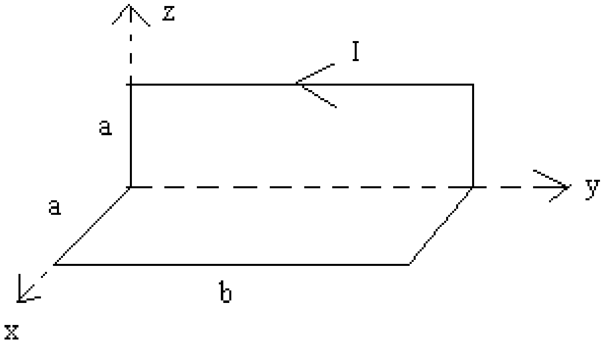
4 An example:

Consider a current loop made of of two rectangles: One in the $x - y$ plane with dimensions a by b, and one in the $y - z$ plane with dimensions b by a. Imagine forming this thing by bending a rectangle $2a$ by b through 90°. Then the magnetic moment tensor has components:

$$M_{12} = I \oint x dy = I \int_0^b a dy = Iab$$

$$M_{13} = I \oint x dz = 0$$

$$M_{23} = I \oint y dz = I \int_0^a b dz = Iab$$



Thus the tensor is:

$$M = Iab \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

The magnetic moment vector has components:

$$m_1 = \frac{1}{2}\varepsilon_{1jk} Mjk = \frac{1}{2}\left(M_{23} - M_{32}\right) = Iab$$
$$m_2 = \frac{1}{2}\varepsilon_{2jk} Mjk = \frac{1}{2}\left(M_{31} - M_{13}\right) = 0$$
$$m_3 = \frac{1}{2}\varepsilon_{3jk} Mjk = \frac{1}{2}\left(M_{12} - M_{21}\right) = Iab$$

corresponding to the two planar parts of the loop. (Notice we can make up the bent loop from two planar loops stuck together.) Thus the magnetic field produced by this loop, at a large distance from the loop, is:

$$\begin{aligned} \vec{B} &= \frac{\mu_0}{4\pi}\frac{1}{|\vec{x}|^3}\left(3\hat{r}\left(\vec{m}\cdot\hat{r}\right) - \vec{m}\right) \\ &= \frac{\mu_0}{4\pi}\frac{Iab}{|\vec{x}|^3}\left(3\vec{x}\,\frac{x+z}{|\vec{x}|^2} - \hat{x} - \hat{z}\right) \end{aligned}$$

Now we introduce the external magnetic field

$$\vec{B}_{\text{ext}} = B_0\left(1 + \alpha x\right)\hat{x} + B_0\left(1 - \alpha y\right)\hat{y}$$

(Notice that $\vec{\nabla}\cdot\vec{B}_{\text{ext}} = 0$ and $\vec{\nabla}\times\vec{B}_{\text{ext}} = 0$.) The force on the loop in this magnetic field is:

$$\vec{F} = \vec{\nabla}\left(\vec{m}\cdot\vec{B}_{\text{ext}}\right)\Big|_0 = B_0 Iab\vec{\nabla}\left(1 + \alpha x\right) = B_0 Iab\alpha\,\hat{x}$$

Notice that this result is exact since the magnetic field has no higher order derivatives.

The leading term in the torque is:

$$\begin{aligned} \vec{\tau} &= \vec{m}\times\vec{B}\left(0\right) \\ &= B_0 Iab\left(\hat{x} + \hat{z}\right)\times\left(\hat{x} + \hat{y}\right) \\ &= B_0 Iab\left(\hat{z} + \hat{y} - \hat{x}\right) \end{aligned}$$

The next term involves the tensor

$$M_{ijk} = I \oint x_i x_j dx_k$$

and the first derivatives of $\vec{B}$. The only non-zero derivatives are $\partial B_x/\partial x = B_0\alpha$ and $\partial B_y/\partial y = -B_0\alpha$. The correction terms (14) are thus:

$$\begin{aligned}
\tau_1 \,(\text{correction terms}) &= \left.\frac{\partial B_k}{\partial x_m}\right|_0 M_{km1} - \left.\frac{\partial B_1}{\partial x_m}\right|_0 M_{mkk} \\
&= B_0\alpha \left(M_{111} - M_{221} - M_{111} - M_{122} - M_{133}\right) \\
&= -B_0\alpha \left(M_{221} + M_{122} + M_{133}\right) \\
\tau_2 \,(\text{correction terms}) &= \left.\frac{\partial B_k}{\partial x_m}\right|_0 M_{km2} - \left.\frac{\partial B_2}{\partial x_m}\right|_0 M_{mkk} \\
&= B_0\alpha \left(M_{112} - M_{222} + M_{211} + M_{222} + M_{233}\right) \\
&= B_0\alpha \left(M_{112} + M_{211} + M_{233}\right)
\end{aligned}$$

and

$$\begin{aligned}
\tau_3 \,(\text{correction terms}) &= \left.\frac{\partial B_k}{\partial x_m}\right|_0 M_{km3} - \left.\frac{\partial B_3}{\partial x_m}\right|_0 M_{kkm} \\
&= B_0\alpha \left(M_{113} - M_{223}\right)
\end{aligned}$$

Let's look at the values of M that we need:

$$\begin{aligned}
M_{112} &= I \oint xxdy = I \int_0^b a^2 dy = Ia^2 b \\
M_{113} &= I \oint xxdz = 0 \\
M_{122} &= I \oint xydy = I \int_0^b aydy = I\frac{ab^2}{2}
\end{aligned}$$

$$M_{133} = I \oint xzdz = 0$$

$$\begin{aligned}
M_{211} &= I \oint yxdx = I \int_a^0 bxdx = -I\frac{a^2 b}{2} \\
M_{221} &= I \oint yydx = I \int_a^0 b^2 dx = -Iab^2 \\
M_{223} &= I \oint yydz = I \int_0^a b^2 dz = Iab^2 \\
M_{233} &= I \oint yzdz = I \int_0^a bzdz = I\frac{a^2 b}{2}
\end{aligned}$$

Thus the correction terms are:

$$\begin{aligned}
\tau_1\,(\text{correction terms}) &= -B_0\alpha\left(M_{221}+M_{122}+M_{133}\right) = -B_0\alpha\left(-Iab^2+I\frac{ab^2}{2}+0\right)\\
&= \frac{1}{2}B_0\alpha Iab^2\\
\tau_2\,(\text{correction terms}) &= B_0\alpha\left(M_{112}+M_{211}+M_{233}\right) = B_0\alpha\left(I\frac{a^2b}{2}-I\frac{a^2b}{2}+I\frac{a^2b}{2}\right)\\
&= \frac{1}{2}B_0\alpha Ia^2b
\end{aligned}$$

and

$$\tau_3\,(\text{correction terms}) = B_0\alpha\left(M_{113}-M_{223}\right) = B_0\alpha\left(0-Iab^2\right) = -B_0\alpha Iab^2$$

Thus:

$$\vec{\tau} = B_0Iab\left[\hat{z}\left(1-\alpha b\right)+\hat{y}\left(1+\frac{\alpha a}{2}\right)-\hat{x}\left(1-\frac{\alpha b}{2}\right)\right]$$

Again this result is exact as there are no higher derivatives of $\vec{B}$. Check the dimensions of the result.

If you need more terms, it is probably wise to choose a different approach.

5 Connection between magnetic moment and angular momentum

If the current distribution is made up of N particles, where particle i has position $\vec{x}_i$, charge q_i, mass μ_i, and moves with velocity $\vec{v}_i$, then the current is due to the particles' motion

$$\vec{j} = \sum_{i=1}^{N} q_i\vec{v}_i\delta\left(\vec{x}-\vec{x}_i\right)$$

and then

$$M_{pq} = \int x_p j_q dV = \sum_{i=1}^{N} q_i \int x_p v_{i,q}\delta\left(\vec{x}-\vec{x}_i\right)dV$$

and (eqn 4)

$$\begin{aligned}
m_k &= \frac{1}{2}\varepsilon_{kpq}M_{pq} = \frac{1}{2}\sum_{i=1}^{N} q_i\int \varepsilon_{kpq}x_p v_{i,q}\delta\left(\vec{x}-\vec{x}_i\right)dV\\
\vec{m} &= \frac{1}{2}\sum_{i=1}^{N} q_i\vec{x}_i\times\vec{v}_i = \sum_{i=1}^{N}\frac{q_i}{2\mu_i}\vec{L}_i
\end{aligned}$$

where

$$\vec{L}_i = \mu_i\vec{x}_i\times\vec{v}_i$$

is the angular momentum of particle i about the origin. If all the particles are electrons, for example, then the magnetic moment is

$$\vec{m} = -\frac{e}{2m_e}\sum_{i=1}^{N}\vec{L}_i = -\frac{e}{2m_e}\vec{L} \tag{15}$$

where $\vec{L}$ is the total angular momentum of the collection of electrons.

Relation (15) is very important, and holds even on the atomic scale. However, it needs modification when applied to the internal angular momentum of individual particles, when quantum mechanics plays an important role. For an electron, for example,

$$\vec{m} = -g\frac{e}{2m_e}\vec{s}$$

where $\vec{s}$ is the electron spin and $g \simeq 2$. See Jackson page 565 for precise values of g.

See Jackson problem 6.5 for the relation between $\vec{m}$ and the electromagnetic field momentum.