Example using spherical harmonics– Sp 2012

Magnetic field due to a current loop.

A circular loop of radius a carries current I. We place the origin at the center of the loop, with polar axis perpendicular to the plane of the loop. Then the current density is

$$\vec{j} = I\frac{\delta(r-a)}{a}\delta(\mu)\ \hat{\phi}$$

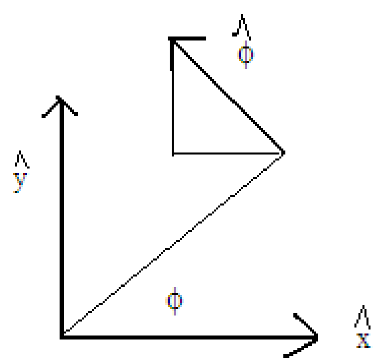
(You can get this most easily by starting with the expression in cylindrical coordinates

$$\vec{j} = I\delta(z)\,\delta(r-a)\,\hat{\phi}$$

and using $z = r\cos\theta$. See also Lea pg 315 Example 6.7.) Then the magnetic potential is (Notes 1 eqn 17)

$$\begin{aligned} \vec{A}(\vec{x}) &= \frac{\mu_0}{4\pi}\int\frac{\vec{j}(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3\vec{x}' \qquad (1) \\ &= \frac{\mu_0}{4\pi a}I\int\frac{\delta(r'-a)\,\delta(\mu')\ \hat{\phi}(\phi')}{|\vec{x}-\vec{x}'|}d^3\vec{x}' \end{aligned}$$

We must take care here, because the unit vector $\hat{\phi}$ is not a constant. We must re-express it in terms of the constant Cartesian unit vectors,



$$\hat{\phi}(\phi') = -\sin\phi'\hat{x} + \cos\phi'\hat{y}$$

and thus:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi a}I\int\frac{\delta(r'-a)\,\delta(\mu')}{|\vec{x}-\vec{x}'|}\left(-\sin\phi'\hat{x}+\cos\phi'\hat{y}\right)\ d^3\vec{x}'$$

Now we use our **most useful result** (J eqn 3.70) to expand the $1/R$ in the integrand. With $r_< = \min(r, r')$ and similarly for $r_>$,

$$\begin{aligned}
\vec{A}(\vec{x}) &= \frac{\mu_0}{4\pi a} I \int \delta(r'-a)\,\delta(\mu') \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} Y_{lm}(\theta,\phi)\, Y_{lm}^*(\theta',\phi') \left(-\sin\phi'\hat{x} + \cos\phi'\hat{y}\right) d^3\vec{x}' \\
&= \frac{\mu_0}{4\pi a} I \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} Y_{lm}(\theta,\phi) \times \\
&\quad \int_0^{2\pi} \int_{-1}^{+1} \int_0^{\infty} \delta(r'-a)\,\delta(\mu') \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta',\phi') \left(-\sin\phi'\hat{x} + \cos\phi'\hat{y}\right) (r')^2\, dr' d\mu' d\phi' \\
&= \mu_0 a I \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{Y_{lm}(\theta,\phi)}{2l+1} \frac{r_<^l}{r_>^{l+1}} \int_0^{2\pi} Y_{lm}^*\left(\frac{\pi}{2},\phi'\right) \left(-\sin\phi'\hat{x} + \cos\phi'\hat{y}\right)\, d\phi'
\end{aligned}$$

where we used the sifting property to evaluate the integrals over r' and μ'. We now interpret $r_<$ as the min of r and a, and similarly for $r_>$.

To do the integral over ϕ', we rewrite the sines and cosines in terms of exponentials:

$$\begin{aligned}
I_{lm} &= \int_0^{2\pi} Y_{lm}^*\left(\frac{\pi}{2},\phi'\right) \left(-\sin\phi'\hat{x} + \cos\phi'\hat{y}\right) d\phi' \\
&= \int_0^{2\pi} \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(0)\, e^{-im\phi'} \left[e^{i\phi'} \left(\frac{\hat{y}+i\hat{x}}{2}\right) + e^{-i\phi'} \left(\frac{\hat{y}-i\hat{x}}{2}\right) \right] d\phi'
\end{aligned}$$

The integral is zero unless $m = \pm 1$. (Be alert here– if you use the mantra "axisymmetry so $m = 0$" you will get into big trouble!) With $m = +1$ we get:

$$I_{l1} = \sqrt{\frac{2l+1}{4\pi}\frac{(l-1)!}{(l+1)!}} P_l^1(0) \left(\frac{\hat{y}+i\hat{x}}{2}\right) 2\pi$$

and with $m = -1$

$$\begin{aligned}
I_{l,-1} &= \pi \sqrt{\frac{2l+1}{4\pi}\frac{(l+1)!}{(l-1)!}} P_l^{-1}(0)\,(\hat{y}-i\hat{x}) \\
&= \pi \sqrt{\frac{2l+1}{4\pi}\frac{(l+1)!}{(l-1)!}} (-1) \frac{(l-1)!}{(l+1)!} P_l^1(0)\,(\hat{y}-i\hat{x}) \\
&= -\pi \sqrt{\frac{2l+1}{4\pi}\frac{(l-1)!}{(l+1)!}} P_l^1(0)\,(\hat{y}-i\hat{x})
\end{aligned}$$

So

$$\vec{A}(\vec{x}) = \mu_0 a I \sum_{l=1}^{\infty} \frac{\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} \sqrt{\frac{2l+1}{4\pi}\frac{(l-1)!}{(l+1)!}} P_l^1(0) \left[(\hat{y}+i\hat{x})\, Y_{l1}(\theta,\phi) - (\hat{y}-i\hat{x})\, Y_{l,-1} \right]$$

The sum over l starts at 1 because with $l = 0$ there is no $m = \pm 1$ term. Then, using $Y_{l,-m} = (-1)^m Y_{lm}^*$, we get

$$\begin{aligned}
\vec{A}(\vec{x}) &= \mu_0 a I \sum_{l=1}^{\infty} \frac{\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} \frac{2l+1}{4\pi} \frac{(l-1)!}{(l+1)!} P_l^1(0) P_l^1(\mu) \left[(\hat{y} + i\hat{x}) e^{i\phi} + (\hat{y} - i\hat{x}) e^{-i\phi}\right] \\
&= \frac{\mu_0 a I}{4} \sum_{l=1}^{\infty} \frac{r_<^l}{r_>^{l+1}} \frac{(l-1)!}{(l+1)!} P_l^1(0) P_l^1(\mu) (2\hat{y}\cos\phi - 2\hat{x}\sin\phi) \\
&= \frac{\mu_0 a I}{2} \sum_{l=1}^{\infty} \frac{r_<^l}{r_>^{l+1}} \frac{(l-1)!}{(l+1)!} P_l^1(0) P_l^1(\mu)\ \hat{\phi}
\end{aligned}$$

We might have expected to find that $\vec{A}$ is in the ϕ direction. Check dimensions: $\vec{A}$ is current times μ_0, which is consistent with (1).

We can simplify a bit by inserting the value of $P_l^1(0)$. First note that P_l is even if l is even, and odd if l is odd. Since P_l^m is the mth derivative, P_l^m will be odd if $l+m$ is odd and even if $l+m$ is even. So $P_l^m(0) = 0$ unless $l+m$ is even, or, in this case, l is odd.

Now we can use the recursion relation (J3.29 or Lea 8.37)

$$\begin{aligned}
l P_l(\mu) &= \mu P_l'(\mu) - P_{l-1}'(\mu) \\
l P_l(0) &= -P_{l-1}'(0) = P_{l-1}^1(0)
\end{aligned}$$

Thus, using Lea 8.47, with $l = 2n+1$,

$$\begin{aligned}
P_l^1(0) &= (l+1) P_{l+1}(0) = (l+1)(-1)^{(l+1)/2} \frac{l!!}{(l+1)!!} \\
P_{2n+1}^1 &= (-1)^{n+1} \frac{(2n+1)!!}{(2n)!!} \\
&= (-1)^{n+1} \frac{(2n+1)!!}{2^n n!}
\end{aligned}$$

Thus:

$$\begin{aligned}
\vec{A}(\vec{x}) &= \frac{\mu_0 a I}{2} \sum_{n=0}^{\infty} \frac{r_<^{2n+1}}{r_>^{2n+2}} \frac{(2n)!}{(2n+2)!} (-1)^{n+1} \frac{(2n+1)!!}{2^n n!} P_{2n+1}^1(\mu)\ \hat{\phi} \\
&= \frac{\mu_0 a I}{2 r_>} \sum_{n=0}^{\infty} \left(\frac{r_<}{r_>}\right)^{2n+1} \frac{(2n+1)!!}{2(n+1)(2n+1)} (-1)^{n+1} \frac{1}{2^n n!} P_{2n+1}^1(\mu)\ \hat{\phi} \\
&= \frac{\mu_0 a I}{4 r_>} \hat{\phi} \left[-\frac{r_<}{r_>} P_1^1(\mu) + \sum_{n=1}^{\infty} \left(\frac{r_<}{r_>}\right)^{2n+1} \frac{(2n-1)!!}{2^n (n+1)!} (-1)^{n+1} P_{2n+1}^1(\mu)\right]
\end{aligned}$$

Outside the loop, $r > a$, and

$$\vec{A}(\vec{x}) = -\frac{\mu_0 a I}{4r} \left[\frac{a}{r} P_1^1(\mu) + \sum_{n=1}^{\infty} \left(\frac{a}{r}\right)^{2n+1} \frac{(2n-1)!!}{2^n (n+1)!} (-1)^n P_{2n+1}^1(\mu)\right] \hat{\phi} \quad (2)$$

For $r \gg a$, $n = 0$ ($\ell = 1$) is the dominant term:

$$\vec{A}(\vec{x}) = -\frac{\mu_0 I}{4}\frac{a^2}{r^2}P_1^1(\mu)\ \hat{\phi}$$

and (Lea 8.53)

$$P_1^1 = -\sin\theta\frac{d}{d\mu}(\mu) = -\sin\theta$$

so

$$\vec{A}(\vec{x}) = \frac{\mu_0 a^2 I}{4r^2}\sin\theta\ \hat{\phi} = \frac{\mu_0}{4\pi}\frac{m}{r^2}\sin\theta\ \hat{\phi}$$

where $m = \pi a^2 I$ is the magnetic moment of the loop. Compare with Jackson equation 5.55. Then, in this limit

$$\begin{aligned}
\vec{B} &= \vec{\nabla}\times\vec{A} = \frac{\hat{r}}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\mu_0}{4\pi}\frac{m}{r^2}\sin^2\theta\right) - \frac{\hat{\theta}}{r}\frac{\partial}{\partial r}\frac{\mu_0}{4\pi}\frac{m}{r}\sin\theta \\
&= \frac{\hat{r}}{r\sin\theta}\frac{\mu_0}{4\pi}\frac{m}{r^2}2\sin\theta\cos\theta + \frac{\hat{\theta}}{r}\frac{\mu_0}{4\pi}\frac{m}{r^2}\sin\theta \\
&= \frac{\mu_0}{4\pi}\frac{m}{r^3}\left(\hat{r}\ 2\cos\theta + \hat{\theta}\ \sin\theta\right)
\end{aligned}$$

This is a dipole field.

Inside the loop, $r < a$ and we have:

$$\vec{A}(\vec{x}) = -\frac{\mu_0 a I}{4a}\left[\frac{r}{a}P_1^1(\mu) + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^{2n+1}\frac{(2n-1)!!}{2^n(n+1)!}(-1)^n P_{2n+1}^1(\mu)\right]\hat{\phi} \quad (3)$$

Near the center, $r \ll a$, the $n = 0$ term dominates again, and we have:

$$\begin{aligned}
\vec{A} &= -\frac{\mu_0 I}{4}\frac{r}{a}P_1^1(\mu)\ \hat{\phi} \\
&= \frac{\mu_0 I}{4}\frac{r}{a}\sin\theta\ \hat{\phi}
\end{aligned}$$

and

$$\begin{aligned}
\vec{B}(\vec{x}) &= \frac{\mu_0}{4a}I\left[\frac{\hat{r}}{r\sin\theta}\frac{\partial}{\partial\theta}\left(r\sin^2\theta\right) - \frac{\hat{\theta}}{r}\frac{\partial}{\partial r}r^2\sin\theta\right] \\
&= \frac{\mu_0}{2a}I\left(\hat{r}\cos\theta - \hat{\theta}\sin\theta\right) \\
&= \frac{\mu_0}{2a}I\ \hat{z}
\end{aligned}$$

a uniform field, as expected. Compare with Lea and Burke equation 28.7 with $z = 0$.

Field on axis:

From LB 28.7, the field on the polar ($z-$) axis is

$$\vec{B}(z) = \frac{\mu_0 I a^2}{2\left(z^2 + a^2\right)^{3/2}}\hat{z}$$

So for $z > a$ we have

$$\begin{aligned}
\vec{B}(z) &= \frac{\mu_0 I a^2}{2z^3}\left(1 - \frac{3}{2}\frac{a^2}{z^2} + \left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)\frac{1}{2}\frac{a^4}{z^4} + \cdots\right)\hat{z} \\
&= \frac{\mu_0 I a^2}{2z^3}\left(1 - \frac{3}{2}\frac{a^2}{z^2} + \frac{15}{8}\frac{a^4}{z^4} + \cdots\right)\hat{z} \qquad (4)
\end{aligned}$$

Our solution for $\mu = 1$ ($\theta = 0$) is

$$\begin{aligned}
\vec{B} &= \vec{\nabla} \times \vec{A} \\
&= \frac{\mu_0 a I}{4}\left\{\frac{\hat{r}}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\left[-\frac{a}{r^2}P_1^1(\mu) + \sum_{n=1}^{\infty}\frac{a^{2n+1}}{r^{2n+2}}\frac{(2n-1)!!}{2^n(n+1)!}(-1)^{n+1}P_{2n+1}^1(\mu)\right]\right)\right. \\
&\qquad \left. -\frac{\hat{\theta}}{r}\left[\frac{\partial}{\partial r}\left(-\frac{a}{r}P_1^1(\mu) + \sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^{2n+1}\frac{(2n-1)!!}{2^n(n+1)!}(-1)^{n+1}P_{2n+1}^1(\mu)\right)\right]\right\}
\end{aligned}$$

The theta component is $\frac{\mu_0 a I}{4r}$ times

$$\left(-\frac{a}{r^2}P_1^1(\mu) + \sum_{n=1}^{\infty}(2n+1)\left(\frac{a}{r}\right)^{2n+1}\frac{(2n-1)!!}{2^n(n+1)!}(-1)^{n+1}P_{2n+1}^1(1)\right)$$

But

$$P_{2n+1}^1(1) = -\sqrt{1-\mu^2}\frac{d}{d\mu}P_{2n+1}(\mu)\bigg|_{\mu=1} = 0$$

So the theta component is zero.

The $r-$component is

$$\begin{aligned}
B_r &= -\frac{\mu_0 a I}{4}\frac{\hat{r}}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\left[\frac{a}{r^2}P_1^1(\mu) + \sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^{2n+1}\frac{(2n-1)!!}{2^n(n+1)!}(-1)^n P_{2n+1}^1(\mu)\right]\right) \\
&= \frac{\mu_0 a I}{4}\hat{r}\frac{\partial}{\partial\mu}\left(\sqrt{1-\mu^2}\left[\frac{a}{r^3}P_1^1(\mu) + \sum_{n=1}^{\infty}\frac{a^{2n+1}}{r^{2n+3}}\frac{(2n-1)!!}{2^n(n+1)!}(-1)^n P_{2n+1}^1(\mu)\right]\right)
\end{aligned}$$

Note that

$$\begin{aligned}
\frac{d}{d\mu}\sqrt{1-\mu^2}P_{2n+1}^1(\mu) &= -\frac{d}{d\mu}\left(1-\mu^2\right)\frac{d}{d\mu}P_{2n+1}(\mu) \\
&= (2n+1)(2n+2)P_{2n+1}(\mu)
\end{aligned}$$

from the differential equation for P_{2n+1}. Then we evaluate at $\mu = 1$ where $P_{2n+1}(1) = 1$

$$B_r(r,1) = \frac{\mu_0 a^2 I}{4}\frac{\hat{r}}{r^3}\left(\left[2 + \sum_{n=1}^{\infty}\frac{a^{2n}}{r^{2n}}\frac{(2n-1)!!}{2^n(n+1)!}(-1)^n(2n+1)(2n+2)\right]\right)$$

Changing to the $z-$coordinate, we get

$$\begin{aligned} B_z\left(z\right) &= \frac{\mu_0 a^2 I}{2z^3}\hat{z}\left(1+\frac{1}{2}\sum_{n=1}^{\infty}\frac{a^{2n}}{z^{2n}}\frac{(2n+1)!!}{2^{n-1}n!}\left(-1\right)^n\right) \\ &= \frac{\mu_0 a^2 I}{2z^3}\hat{z}\left(1-\frac{3}{2}\frac{a^2}{z^2}+\frac{a^4}{2z^4}\frac{5\times 3}{2\times 2}+\cdots\right) \\ &= \frac{\mu_0 a^2 I}{2z^3}\hat{z}\left(1-\frac{3}{2}\left(\frac{a}{z}\right)^2+\frac{15}{8}\left(\frac{a}{z}\right)^4+\cdots\right) \end{aligned}$$

which agrees with (4).