

Motion using Hamiltonians

S.M.Lea

Let's look at a particle moving in a uniform magnetic field $\vec{B} = B_0 \hat{z}$. We can satisfy the Lorentz gauge condition

$$\partial_\alpha A^\alpha = 0$$

and the relation $\vec{B} = \vec{\nabla} \times \vec{A}$ with the potential

$$A^\alpha = (0, 0, B_0 x, 0)$$

The corresponding canonical momentum is:

$$P^\alpha = p^\alpha + \frac{q}{c} A^\alpha = \left(p^0, p^1, p^2 + \frac{q}{c} B_0 x, p^3 \right)$$

and Hamilton's equations are:

$$\frac{dP^\alpha}{d\tau} = \frac{q}{mc} \left(P_\beta - \frac{q}{c} A_\beta \right) \partial^\alpha A^\beta \quad (1)$$

where the only non-zero component of $\partial^\alpha A^\beta$ is $\partial^1 A^2 = -B_0$, since $\partial^1 \equiv -\frac{\partial}{\partial x}$. Then from Hamilton's equations (1), we get:

$$\frac{dP^0}{d\tau} = 0 \Rightarrow P^0 = p^0 = \gamma mc = \text{constant}$$

Thus the particle's energy remains constant.

$$\frac{dP^2}{d\tau} = 0 \Rightarrow P^2 = p^2 + \frac{q}{c} B_0 x = mu_y + \frac{q}{c} B_0 x = \text{constant}$$

where u_y is the y -component of the 4-velocity, $= \gamma v_y$ where \vec{v} is the 3-velocity. We may choose our origin so that $u_y = 0$ when $x = 0$, and then:

$$u_y = -\frac{q B_0}{mc} x \quad (2)$$

The next equation is:

$$\frac{dP^3}{d\tau} = 0 \Rightarrow P^3 = p^3 = mu_z = \gamma m v_z = \text{constant}$$

Thus the particle's velocity component along the field remains constant. The last equation is:

$$\begin{aligned} \frac{dP^1}{d\tau} &= \frac{q}{c} (u_2) (-B_0) \\ \frac{dp^1}{d\tau} &= \frac{d}{d\tau} (mu^1) = \frac{q}{c} (-u_y) (-B_0) = \frac{q}{c} u_y B_0 \end{aligned}$$

We may insert the result (2) on the right hand side:

$$\frac{du^1}{d\tau} = \frac{d^2x}{d\tau^2} = - \left(\frac{qB_0}{mc} \right)^2 x$$

which we may integrate immediately to get:

$$x = A \cos \tau + B \sin \tau$$

with ω equal to the cyclotron frequency:

$$\omega = \frac{qB_0}{mc}$$

Then from equation (2), we have

$$\frac{dy}{d\tau} = u_y = - \omega (A \cos \tau + B \sin \tau)$$

and thus

$$y = -A \sin \tau + B \cos \tau + C$$

Now since we have already established that γ remains constant, we may write $\tau = t/\gamma$. We may also choose the origin of τ (and t) so that:

$$x = A \cos \frac{t}{\gamma}$$

(i.e. $B = 0$), and again, careful choice of origin allows us to take $C = 0$, so

$$y = -A \sin \frac{t}{\gamma}$$

which is circular motion with angular frequency $\omega = 1/\gamma$ and radius A . No surprises.