Green's function notes 2012

1 Introduction

Back in notes 2.5 we derived the potential in terms of the Green's function.

Dirichlet problem: Equation (6) in notes 2.5 is

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V G_D(\vec{x},\vec{x}')\,\rho(\vec{x}')\ d^3\vec{x}' - \frac{1}{4\pi}\int_S \Phi(\vec{x}')\frac{\partial G_D(\vec{x},\vec{x}')}{\partial n'}\,dA' \qquad (1)$$

which is also Jackson 1.44. The Dirichlet Green's function is symmetric in $\vec{x}$ and $\vec{x}'$ (see Lea §C.7.1).

$$G_D(\vec{x},\vec{x}') = G_D(\vec{x}',\vec{x})$$

See also Jackson problem 1.14.

Neumann problem: Writing $<\Phi>_S = \frac{1}{A}\int_S \Phi\ dA$, the average value of the potential over the surface S, equation (7) in notes 2.5 is

$$\Phi(\vec{x}) - \ <\Phi>_S = \frac{1}{4\pi\varepsilon_0}\int_V G_N(\vec{x},\vec{x}')\,\rho(\vec{x}')\,d^3\vec{x}' + \frac{1}{4\pi}\int_S G_N\frac{\partial\Phi}{\partial n'}dA' \qquad (2)$$

In order to make use of these expressions, we need a form for the Green's function that is relatively easy to integrate. That suggests that we expand the Green's function in orthogonal functions. The Dirichlet Green's function is symmetric in the two sets of coordinates $\vec{x}$ and $\vec{x}'$. (See Lea pg 514 and J problem 1.14.) In the Neumann case it is possible to impose the symmetry as an additional condition. (In problem 3.27 you will see why this does not affect the computed potential). So it does not matter whether we compute $G(\vec{x},\vec{x}')$ or $G(\vec{x}',\vec{x})$. I find it easier to use $\vec{x}$ as the variable and $\vec{x}'$ as fixed while finding G to reduce the number of primes I have to write. Then the differential equation satisfied by G is (notes 2.5 eqn 3):

$$\nabla^2 G(\vec{x},\vec{x}') = -4\pi\delta(\vec{x}-\vec{x}') \qquad (3)$$

This means that G represents the potential at $\vec{x}$ due to a unit point charge at $\vec{x}'$, multiplied by $4\pi\varepsilon_0$. The boundary conditions are

Dirichlet case:

$$G_D(\vec{x},\vec{x}') = 0 \ \text{ for } \vec{x} \ \text{ on } S \qquad (4)$$

and

Neumann case:

$$\frac{\partial G_N}{\partial n} = \hat{n}\cdot\vec{\nabla}G_N = -\frac{4\pi}{A} \ \text{ for } \vec{x} \text{ on } S \qquad (5)$$

where A is the *total* area of the surface S bounding the volume V.

2 Division of region method

As usual we begin by (**1**) drawing the region under consideration and (**2**) choosing a coordinate system so that the boundaries are represented by constant values of one or more of the coordinates. In this method (**3**) we place the unit point charge at an arbitrary point $\vec{x}'$ within the volume, and (**5**) use this point to divide the region into two separate regions, I and II, with $\vec{x}'$ on the boundary between them. Then G satisifies the simpler differential equation

$$\nabla^2 G(\vec{x}, \vec{x}') = 0$$

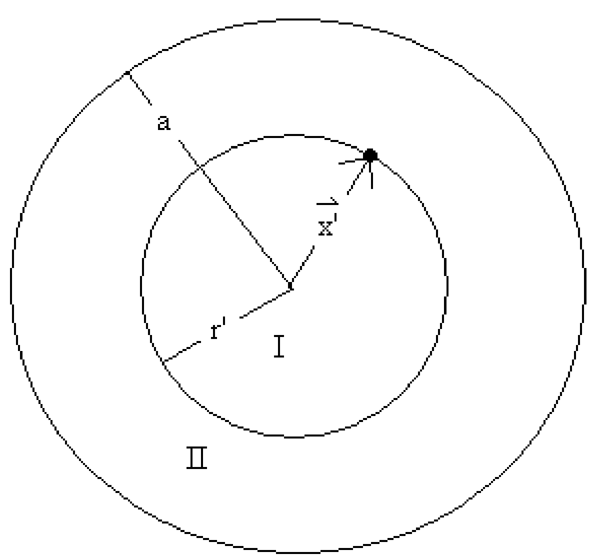
within each of regions I and II, (but not on the boundary between them). This means that we can (**4**) make use of the eigenfunctions of Laplace's equation. The method we use is the 10-step method in Lea (pg 515). (The first 5 steps are labelled above.)

2.1 Dirichlet Green's Function in Spherical Coordinates

Suppose our volume is the interior of a sphere of radius a. We place our unit point charge at an arbitrary (but, for the moment, fixed) point in the region with coordinates r', θ', ϕ'. This point then divides the volume into two regions:

Region I: $0 \leq r < r'$

Region II: $r' < r \leq a$



(**6**) Then in region I the appropriate solution to Laplace's equation is

$$G_I(\vec{x},\vec{x}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} A_{lm} r^l Y_{lm}(\theta,\phi)$$

where we have used only the functions r^l and excluded $r^{-(l+1)}$ because the potential should be finite at $r = 0$. The dependence on the coordinates r', θ' and ϕ' is contained in the coefficient A_{lm}.

Region two contains neither $r = 0$ nor $r \to \infty$, so we need both functions of r :

$$G_{II}(\vec{x},\vec{x}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\left(B_{lm} r^l + \frac{C_{lm}}{r^{l+1}}\right) Y_{lm}(\theta,\phi)$$

However, here we have to deal with the boundary at $r = a$ where G has to be zero:

$$G_{II}(\vec{x}_{\text{on } S},\vec{x}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\left(B_{lm} a^l + \frac{C_{lm}}{a^{l+1}}\right) Y_{lm}(\theta,\phi) = 0$$

We make use of the orthogonality of the Y_{lm} to argue that each term must separately equal zero:

$$B_{lm} a^l + \frac{C_{lm}}{a^{l+1}} = 0 \Rightarrow C_{lm} = -B_{lm} a^{2l+1}$$

Then

$$G_{II}(\vec{x},\vec{x}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} B_{lm}\left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right) Y_{lm}(\theta,\phi)$$

We still have two sets of unknown constants: the A_{lm} and B_{lm}. But we have one more boundary to consider at $r = r'$. (**7**)The first boundary condition we need is continuity of the potential (G) at $r = r'$.

$$\begin{aligned} G_I(\vec{x},\vec{x}')_{r=r'} &= \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} A_{lm}(r')^l Y_{lm}(\theta,\phi) \\ &= \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} B_{lm}\left((r')^l - \frac{a^{2l+1}}{(r')^{l+1}}\right) Y_{lm}(\theta,\phi) = G_{II}(\vec{x},\vec{x}')_{r=r'} \end{aligned}$$

Again we make use of the orthogonality of the Y_{lm} to argue that

$$A_{lm}(r')^l = B_{lm}\left((r')^l - \frac{a^{2l+1}}{(r')^{l+1}}\right)$$

For each l, m. Thus

$$\begin{aligned} G_I(\vec{x},\vec{x}') &= \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} B_{lm}\left(1 - \frac{a^{2l+1}}{(r')^{2l+1}}\right) r^l Y_{lm}(\theta,\phi) \\ G_{II}(\vec{x},\vec{x}') &= \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} B_{lm} r^l \left(1 - \frac{a^{2l+1}}{r^{2l+1}}\right) Y_{lm}(\theta,\phi) \end{aligned}$$

We'd like to display the symmetry in r and r' more obviously, and we can do that by relabeling

$$B_{lm} = \beta_{lm} \left(r'\right)^l$$

(Remember that for the moment r' is a constant.) Then

$$\begin{aligned}
G_I\left(\vec{x}, \vec{x}'\right) &= \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \beta_{lm} \left(r'\right)^l r^l \left(1 - \frac{a^{2l+1}}{\left(r'\right)^{2l+1}}\right) Y_{lm}(\theta, \phi) \\
G_{II}\left(\vec{x}, \vec{x}'\right) &= \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \beta_{lm} \left(r'\right)^l r^l \left(1 - \frac{a^{2l+1}}{r^{2l+1}}\right) Y_{lm}(\theta, \phi)
\end{aligned}$$

or

$$G\left(\vec{x}, \vec{x}'\right) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \beta_{lm} \left(r'\right)^l r^l \left(1 - \frac{a^{2l+1}}{r_>^{2l+1}}\right) Y_{lm}(\theta, \phi)$$

where $r_> = \max\left(r, r'\right)$ and the same expression holds in both regions.

We still need to find the β_{lm} and to do this (**8**) we make use of the differential equation (3). First we insert our expressions for G and evaluate the derivatives in θ and ϕ. We express the delta function in terms of the spherical coordinates using the result of problem 1.2.

$$\begin{aligned}
\nabla^2 G &= \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} G\right) + \nabla^2_{\text{ang}} \sum_{l=0}^{\infty} \beta_{lm} \left(r'\right)^l r^l \left(1 - \frac{a^{2l+1}}{r_>^{2l+1}}\right) Y_{lm}(\theta, \phi) \\
&= \frac{1}{r^2} \frac{\partial}{\partial r}\left[r^2 \frac{\partial}{\partial r} \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \beta_{lm} \left(r'\right)^l r^l \left(1 - \frac{a^{2l+1}}{r_>^{2l+1}}\right) Y_{lm}(\theta, \phi)\right] \\
&\quad - \frac{1}{r^2} \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \beta_{lm} \left(r'\right)^l r^l \left(1 - \frac{a^{2l+1}}{r_>^{2l+1}}\right) l\,(l+1)\, Y_{lm}(\theta, \phi) \\
&= -\frac{4\pi}{r^2} \delta\left(r - r'\right) \delta\left(\mu - \mu'\right) \delta\left(\phi - \phi'\right)
\end{aligned}$$

Now multiply both sides by $Y^*_{l'm'}(\theta, \phi)$ and integrate over the whole solid angle of the sphere. On the LHS we use the orthogonality of the Y_{lm}. On the RHS we use the sifting property.

$$\begin{aligned}
&\frac{1}{r^2} \frac{\partial}{\partial r}\left[r^2 \frac{\partial}{\partial r} \beta_{l'm'} \left(r'r\right)^{l'} \left(1 - \frac{a^{2l'+1}}{r_>^{2l'+1}}\right)\right] - \beta_{l'm'} \frac{\left(r'r\right)^{l'}}{r^2} \left(1 - \frac{a^{2l'+1}}{r_>^{2l'+1}}\right) l'\left(l'+1\right) \qquad (6) \\
&= -\frac{4\pi}{r^2} \delta\left(r - r'\right) Y^*_{l'm'}\left(\theta', \phi'\right)
\end{aligned}$$

Thus each β_{lm} contains a factor $Y^*_{lm}\left(\theta', \phi'\right)$.

$$\beta_{lm} = \gamma_{lm} Y^*_{lm}\left(\theta', \phi'\right)$$

and

$$G(\vec{x},\vec{x}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\gamma_{lm}(r'r)^l\left(1-\frac{a^{2l+1}}{r_>^{2l+1}}\right)Y_{lm}(\theta,\phi)\,Y_{lm}^*(\theta',\phi')$$

Now we have symmetry in all the coordinates, and the remaining set of constants γ_{lm} should be independent of all coordinates. Our remaining equation (6) now takes the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\gamma_{lm}(r'r)^l\left(1-\frac{a^{2l+1}}{r_>^{2l+1}}\right)\right]-\gamma_{lm}(r'r)^l\left(1-\frac{a^{2l+1}}{r_>^{2l+1}}\right)l(l+1) = -\frac{4\pi}{r^2}\delta(r-r')$$

(**9**) To make use of this equation we multiply both sides by r^2, then integrate across the internal boundary from $r=r'-\varepsilon$ to $r=r'+\varepsilon$.

$$\begin{aligned}&\int_{r'-\varepsilon}^{r'+\varepsilon}\left\{\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\gamma_{lm}(r'r)^{l'}\left(1-\frac{a^{2l+1}}{r_>^{2l+1}}\right)\right]-\gamma_{lm}(r'r)^{l'}\left(1-\frac{a^{2l+1}}{r_>^{2l+1}}\right)l(l+1)\right\}dr\\ = \quad &-4\pi\int_{r'-\varepsilon}^{r'+\varepsilon}\delta(r-r')\,dr\end{aligned}$$

The integral on the RHS equals 1. The second term on the LHS $\to 0$ as $\varepsilon\to 0$ because the integrand is continuous by construction and has no singularities in the range of integration. Thus we are left with

$$\lim_{\varepsilon\to 0}\gamma_{lm}r^2(r')^l\frac{\partial}{\partial r}\left[r^l\left(1-\frac{a^{2l+1}}{r_>^{2l+1}}\right)\right]\Bigg|_{r'-\varepsilon}^{r'+\varepsilon} = -4\pi$$

At the upper limit, $r=r'+\varepsilon$ is $>r'$, so $r_>=r$. At the lower limit, $r=r'-\varepsilon$ is $<r'$, so $r_>=r'$. Thus

$$\gamma_{lm}(r')^l\left\{r^2\,\frac{\partial}{\partial r}\left[r^l-\frac{a^{2l+1}}{r^{l+1}}\right]\Bigg|_{r'}-r^2\frac{\partial}{\partial r}\left[r^l\left(1-\frac{a^{2l+1}}{(r')^{2l+1}}\right)\right]\Bigg|_{r'}\right\} = -4\pi$$

$$\begin{aligned}\gamma_{lm}(r')^{l+2}\left[l(r')^{l-1}+(l+1)\frac{a^{2l+1}}{(r')^{l+2}}-l(r')^{l-1}\left(1-\frac{a^{2l+1}}{(r')^{2l+1}}\right)\right] &= -4\pi\\ \gamma_{lm}(r')^{l+2}\left[(l+1)\frac{a^{2l+1}}{(r')^{l+2}}+l\frac{a^{2l+1}}{(r')^{l+2}}\right] &= -4\pi\\ \gamma_{lm} = -\frac{4\pi}{2l+1}\frac{1}{a^{2l+1}}\end{aligned}$$

and thus

$$G(\vec{x},\vec{x}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\frac{4\pi}{2l+1}\frac{(r')^l r^l}{a^{2l+1}}\left(\frac{a^{2l+1}}{r_>^{2l+1}}-1\right)Y_{lm}(\theta,\phi)\,Y_{lm}^*(\theta',\phi') \qquad (7)$$

(**10**) G has dimensions of 1/length, as required, and displays the necessary symmetry. It is also positive, since $a\geq r_>$ throughout the region.

2.2 Use of the result

Suppose we have a sphere of radius a with potential V on the top and $-V$ on the bottom. Inside the sphere, a uniform line charge of length $2b$ ($b < a$) with uniform line charge density λ runs along the diameter of the dividing plane and is centered at the center of the sphere. Find the potential inside.

First we have to choose coordinates. No matter what we do, we do not have azimuthal symmetry. I'm going to put the polar axis along the line charge, giving a charge density

$$\rho(\vec{x}) = \frac{\lambda}{2\pi r^2}\left[\delta(\mu - 1) + \delta(\mu + 1)\right] S(b - r)$$

We can check this by finding the total charge on a differential piece of the line that runs from r to $r + dr$ with $r < b$. Then

$$\begin{aligned} dq &= 2\lambda dr = \int_0^{2\pi}\int_{-1}^{+1} \frac{\lambda}{2\pi r^2}\left[\delta(\mu - 1) + \delta(\mu + 1)\right] r^2 dr d\mu d\phi \\ &= \frac{\lambda}{2\pi}(2)\, 2\pi dr = 2\lambda dr \qquad \checkmark \end{aligned}$$

(Do you understand the factor of 2? Draw a picture and see!) Thus our potential is (eqn 1)

$$\begin{aligned} \Phi(\vec{x}) &= \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}\int_{-1}^{+1}\int_0^a G_D(\vec{x}, \vec{x}')\frac{\lambda}{2\pi (r')^2}\left[\delta(\mu' - 1) + \delta(\mu' + 1)\right] S(b - r')\ (r')^2 dr' d\mu' d\phi' \\ &\quad - \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{+1}\Phi(\vec{x}')\frac{\partial G_D(\vec{x}, \vec{x}')}{\partial n'} dA' \end{aligned} \tag{8}$$

Now the field point $\vec{x}$ is fixed and $\vec{x}'$ is variable. The first integral is

$$\begin{aligned} 4\pi\varepsilon_0 \Phi_1(\vec{x}) = \int_0^{2\pi}\int_{-1}^{+1}\int_0^a \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} \frac{4\pi}{2l+1}\frac{(r')^l r^l}{a^{2l+1}}\left(\frac{a^{2l+1}}{r_>^{2l+1}} - 1\right) Y_{lm}(\theta, \phi)\, Y_{lm}^*(\theta', \phi') \\ \times \frac{\lambda}{2\pi (r')^2}\left[\delta(\mu' - 1) + \delta(\mu' + 1)\right] S(b - r')\ (r')^2 dr' d\mu' d\phi' \end{aligned}$$

The integral over ϕ' gives zero unless $m = 0$, in which case we get 2π, so we have

$$\int_{-1}^{+1}\int_0^a \sum_{l=0}^{\infty} \frac{4\pi\lambda}{2l+1}\frac{(r')^l r^l}{a^{2l+1}}\left(\frac{a^{2l+1}}{r_>^{2l+1}} - 1\right) N_{l0}^2 P_l(\mu)\, P_l(\mu')\left[\delta(\mu' - 1) + \delta(\mu' + 1)\right] S(b - r')\ dr' d\mu'$$

Next we use the sifting property of the delta-functions to do the integral over μ'

$$\int_0^a \sum_{l=0}^{\infty} \frac{4\pi\lambda}{2l+1}\frac{(r')^l r^l}{a^{2l+1}}\left(\frac{a^{2l+1}}{r_>^{2l+1}} - 1\right) N_{l0}^2 P_l(\mu)\left[P_l(1) + P_l(-1)\right] S(b - r')\ dr'$$

Since $P_l(-1) = (-1)^l$, the result is zero unless l is even, when the square bracket equals 2. Finally we do the integral over r'. The integrand is zero for $r' > b$, so we have

$$\sum_{l=0,\text{ even}}^{\infty} \frac{8\pi\lambda}{2l+1} \frac{r^l}{a^{2l+1}} \int_0^b (r')^l \left(\frac{a^{2l+1}}{r_>^{2l+1}} - 1\right) dr' N_{l0}^2 P_l(\mu)$$

Looking at the integral over r' by itself, we note that if $r < b$ we have to split the integral into two parts: $r' < r$ and $r' > r$.

$$\begin{aligned}
\int_0^b (r')^l \left(\frac{a^{2l+1}}{r_>^{2l+1}} - 1\right) dr' &= \int_0^r (r')^l \left(\frac{a^{2l+1}}{r^{2l+1}} - 1\right) dr' + \int_r^b (r')^l \left(\frac{a^{2l+1}}{(r')^{2l+1}} - 1\right) dr' \\
&= \frac{r^{l+1}}{l+1}\left(\frac{a^{2l+1}}{r^{2l+1}} - 1\right) - \frac{a^{2l+1}}{l}\left(\frac{1}{b^l} - \frac{1}{r^l}\right) - \left(\frac{b^{l+1} - r^{l+1}}{l+1}\right) \\
&= \frac{1}{l+1}\left(\frac{a^{2l+1}}{r^l} - r^{l+1}\right) - \frac{a^{2l+1}}{l}\left(\frac{1}{b^l} - \frac{1}{r^l}\right) - \left(\frac{b^{l+1} - r^{l+1}}{l+1}\right) \\
&= \frac{a^{2l+1}}{r^l}\frac{(2l+1)}{l(l+1)} - \frac{a^{2l+1}}{lb^l} - \frac{b^{l+1}}{l+1} \qquad \text{for} \quad r < b, \quad l > 0
\end{aligned}$$

Note that this fails for $l = 0$ because of the l in the denominator. For $l = 0$, we get

$$\begin{aligned}
\int_0^b \left(\frac{a}{r_>} - 1\right) dr' &= \int_0^r \left(\frac{a}{r} - 1\right) dr' + \int_r^b \left(\frac{a}{r'} - 1\right) dr' \\
&= a - r + a\ln\frac{b}{r} - (b - r) \\
&= a - b + a\ln\frac{b}{r}
\end{aligned}$$

If $r > b$ we have, $r = r_>$ throughout the range of integration, and then we have

$$\int_0^b (r')^l \left(\frac{a^{2l+1}}{r^{2l+1}} - 1\right) dr' = \frac{b^{l+1}}{l+1}\left(\frac{a^{2l+1}}{r^{2l+1}} - 1\right) \qquad \text{for} \quad r > b$$

Including the factor of $1/4\pi\varepsilon_0$, the first term in the potential for $r < b$ is

$$\begin{aligned}
\Phi_1(\vec{x}) &= \frac{2\lambda}{\varepsilon_0}\left\{\frac{1}{a}\left(a - b + a\ln\frac{b}{r}\right) N_{00}^2 P_0(\mu) + \right. \\
&\qquad \left. \sum_{l=2,\text{ even}}^{\infty} \frac{N_{l0}^2}{2l+1}\frac{r^l}{a^{2l+1}}\left[\frac{a^{2l+1}}{r^l}\frac{(2l+1)}{l(l+1)} - \frac{a^{2l+1}}{lb^l} - \frac{b^{l+1}}{l+1}\right] P_l(\mu)\right\} \\
&= \frac{\lambda}{2\pi\varepsilon_0}\left\{\left(1 - \frac{b}{a} + \ln\frac{b}{r}\right) + \sum_{l=2,\text{ even}}^{\infty}\left[\frac{(2l+1)}{l(l+1)} - \frac{r^l}{lb^l} - \frac{r^l}{a^{2l+1}}\frac{b^{l+1}}{l+1}\right] P_l(\mu)\right\}
\end{aligned}$$

and for $r > b$

$$\Phi_1(\vec{x}) = \sum_{l=0,\text{even}}^{\infty} \frac{\lambda}{2\pi\varepsilon_0} \frac{r^l}{a^{2l+1}} \frac{b^{l+1}}{l+1} \left(\frac{a^{2l+1}}{r^{2l+1}} - 1 \right) P_l(\mu)$$

The log term is expected near a line charge, and even l denotes the reflection symmetry about the equatorial ($z = 0$) plane.. However, for $r > b$, we get

$$\frac{2\lambda b}{4\pi\varepsilon_0} \left(\frac{1}{r} - \frac{1}{a} \right) + \frac{\lambda}{2\pi\varepsilon_0} \sum_{l=2,\text{ even}}^{\infty} \frac{r^l}{a^{2l+1}} \frac{b^{l+1}}{l+1} \left(\frac{a^{2l+1}}{r^{2l+1}} - 1 \right) P_l(\mu)$$

The first term is the potential due to a point charge $q = 2\lambda b$ at the center of a grounded sphere of radius a. Again this is what we would expect. Finally we note that the potential is continuous at $r = b$.

We still need to evaluate the second term in (8). It contains

$$\frac{\partial G}{\partial n'} = \hat{n}' \cdot \vec{\nabla} G = \frac{\partial G}{\partial r'}$$

where the outward normal from the volume at the surface is $\hat{n} = \hat{r}$. Thus we have

$$\begin{aligned} -4\pi\Phi_2(\vec{x}) &= \int_0^{2\pi} \int_{-1}^{+1} \Phi(\vec{x}') \frac{\partial G_D(\vec{x}, \vec{x}')}{\partial n'} dA' \\ &= V \left(\int_0^{\pi} - \int_{\pi}^{2\pi} \right) \int_{-1}^{+1} \frac{\partial}{\partial r'} \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \frac{4\pi}{2l+1} \frac{(r')^l r^l}{a^{2l+1}} \left(\frac{a^{2l+1}}{r_>^{2l+1}} - 1 \right) \Bigg|_{r'=a} \\ &\qquad \times Y_{lm}(\theta, \phi) Y_{lm}^*(\theta', \phi') a^2 d\mu' d\phi' \end{aligned}$$

On the outer surface, $r' = a > r$, so $r_> = r'$ and we have

$$V \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \frac{4\pi}{2l+1} Y_{lm}(\theta, \phi) \left(\int_0^{\pi} - \int_{\pi}^{2\pi} \right) \int_{-1}^{+1} \frac{\partial}{\partial r'} \frac{(r')^l r^l}{a^{2l+1}} \left(\frac{a^{2l+1}}{(r')^{2l+1}} - 1 \right) \Bigg|_{r'=a} \times Y_{lm}^*(\theta', \phi') a^2 d\mu' d\phi'$$

Doing the integral over ϕ' first, we get zero if $m = 0$ and for $m \neq 0$

$$\begin{aligned} \left(\int_0^{\pi} - \int_{\pi}^{2\pi} \right) e^{-im\phi'} d\phi' &= \frac{e^{-im\phi'}}{-im} \Bigg|_0^{\pi} - \frac{e^{-im\phi'}}{-im} \Bigg|_{\pi}^{2\pi} \\ &= \frac{e^{-im\pi} - 1}{-im} - \frac{e^{-im2\pi} - e^{-im\pi}}{-im} \\ &= \frac{2}{im} \left(1 - (-1)^m \right) \end{aligned}$$

So we get zero for m even and $4/im$ for m odd. Since $l \geq m$, the sum over l now starts at 1. Thus the integral has reduced to

$$V \sum_{l=1}^{\infty} \sum_{m=-l,\text{ odd}}^{+l} \frac{16\pi N_{lm}}{2l+1} \frac{Y_{lm}(\theta, \phi)}{im} \frac{\partial}{\partial r'} \frac{r^l}{a^{2l+1}} \left(\frac{a^{2l+1}}{(r')^{l+1}} - (r')^l \right) \Bigg|_{r'=a} \int_{-1}^{+1} P_l^m(\mu') a^2 d\mu'$$

$P_l^m(\mu)$ is an even function of μ if $l+m$ is even and an odd function if $l+m$ is odd. Thus for the integral over μ' to be non-zero, we need $l+m$ to be even, and thus l odd. Let's call the integral I_{lm}.The derivative is

$$\frac{\partial}{\partial r'}\frac{r^l}{a^{2l+1}}\left(\frac{a^{2l+1}}{(r')^{l+1}}-(r')^l\right)\Bigg|_{r'=a}=-(l+1)\frac{r^l}{a^{l+2}}-l\frac{r^l}{a^{2l+1}}a^{l-1}=-(2l+1)\frac{r^l}{a^{l+2}}$$

and thus we have

$$-4\pi\Phi_2(\vec{x})=-V\sum_{l=1,\text{odd}}^{\infty}\sum_{m=-l,\text{ odd}}^{+l}(2l+1)\left(\frac{r}{a}\right)^l\frac{16\pi N_{lm}}{2l+1}\frac{Y_{lm}(\theta,\phi)}{im}I_{lm}$$

Finally we should tidy this up by combining the positive and negative m terms. Remember that $Y_{l,-m}=(-1)^m Y_{lm}^*$ and $N_{l,-m}P_l^{-m}=(-1)^m N_{lm}P_l^m$. That makes $N_{l,-m}I_{l,-m}=(-1)^m N_{lm}I_{lm}$. Thus

$$\begin{aligned}N_{lm}Y_{lm}(\theta,\phi)\frac{I_{lm}}{im}+\frac{N_{l,-m}Y_{l,-m}}{-im}I_{l,-m} &= N_{lm}^2P_l^m(\mu)e^{im\phi}\frac{I_{lm}}{im}+\frac{N_{l,m}^2P_l^m(\mu)}{-im}I_{lm}e^{-im\phi}\\ &= 2N_{lm}^2P_l^m(\mu)\frac{I_{lm}}{m}\sin m\phi\end{aligned}$$

and so the second term is

$$\begin{aligned}\Phi_2(\vec{x}) &= \frac{32\pi V}{4\pi}\sum_{l=1,\text{odd}}^{\infty}\sum_{m=1,\text{ odd}}^{+l}\left(\frac{r}{a}\right)^l N_{lm}^2P_l^m(\mu)\frac{I_{lm}}{m}\sin m\phi\\ &= 8V\sum_{l=1,\text{odd}}^{\infty}\sum_{m=1,\text{ odd}}^{+l}\left(\frac{r}{a}\right)^l\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}P_l^m(\mu)\frac{I_{lm}}{m}\sin m\phi\\ &= \frac{2V}{\pi}\sum_{l=1,\text{odd}}^{\infty}\sum_{m=1,\text{ odd}}^{+l}\left(\frac{r}{a}\right)^l\left(\frac{2l+1}{m}\right)\frac{(l-m)!}{(l+m)!}I_{lm}P_l^m(\mu)\sin m\phi\end{aligned}$$

The total potential is the sum of the two terms:

$$\begin{aligned}\Phi(\vec{x}) &= \frac{\lambda}{2\pi\varepsilon_0}\left\{\left(1-\frac{b}{a}+\ln\frac{b}{r}\right)+\sum_{l=2,\text{ even}}^{\infty}\left[\frac{(2l+1)}{l(l+1)}-\frac{r^l}{lb^l}-\frac{r^l}{a^{2l+1}}\frac{b^{l+1}}{l+1}\right]P_l(\mu)\right\}\\ &\quad+\frac{2V}{\pi}\sum_{l=1,\text{odd}}^{\infty}\sum_{m=1,\text{ odd}}^{+l}\frac{(2l+1)}{m}\frac{(l-m)!}{(l+m)!}I_{lm}\left(\frac{r}{a}\right)^l P_l^m(\mu)\sin m\phi\ \ ...\ r<b\end{aligned}$$

and

$$\begin{aligned}\Phi(\vec{x}) &= \frac{\lambda b}{2\pi\varepsilon_0}\left(\frac{1}{r}-\frac{1}{a}\right)+\frac{\lambda}{2\pi\varepsilon_0}\sum_{l=2,\text{ even}}^{\infty}\frac{r^l}{a^{2l+1}}\frac{b^{l+1}}{l+1}\left(\frac{a^{2l+1}}{r^{2l+1}}-1\right)P_l(\mu)\\ &\quad+\frac{2V}{\pi}\sum_{l=1,\text{odd}}^{\infty}\sum_{m=1,\text{ odd}}^{+l}\frac{(2l+1)}{m}\frac{(l-m)!}{(l+m)!}I_{lm}\left(\frac{r}{a}\right)^l P_l^m(\mu)\sin m\phi\ \ ...\ r>b\end{aligned}$$

Compare the second term, in method and result, with pages 393-395 in Lea.

Let's evaluate the first few terms in our result.

$$I_{11} = \int_0^\pi (-\sin\theta)\sin\theta d\theta = -\frac{\pi}{2}$$

Thus

$$\begin{aligned}
\Phi(\vec{x}) &= \frac{\lambda}{2\pi\varepsilon_0}\left\{\left(1-\frac{b}{a}+\ln\frac{b}{r}\right)+\left[\frac{5}{6}-\frac{r^2}{2b^2}-\frac{r^2}{a^5}\frac{b^3}{3}\right]\frac{1}{2}\left(3\mu^2-1\right)\right\} \\
&\quad -\frac{2V}{\pi}3\frac{1}{2}\frac{\pi}{2}\left(\frac{r}{a}\right)(-\sin\theta)\sin\phi+\cdots \\
&= \frac{\lambda}{4\pi\varepsilon_0}\left\{\left(1-\frac{b}{a}+\ln\frac{b}{r}\right)+\left[5-\frac{r^2}{b^2}\left(3-2\frac{b^3}{a^3}\right)\right]\frac{\left(3\cos^2\theta-1\right)}{12}\right\} \\
&\quad +\frac{3V}{2}\frac{r}{a}\sin\theta\sin\phi+\cdots \quad ...\ r<b
\end{aligned}$$

and for $r > b$

$$\begin{aligned}
\Phi(\vec{x}) &= \frac{\lambda b}{2\pi\varepsilon_0}\left(\frac{1}{r}-\frac{1}{a}\right)+\frac{\lambda}{2\pi\varepsilon_0}\frac{r^2}{a^5}\frac{b^3}{3}\left(\frac{a^5}{r^5}-1\right)\frac{1}{2}\left(3\mu^2-1\right)+\frac{3V}{2}\frac{r}{a}\sin\theta\sin\phi+\cdots \\
&= \frac{\lambda b}{2\pi\varepsilon_0}\left\{\frac{1}{r}-\frac{1}{a}+\frac{r^2}{a^5}\frac{b^3}{6}\left(\frac{a^5}{r^5}-1\right)\left(3\cos^2\theta-1\right)\right\}+\frac{3V}{2}\frac{r}{a}\sin\theta\sin\phi+\cdots
\end{aligned}$$

2.3 Dirichlet Green's function in Cylindrical coordinates

Here we will find the Green's function for the interior of an infinitely long tube of radius a. Because we are going to use the result to find a potential that does depend on z, we need the three-dimensional Green's function. ((**1**) Lea Figure C.6) (**2**) We choose cylindrical coordinates with z-axis along the axis of the tube, and (**3**) place our unit point charge at $\vec{x}'$ with (fixed) coordinates (ρ', ϕ', z'). (**4**) The solutions of Laplace's equation in this coordinate system are

$$e^{im\phi}\left\{\begin{array}{c} J_m(k\rho) \\ N_m(k\rho) \end{array}\right\}e^{\pm kz}$$

or

$$e^{im\phi}\left\{\begin{array}{c} I_m(k\rho) \\ K_m(k\rho) \end{array}\right\}\left\{\begin{array}{c} \sin kz \\ \cos kz \end{array}\right\}$$

The sets of functions $\{e^{im\phi}$ times $J_m(k\rho)$ or $N_m(k\rho)\}$ and $\{e^{im\phi}$ times $\sin kz$ or $\cos kz\}$ form complete orthogonal sets of functions, so we may divide our space in either z or ρ. (**5**) Let's divide in ρ. (The other choice is in Lea pg 521-525.) (**6**)Then in region I, $\rho < \rho'$, we need a function that is finite at $\rho = 0$, which is I. Here we have no boundaries at finite z, so we have no way to determine specific values for k. So we have

$$G_I(\vec{x}, \vec{x}') = \sum_{m=-\infty}^{+\infty}\int_{-\infty}^{\infty} A_m(k)\, e^{ikz} I_m(k\rho)\, e^{im\phi} dk$$

In region II, $a > \rho > \rho'$, we have both functions.

$$G_{II}(\vec{x},\vec{x}') = \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{\infty} B_m(k)\, e^{ikz} \left[I_m(k\rho) + C_m K_m(k\rho) \right] e^{im\phi} dk$$

At $\rho = a$, $G_{II} = 0$, so

$$0 = \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{\infty} B_m(k)\, e^{ikz} \left[I_m(ka) + C_m K_m(ka) \right] e^{im\phi} dk$$

By orthogonality of the $e^{\pm ikz} e^{im\phi}$ in z and ϕ, we can equate each term separately to zero, so we get

$$C_m = -\frac{I_m(ka)}{K_m(ka)}$$

and thus

$$G_{II}(\vec{x},\vec{x}') = \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{\infty} B_m(k)\, e^{ikz} \left[I_m(k\rho) - \frac{I_m(ka)}{K_m(ka)} K_m(k\rho) \right] e^{im\phi} dk$$

(**7**) Now we use continuity of the potential at $\rho = \rho'$ to get

$$\begin{aligned} & \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{\infty} A_m(k)\, e^{ikz} I_m(k\rho')\, e^{im\phi} dk \\ = & \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{\infty} B_m(k)\, e^{ikz} \left[I_m(k\rho') - \frac{I_m(ka)}{K_m(ka)} K_m(k\rho') \right] e^{im\phi} dk \end{aligned}$$

and again by orthogonality we have

$$A_m = B_m(k) \left[1 - \frac{I_m(ka)}{I_m(k\rho')} \frac{K_m(k\rho')}{K_m(ka)} \right]$$

Thus

$$G_I(\vec{x},\vec{x}') = \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{\infty} B_m(k)\, e^{ikz} \left[1 - \frac{I_m(ka)}{I_m(k\rho')} \frac{K_m(k\rho')}{K_m(ka)} \right] I_m(k\rho)\, e^{im\phi} dk$$

(**8**) Next we make use of the differential equation, expressing the delta function on the right in cylindrical coordinates:

$$\nabla^2 G = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho G + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} G + \frac{\partial^2}{\partial z^2} G = -\frac{4\pi}{\rho}\delta(\rho-\rho')\,\delta(\phi-\phi')\,\delta(z-z')$$

$$\begin{aligned} & \sum_{m=-\infty}^{+\infty} \int_{0}^{\infty} B_m(k)\, e^{ikz} e^{im\phi} \left\{ \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial g(\rho)}{\partial\rho} - \frac{m^2}{\rho^2} g + k^2 g \right\} dk \\ = & -\frac{4\pi}{\rho}\delta(\rho-\rho')\,\delta(\phi-\phi')\,\delta(z-z') \end{aligned}$$

where

$$g_I \equiv B_m(k)\left[1 - \frac{I_m(ka)}{I_m(k\rho')}\frac{K_m(k\rho')}{K_m(ka)}\right] I_m(k\rho)$$

and

$$g_{II} \equiv B_m(k)\left[I_m(k\rho) - \frac{I_m(ka)}{K_m(ka)} K_m(k\rho)\right]$$

Now we multiply both sides by $e^{-im'\phi}$ and integrate from 0 to 2π in ϕ. On the RHS we use the sifting property, and on the left hand side, we use orthogonality of the $e^{im\phi}$.

$$2\pi \int_{-\infty}^{\infty} B_{m'}(k)\, e^{ikz} \left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial g(\rho)}{\partial\rho} - \frac{(m')^2}{\rho^2} g + k^2 g\right\} dk = -\frac{4\pi}{\rho}\delta(\rho-\rho')\,\delta(z-z')\, e^{-im'\phi'}$$

Now we can drop the prime on m. Similarly we multiply by $e^{-ik'z}$ and integrate over all z to get

$$\begin{aligned}
\int_{-\infty}^{\infty} B_m(k)\, 2\pi\delta(k-k') \left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial g(\rho)}{\partial\rho} - \frac{m^2}{\rho^2} g + k^2 g\right\} dk &= -\frac{2}{\rho}\delta(\rho-\rho')\, e^{-ik'z'} e^{-im\phi'} \\
B_m(k')\,\pi \left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial g(\rho)}{\partial\rho} - \frac{m^2}{\rho^2} g + (k')^2 g\right\} &= -\frac{1}{\rho}\delta(\rho-\rho')\, e^{-ik'z'} e^{-im\phi'}
\end{aligned}$$

Drop the primes on the k', and relabel, remembering that z' and ϕ' are fixed for the moment.

$$B_m(k) = \beta_m(k)\, e^{-ikz'} e^{-im\phi'}$$

(**9**) Finally we multiply both sides by ρ and integrate across the boundary at $\rho = \rho'$

$$\int_{\rho'-\varepsilon}^{\rho'+\varepsilon} \beta_m(k)\,\pi \left\{\frac{\partial}{\partial\rho}\rho\frac{\partial g(\rho)}{\partial\rho} - \frac{m^2}{\rho} g + k^2\rho g\right\} d\rho = -\int_{\rho'-\varepsilon}^{\rho'+\varepsilon} \delta(\rho-\rho')\, d\rho$$

Making use of the continuity of g, we get

$$\begin{aligned}
\beta_m(k)\ \rho\frac{\partial g(\rho)}{\partial\rho}\bigg|_{\rho'-\varepsilon}^{\rho'+\varepsilon} &= -\frac{1}{\pi} \\
\beta_m(k)\, k\rho' \left\{\left[I'_m(k\rho') - \frac{I_m(ka)}{K_m(ka)} K'_m(k\rho')\right] - \left[1 - \frac{I_m(ka)}{I_m(k\rho')}\frac{K_m(k\rho')}{K_m(ka)}\right] I'_m(k\rho')\right\} &= -\frac{1}{\pi} \\
\beta_m(k)\, k\rho' \left\{-\frac{I_m(ka)}{K_m(ka)} K'_m(k\rho') + I_m(ka)\,\frac{K_m(k\rho')}{K_m(ka)}\frac{I'_m(k\rho')}{I_m(k\rho')}\right\} &= -\frac{1}{\pi}
\end{aligned}$$

Thus

$$\beta_m(k) = \frac{1}{\pi k\rho'}\frac{K_m(ka)\, I_m(k\rho')}{I_m(ka)\left[K'_m(k\rho')\, I_m(k\rho') - I'_m(k\rho')\, K_m(k\rho')\right]}$$

The denominator is the Wronskian of the modified Bessel differential equation. To evaluate it, let's use the large argument form of the functions.

$$K_m(x) \simeq \sqrt{\frac{\pi}{2x}} e^{-x}; \quad I_m(x) \simeq \frac{1}{\sqrt{2\pi x}} e^{x}$$

$$\begin{aligned} & K_m'(x) I_m(x) - I_m'(x) K_m(x) \\ = & \sqrt{\frac{\pi}{2}} e^{-x} \left(-\frac{1}{\sqrt{x}} - \frac{1}{2x^{3/2}} \right) \sqrt{\frac{1}{2\pi x}} e^{x} - \sqrt{\frac{\pi}{2}} e^{x} \left(\frac{1}{\sqrt{x}} - \frac{1}{2x^{3/2}} \right) \frac{e^{-x}}{\sqrt{2\pi x}} \\ = & -\frac{1}{x} \end{aligned}$$

Thus

$$\beta_m(k) = -\frac{1}{\pi} \frac{K_m(ka) I_m(k\rho')}{I_m(ka)}$$

and then

$$\begin{aligned} G_I(\vec{x}, \vec{x}') &= \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{\infty} \frac{-1}{\pi} \frac{K_m(ka) I_m(k\rho')}{I_m(ka)} e^{ik(z-z')} \\ & \quad \times \left[1 - \frac{I_m(ka)}{I_m(k\rho')} \frac{K_m(k\rho')}{K_m(ka)} \right] I_m(k\rho) e^{im(\phi-\phi')} dk \\ & \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{\infty} \frac{1}{\pi} e^{ik(z-z')} \left[I_m(ka) K_m(k\rho') - K_m(ka) I_m(k\rho') \right] \frac{I_m(k\rho)}{I_m(ka)} e^{im(\phi-\phi')} dk \end{aligned}$$

and

$$\begin{aligned} G_{II}(\vec{x}, \vec{x}') &= \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{\infty} \frac{-1}{\pi} \frac{K_m(ka) I_m(k\rho')}{I_m(ka)} e^{ik(z-z')} \left[I_m(k\rho) - \frac{I_m(ka)}{K_m(ka)} K_m(k\rho) \right] e^{im(\phi-\phi')} dk \\ &= \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{I_m(k\rho')}{I_m(ka)} e^{ik(z-z')} \left[I_m(ka) K_m(k\rho) - K_m(ka) I_m(k\rho) \right] e^{im(\phi-\phi')} dk \end{aligned}$$

or, in both cases,

$$G(\vec{x}, \vec{x}') = \frac{1}{\pi} \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{\infty} \frac{I_m(k\rho_<)}{I_m(ka)} \left[I_m(ka) K_m(k\rho_>) - K_m(ka) I_m(k\rho_>) \right] e^{ik(z-z')} e^{im(\phi-\phi')} dk$$

(**10**) TheGreen's function has dimensions 1/length (from the integral over k) as expected. It also displays the necessary symmetry.

Now let's solve the same problem that is in the book with V_0 on the wall between $z = -a$ and $z = +a$. The outward normal is $\hat{n} = +\hat{\rho}$, and at $\rho' = a$,

$\rho_> = \rho'$, so

$$\begin{aligned}
\Phi(\vec{x}) &= -\frac{V_0}{4\pi}\int_0^{2\pi}\int_{-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}\int_{-a}^{a}\frac{1}{\pi}\frac{I_m(k\rho)}{I_m(ka)}e^{ik(z-z')}e^{im(\phi-\phi')} \\
&\qquad \times \frac{\partial}{\partial\rho'}\left[I_m(ka)K_m(k\rho') - K_m(ka)I_m(k\rho')\right]\Big|_{\rho'=a} dkad\phi'dz' \\
&= -\frac{V_0}{4\pi}2\pi\int_{-\infty}^{+\infty}\int_{-a}^{a}\frac{1}{\pi}\frac{I_0(k\rho)}{I_0(ka)}e^{ik(z-z')}k\left[I_0(ka)K_0'(ka) - K_0(ka)I_0'(ka)\right]dkadz'
\end{aligned}$$

The integral over ϕ' gives zero unless $m = 0$, so Φ is independent of ϕ, as expected. Again we have the Wronskian of the modified Bessel functions, so

$$\begin{aligned}
\Phi(\vec{x}) &= +\frac{V_0}{2\pi}\int_{-\infty}^{+\infty}\int_{-a}^{a}k\frac{I_0(k\rho)}{I_0(ka)}e^{ik(z-z')}\frac{1}{ka}dkadz' \\
&= \frac{V_0}{2\pi}\int_{-\infty}^{+\infty}\frac{I_0(k\rho)}{I_0(ka)}\left.\frac{e^{ik(z-z')}}{-ik}\right|_{-a}^{+a}dk \\
&= \frac{V_0}{2\pi}\int_{-\infty}^{+\infty}\frac{I_0(k\rho)}{I_0(ka)}\frac{e^{ikz}\left(e^{-ika} - e^{ika}\right)}{-ik}dk \\
&= \frac{V_0}{\pi}\int_{-\infty}^{+\infty}\frac{I_0(k\rho)}{I_0(ka)}\frac{e^{ikz}\sin ka}{k}dk \\
&= \frac{2V_0}{\pi}\int_0^{+\infty}\frac{I_0(k\rho)}{I_0(ka)}\frac{\sin ka\cos kz}{k}dk
\end{aligned}$$

The result is dimensionally correct. Let's see how this looks for $\rho \to a$.

$$\begin{aligned}
\Phi(a,z) &= \frac{V_0}{\pi}\int_{-\infty}^{+\infty}\frac{e^{ikz}\sin ka}{k}dk = \frac{V_0}{\pi}\int_0^{a}\int_{-\infty}^{+\infty}e^{ikz}\cos ku\ dkdu \\
&= \frac{V_0}{\pi}2\pi\int_0^{a}\frac{\delta(z-u)+\delta(z+u)}{2}du \\
&= V_0 \text{ if } 0 \le z \le a \ \ \text{(from the first delta function)} \\
&= V_0 \ \text{ if } -a \le z \le 0 \ \ \text{(from the second delta function)} \\
&= 0 \ \text{ otherwise}
\end{aligned}$$

Comparing with the result in the book,

$$\Phi(\vec{x}) = 2V_0\sum_{n=1}^{\infty}\frac{J_0(x_{0n}\rho/a)}{x_{0n}J_1(x_{0n})}\left[1 - e^{-x_{0n}}\cosh\left(x_{0n}\frac{z}{a}\right)\right]$$

We see that the main difference is that we have an integral instead of an infinite sum. We can evaluate the integral numerically. For $z = a/2$, we have

$\rho = 0.1a : \frac{2}{\pi}\int_0^{+\infty}\frac{I_0(.1k)}{I_0(k)}\frac{\sin k\cos(k/2)}{k}dk = 0.768\,66$

$\rho = 0.2a : \frac{2}{\pi}\int_0^{+\infty}\frac{I_0(.2k)}{I_0(k)}\frac{\sin k\cos k/2}{k}dk = 0.774\,76$

$\rho = 0.3a : \frac{2}{\pi}\int_0^{+\infty}\frac{I_0(.3k)}{I_0(k)}\frac{\sin k\cos k/2}{k}dk = 0.785\,18$
$\rho = 0.4a : \frac{2}{\pi}\int_0^{+\infty}\frac{I_0(.4k)}{I_0(k)}\frac{\sin k\cos k/2}{k}dk = 0.800\,28$
$\rho = 0.5a : \frac{2}{\pi}\int_0^{+\infty}\frac{I_0(.5k)}{I_0(k)}\frac{\sin k\cos k/2}{k}dk = 0.820\,58$
$\rho = 0.6a : \frac{2}{\pi}\int_0^{+\infty}\frac{I_0(.6k)}{I_0(k)}\frac{\sin k\cos k/2}{k}dk = 0.846\,6$
$\rho = 0.7a : \frac{2}{\pi}\int_0^{+\infty}\frac{I_0(.7k)}{I_0(k)}\frac{\sin k\cos k/2}{k}dk = 0.878\,66$
$\rho = 0.8a : \frac{2}{\pi}\int_0^{+\infty}\frac{I_0(.8k)}{I_0(k)}\frac{\sin k\cos k/2}{k}dk = 0.916\,36$
$\rho = 0.9a : \frac{2}{\pi}\int_0^{+\infty}\frac{I_0(.9k)}{I_0(k)}\frac{\sin k\cos k/2}{k}dk = 0.957\,96$
For $z/a = 2$, $\rho = 0.1a$ we have $\frac{2}{\pi}\int_0^{+\infty}\frac{I_0(.1k)}{I_0(k)}\frac{\sin k\cos(2k)}{k}dk = 6.877\,8\times 10^{-2}$
Compare with Lea Figure C.7. The results are the same.

3 Expansion in eigenfunctions without division.

This method is outlined in Jackson section 3.12 and Lea sections C.2 and C.7.5. It gives us a single function, valid everywhere in the region, at the expense of an extra sum. We'll begin with a one-dimensional problem, and then extend the result to three dimensions.

We start with the Sturm-Liouville equation

$$\frac{d}{dx}\left(f\frac{dy}{dx}\right) - g\left(x\right)y + \lambda w\left(x\right)y = 0 \tag{9}$$

where we either have boundary conditions (Dirichlet or Neumann) that guarantee orthogonality of the solutions, or else $f(x) = 0$ on the boundary. The solutions are the eigenfunctions $y_n(x)$ with eigenvalues λ_n. Let's normalize the eigenfunctions (as we did with the Y_{lm}) so that they satisfy the orthonormality relation

$$\int_a^b w\left(x\right)y_n\left(x\right)y_m\left(x\right)dx = \delta_{nm} \tag{10}$$

Now we look for a Green's functions that satisfies the differential equation

$$\frac{d}{dx}\left(f\frac{dG\left(x,x'\right)}{dx}\right) - g\left(x\right)G\left(x,x'\right) + \lambda w\left(x\right)G\left(x,x'\right) = -4\pi\delta\left(x-x'\right) \tag{11}$$

where the constant λ does not equal λ_n for any n. We expand G in the eigenfunctions y_n :

$$G\left(x,x'\right) = \sum_n \gamma_n\left(x'\right)y_n\left(x\right)$$

and stuff into the differential equation (11).

$$\begin{aligned}&\sum_n \gamma_n\left(x'\right)\frac{d}{dx}\left(f\frac{dy_n\left(x\right)}{dx}\right) - g\left(x\right)\sum_n\gamma_n\left(x'\right)y_n\left(x\right) + \lambda w\left(x\right)\sum_n\gamma_n\left(x'\right)y_n\left(x\right)\\ =\;& -4\pi\delta\left(x-x'\right)\end{aligned}$$

$$\sum_n \gamma_n(x')\left[-\lambda_n w(x) y_n(x)\right] + \lambda w(x) \sum_n \gamma_n(x') y_n(x) = -4\pi\delta(x-x')$$

where we used the differential equation (9) to replace the derivatives of y_n. Now multiply both sides by $y_m(x)$ and integrate over the range a to b. We use orthogonality (eqn 10) on the LHS and the sifting property on the RHS.

$$(\lambda - \lambda_m)\gamma_m(x') = -4\pi y_m(x')$$

Thus

$$\gamma_m(x') = -4\pi \frac{y_m(x')}{(\lambda - \lambda_m)}$$

(now it is clear why λ cannot equal any λ_n) and thus

$$G(x.x') = 4\pi \sum_n \frac{y_n(x') y_n(x)}{\lambda_n - \lambda} \tag{12}$$

To extend this result to potential problems in 3-d, we start with equation (3). The eigenvalue λ in this equation is zero, so we need eigenfunctions that satisfy an equation with non-zero λ, and that is the Helmholtz equation.

$$\nabla^2 f(\vec{x}) + k^2 f(\vec{x}) = 0$$

Let's use this to find the Dirichlet G for the interior of a rectangular box measuring a by b by c. We put the origin at one corner, with Cartesian axes along the three sides. Then the eigenfunctions we want are the solutions $f = X(x)Y(y)Z(z)$ of

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} + k^2 = 0$$

with $X(0) = X(a) = 0$, $Y(0) = Y(b) = 0$ and $Z(0) = Z(c) = 0$. Then $X = \sin n\pi x/a$, $Y = \sin m\pi y/b$ and $Z = \sin p\pi z/c$, and

$$k_{nmp}^2 = \pi^2\left(\frac{n^2}{a^2} + \frac{m^2}{b^2} + \frac{p^2}{c^2}\right)$$

We still need to normalize the eigenfunctions.

$$\int_0^a \sin^2 \frac{n\pi x}{a} dx = \frac{a}{2}$$

so the normalized function is

$$X_n = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}$$

and so

$$f_{nmp} = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} \sqrt{\frac{2}{b}} \sin \frac{m\pi y}{b} \sqrt{\frac{2}{c}} \sin \frac{p\pi z}{c}$$

Now we just stuff this result into (12) (extended to 3-d) with $\lambda = 0$.

$$
\begin{aligned}
G\left(\vec{x}, \vec{x}'\right) &= 4\pi \left(\sqrt{\frac{8}{abc}}\right)^2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \frac{\sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \sin \frac{p\pi z}{c} \sin \frac{n\pi x'}{a} \sin \frac{m\pi y'}{b} \sin \frac{p\pi z'}{c}}{\pi^2 \left(\frac{n^2}{a^2} + \frac{m^2}{b^2} + \frac{p^2}{c^2}\right)} \\
&= \frac{32}{\pi abc} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \frac{\sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \sin \frac{p\pi z}{c} \sin \frac{n\pi x'}{a} \sin \frac{m\pi y'}{b} \sin \frac{p\pi z'}{c}}{\left(\frac{n^2}{a^2} + \frac{m^2}{b^2} + \frac{p^2}{c^2}\right)}
\end{aligned}
$$

Again check that G has dimensions of [1/length]. The benefit of this approach is that we have a single expression that we can use in the whole region. The disadvantage is that we have an extra sum.

Suppose our box, with its walls remaining grounded, contains a sheet of charge with surface charge density σ that extends from $x = a/4$ to $x = 3a/4$ and $y = b/4$ to $y = 3b/4$ at $z = c/2$. Find the potential.

In this case we have the volume integral, but the surface integral is zero. So

$$
\begin{aligned}
\Phi\left(\vec{x}\right) &= \frac{1}{4\pi\varepsilon_0} \int_0^a \int_0^b \int_0^c G\left(\vec{x}, \vec{x}'\right) \rho\left(\vec{x}'\right) dx' dy' dz' \\
&= \frac{\sigma}{4\pi\varepsilon_0} \int_{a/4}^{3a/4} \int_{b/4}^{3b/4} \int_0^c \frac{32}{\pi abc} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \frac{\sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \sin \frac{p\pi z}{c} \sin \frac{n\pi x'}{a} \sin \frac{m\pi y'}{b} \sin \frac{p\pi z'}{c}}{\left(\frac{n^2}{a^2} + \frac{m^2}{b^2} + \frac{p^2}{c^2}\right)} \\
&\qquad \times \delta\left(z' - \frac{c}{2}\right) dx' dy' dz' \\
&= \frac{\sigma}{4\pi\varepsilon_0} \frac{32}{\pi abc} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \frac{\sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \sin \frac{p\pi z}{c} \sin \frac{p\pi}{2}}{\left(\frac{n^2}{a^2} + \frac{m^2}{b^2} + \frac{p^2}{c^2}\right)} \\
&\qquad \times \int_{a/4}^{3a/4} \sin \frac{n\pi x'}{a} dx' \int_{b/4}^{3b/4} \sin \frac{m\pi y'}{b} dy'
\end{aligned}
$$

$$
\begin{aligned}
\Phi\left(\vec{x}\right) &= \frac{\sigma}{4\pi\varepsilon_0} \frac{32}{\pi abc} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \frac{\sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \sin \frac{p\pi z}{c} \sin \frac{p\pi}{2}}{nm\left(\frac{n^2}{a^2} + \frac{m^2}{b^2} + \frac{p^2}{c^2}\right)} \\
&\qquad \times \frac{a}{n\pi} \cos \frac{n\pi x'}{a}\Bigg|_{a/4}^{3a/4} \frac{b}{m\pi} \cos \frac{m\pi x'}{b}\Bigg|_{b/4}^{3b/4} \\
&= \frac{8\sigma}{\varepsilon_0 \pi^4 c} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{p=1,\mathrm{odd}}^{\infty} (-1)^{\left(\frac{p-1}{2}\right)} \frac{\sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \sin \frac{p\pi z}{c}}{nm\left(\frac{n^2}{a^2} + \frac{m^2}{b^2} + \frac{p^2}{c^2}\right)} \\
&\qquad \times \left(\cos \frac{3n\pi}{4} - \cos \frac{n\pi}{4}\right)\left(\cos \frac{3m\pi}{4} - \cos \frac{m\pi}{4}\right)
\end{aligned}
$$

$$\begin{aligned}\Phi(\vec{x}) &= \frac{32\sigma}{\varepsilon_0\pi^4 c}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{p=1,\text{odd}}^{\infty}(-1)^{(p-1)/2}\frac{\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\sin\frac{p\pi z}{c}}{nm\left(\frac{n^2}{a^2}+\frac{m^2}{b^2}+\frac{p^2}{c^2}\right)}\sin\frac{n\pi}{2}\sin\frac{n\pi}{4}\sin\frac{m\pi}{2}\sin\frac{m\pi}{4}\\
&= \frac{32\sigma}{\varepsilon_0\pi^4 c}\sum_{\substack{n=1\\ \text{odd}}}^{\infty}\sum_{\substack{m=1\\ \text{odd}}}^{\infty}\sum_{\substack{p=1\\ \text{odd}}}^{\infty}(-1)^{(p-1)/2+(n-1)/2+(m-1)/2}\frac{\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\sin\frac{p\pi z}{c}}{nm\left(\frac{n^2}{a^2}+\frac{m^2}{b^2}+\frac{p^2}{c^2}\right)}\sin\frac{n\pi}{4}\sin\frac{m\pi}{4}\end{aligned}$$

If $n = 2l + 1$, then

$$\begin{aligned}\sin\frac{n\pi}{4} &= \sin\frac{(2l+1)\pi}{4} = \sin\left(\frac{l\pi}{2}+\frac{\pi}{4}\right) = \sin\frac{l\pi}{2}\cos\frac{\pi}{4}+\cos\frac{l\pi}{2}\sin\frac{\pi}{4}\\
&= \frac{\sqrt{2}}{2}\left(\sin\frac{l\pi}{2}+\cos\frac{l\pi}{2}\right)\\
&= \frac{\sqrt{2}}{2}(-1)^{(l-1)/2}\ \text{ if } l \text{ is odd}\\
&= \frac{\sqrt{2}}{2}(-1)^{l/2}\ \text{ if } l \text{ is even}\end{aligned}$$

Thus

$$\begin{aligned}\Phi(\vec{x}) &= \frac{\sigma}{\varepsilon_0}\frac{16}{\pi^4 c}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^{p+n+m+q}}{\left(\frac{(2n+1)^2}{a^2}+\frac{(2m+1)^2}{b^2}+\frac{(2p+1)^2}{c^2}\right)}\\
&\qquad\times\sin\frac{(2n+1)\pi x}{a}\sin\frac{(2m+1)\pi y}{b}\sin\frac{(2p+1)\pi z}{c}\end{aligned}$$

where

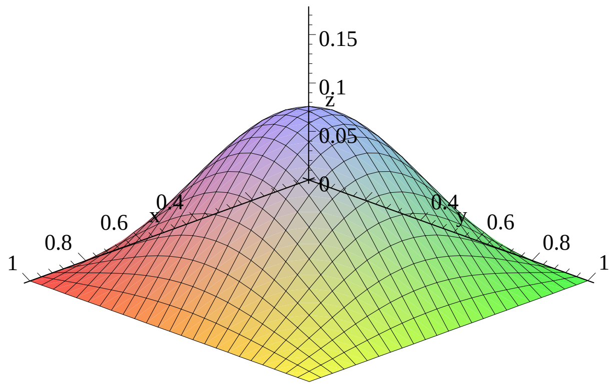
$$q = \text{integer part}\left(\frac{n}{2}\right) + \text{integer part}\left(\frac{m}{2}\right)$$

Check the dimensions!

At $z = c/4$ we have

$$\begin{aligned}\frac{\Phi(x,y)}{16\sigma/(\varepsilon_0\pi^4 c)} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^{p+n+m+q}}{(2n+1)(2m+1)\left(\frac{(2n+1)^2}{a^2}+\frac{(2m+1)^2}{b^2}+\frac{(2p+1)^2}{c^2}\right)}\\
&\qquad\times\sin\frac{(2n+1)\pi x}{a}\sin\frac{(2m+1)\pi y}{b}\sin\frac{(2p+1)\pi}{4}\end{aligned}$$

The diagram shows the dimesnionless potential, compute using values of n, m and $p$from zero to five.


0.15
0.1
z
0.05
0
0.4
x
0.6
0.8
1
0.4
y
0.6
0.8
1