

Grad B Drift

In a single frame with only a magnetic field $\vec{B} = B(x) \hat{z}$ we have the equation of motion:

$$\frac{d\vec{p}}{dt} = q \frac{\vec{v}}{c} \times \vec{B}$$

or

$$\frac{d\vec{v}}{dt} = -\frac{q}{\gamma m c} \vec{B} \times \vec{v}$$

We expect the result to be a gyration plus constant drift, so the equation of motion should take the form

$$\frac{d}{dt} (\vec{v} - \vec{v}_D) = \vec{\omega} \times (\vec{v} - \vec{v}_D) \quad (1)$$

With the usual result

$$\vec{\omega} = -\frac{q}{\gamma m c} \vec{B}_0$$

we may rewrite our equation by Taylor expanding \vec{B} :

$$\begin{aligned} \frac{d\vec{v}}{dt} &= \frac{d}{dt} (\vec{v} - \vec{v}_D) = -\frac{q}{\gamma m c} \left(\vec{B}_0 + [(\vec{x} - \vec{x}_0) \cdot \vec{\nabla}] \vec{B} \Big|_0 + \dots \right) \times \vec{v} \\ &= \vec{\omega} \times \vec{v} - \frac{\omega}{B_0} [(\vec{x} - \vec{x}_0) \cdot \vec{\nabla}] \vec{B} \Big|_0 \times \vec{v} + \dots \end{aligned} \quad (2)$$

Now we make the assumption that the field is slowly varying in the sense that

$$\frac{B}{|\vec{\nabla} B|} \gg r_l = \frac{|\vec{v} - \vec{v}_D|}{\omega}$$

so we may drop higher terms in the expansion. We time average to obtain the drift., since

$$\vec{v} = \vec{\omega} \times \vec{r} + \vec{v}_D$$

and the gyrational part has components that oscillate in time. Similarly

$$\vec{x} = \vec{x}_0 + \vec{r} + \vec{v}_D t$$

where \vec{r} has components that oscilate in time. Thus the second term on the right hand side of (2) is:

$$\begin{aligned} -\vec{\omega} \times \vec{v}_D &= -\left\langle \frac{\omega}{B_0} [(\vec{x} - \vec{x}_0) \cdot \vec{\nabla}] \vec{B} \Big|_0 \times \vec{v}_0 \right\rangle \\ &= \left\langle \vec{\omega} \times \left[\frac{1}{B_0} [(\vec{x} - \vec{x}_0) \cdot \vec{\nabla}] B \Big|_0 \vec{v}_0 \right] \right\rangle \end{aligned}$$

Now we time average to get:

$$\vec{v}_D = -\left\langle \frac{1}{B_0} [(\vec{x} - \vec{x}_0) \cdot \vec{\nabla}] B \Big|_0 \vec{\omega} \times (\vec{x} - \vec{x}_0) \right\rangle$$

The operator

$$\left[(\vec{x} - \vec{x}_0) \cdot \vec{\nabla} \right] \vec{B} \Big|_0$$

selects out only the component of $\vec{x} - \vec{x}_0$ parallel to the gradient of \vec{B} , the x component here, and so only one component of $\vec{\omega} \times \vec{r}$ in \vec{v} will contribute to the time average.

$$\begin{aligned} \vec{v}_D &= - \left\langle \frac{1}{B_0} (x - x_0)^2 \vec{\omega} \times \vec{\nabla} B \right\rangle = - \frac{r_l^2}{2B_0} \vec{\omega} \times \vec{\nabla} B \\ &= \frac{\omega r_l^2}{2B_0^2} \vec{B}_0 \times \vec{\nabla} B \end{aligned}$$

which is Jackson's result.