The field Lagrangian

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1 Lagrangian for a continuous system

Let's start with an example from mechanics to get the big idea. The physical system of interest is a string of length L and mass per unit length μ,fixed at both ends, and under tension T. Choose $x-$axis along the unperturbed string, and $y-$axis perpendicular to it. When the string is vibrating, its kinetic energy is:

$$T=\int_0^L \frac{1}{2}v^2 dm=\int_0^L \frac{1}{2}\left(\frac{\partial y}{\partial t}\right)^2 \mu dx=\frac{\mu}{2}\int_0^L \hat{y}^2 dx$$

To get the potential energy, we use the method of virtual work. The net force on a string segment has components:

$$dF_x=T\cos\theta_1-T\cos\theta_2\approx 0$$

and

$$\begin{aligned} dF_y &= T\sin\theta_1-T\sin\theta_2\approx T\left(\tan\theta_1-\tan\theta_2\right)=T\left(\left.\frac{\partial y}{\partial x}\right|_{x+dx}-\left.\frac{\partial y}{\partial x}\right|_x\right)\\ &= T\frac{\partial^2 y}{\partial x^2}dx \end{aligned}$$

Then the virtual work is

$$\delta W=\int_0^L dF_y\delta y=\int_0^L T\frac{\partial^2 y}{\partial x^2}dx\delta y$$

Now integrate by parts, and make use of the fixed end condition:

$$\delta W=T\left[\left.\delta y y'\right|_0^L-\int_0^L \delta\left(y'\right)y'dx\right]=-\frac{T}{2}\int_0^L \delta\left(y'^2\right)dx$$

Then if $\vec{F}=-\vec{\nabla}V$, then $\vec{F}\cdot d\vec{s}=-\delta V$, and $V=-\int\vec{F}\cdot d\vec{s}$. Here

$$V=-\int\delta W=\frac{T}{2}\int_0^L\left(y'\right)^2dx$$

The Lagrangian for the string is:

$$\begin{aligned} L &= T - V = \int_0^L \frac{1}{2}\left(\mu\hat{y}^2 - T\left(y'\right)^2\right)dx \\ &= \int_0^L \pounds\, dx \end{aligned}$$

where

$$\pounds = \frac{1}{2}\left(\mu\hat{y}^2 - T\left(y'\right)^2\right) \tag{1}$$

is the Lagrangian density for the string.

The action is

$$A = \int L dt = \int_{t_1}^{t_2}\int_0^L \frac{1}{2}\left(\mu\hat{y}^2 - T\left(y'\right)^2\right)dxdt$$

Taking the variation of the action, we get

$$\delta A = \int_{t_1}^{t_2}\int_0^L \left[\frac{\partial\pounds}{\partial\hat{y}}\delta\hat{y} + \frac{\partial\pounds}{\partial y'}\delta y' + \frac{\partial\pounds}{\partial y}\delta y\right]dxdt$$

Integrating by parts gives:

$$\delta A = \int_{t_1}^{t_2}\int_0^L \left[-\frac{d}{dt}\frac{\partial\pounds}{\partial\hat{y}} - \frac{d}{dx}\frac{\partial\pounds}{\partial y'} + \frac{\partial\pounds}{\partial y}\right]\delta y dxdt$$

Thus for the action to be an extremum, we need

$$-\frac{d}{dt}\frac{\partial\pounds}{\partial\hat{y}} - \frac{d}{dx}\frac{\partial\pounds}{\partial y'} + \frac{\partial\pounds}{\partial y} = 0$$

Using equation (1), we find:

$$\frac{d}{dt}\left(2\mu\hat{y}\right) + \frac{d}{dx}\left(-2Ty'\right) - 0 = 0$$

or

$$\mu y - Ty'' = 0$$

which is the wave equation for the string.

Analternative approach is to write the string displacement as a sum over normal modes:

$$y = \sum_n y_n\left(t\right)\sin\frac{n\pi x}{L}$$

Then the Lagrangian density is:

$$\pounds = \sum_n\sum_p \mu\hat{y}_n\hat{y}_p\sin\frac{n\pi x}{L}\sin\frac{p\pi x}{L} - T\frac{\pi^2}{L^2}npy_ny_p\cos\frac{n\pi x}{L}\cos\frac{p\pi x}{L}$$

and then the Lagrangian is

$$L = \int \pounds dx$$

When we integrate over x, The only terms that survive are those with $n = p$.

$$L = \frac{1}{2}\sum_n \mu\hat{y}_n^2 - T\frac{n^2\pi^2}{L^2}y_n^2 \tag{2}$$

The mode amplitudes y_n act as the generalized coordinates for the string. Then Lagrange's equations are

$$\frac{d}{dt}\frac{\partial L}{\partial \hat{y}_n} - \frac{\partial L}{\partial y_n} = \mu \ddot{y}_n - T\frac{n^2\pi^2}{L^2}y_n = 0$$

which is the harmonic oscillator equation with frequency $\omega_n = \frac{T}{\mu}\frac{n\pi}{L}$

2 Lagrangian for the electromagnetic field

Now we want to do a similar treatment for the EM field. We want a Lagrangian density such that the action

$$S = \int \mathcal{L} d^4x$$

is a Lorentz invariant, and where $\mathcal{L}$ is a function of the fields. The "obvious" invariant to try is $\mathcal{L}_{\text{guess}} = F^{\alpha\beta}F_{\alpha\beta}$. (Recall this is proportional to $E^2 - B^2$, an "energy-like" thing.) Here the components of the potential A^α are the "normal modes" - they behave like the y_n in the previous section. Then Lagrange's equations are:

$$\frac{d}{dx^\mu}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial A^\alpha}{\partial x^\mu}\right)} - \frac{\partial \mathcal{L}}{\partial A^\alpha} = 0 \tag{3}$$

To evaluate this, note that

$$\mathcal{L}_{\text{guess}} = \left(\partial^\varepsilon A^\beta - \partial^\beta A^\varepsilon\right) g_{\varepsilon\gamma}g_{\beta\delta}\left(\partial^\gamma A^\delta - \partial^\delta A^\gamma\right)$$

and so

$$\begin{aligned}
\frac{\partial \mathcal{L}_{\text{guess}}}{\partial\left(\partial^\mu A^\alpha\right)} &= \left(\delta^\varepsilon_\mu\delta^\beta_\alpha - \delta^\beta_\mu\delta^\varepsilon_\alpha\right) g_{\varepsilon\gamma}g_{\beta\delta}\left(\partial^\gamma A^\delta - \partial^\delta A^\gamma\right) + \left(\partial^\varepsilon A^\beta - \partial^\beta A^\varepsilon\right) g_{\varepsilon\gamma}g_{\beta\delta}\left(\delta^\gamma_\mu\delta^\delta_\alpha - \delta^\delta_\mu\delta^\gamma_\alpha\right) \\
&= \left(g_{\mu\gamma}g_{a\delta} - g_{\alpha\gamma}g_{\mu\delta}\right)\left(\partial^\gamma A^\delta - \partial^\delta A^\gamma\right) + \left(g_{\varepsilon\mu}g_{\beta\alpha} - g_{\varepsilon\alpha}g_{\beta\mu}\right)\left(\partial^\varepsilon A^\beta - \partial^\beta A^\varepsilon\right) \\
&= \partial_\mu A_\alpha - \partial_\alpha A_\mu - \partial_\alpha A_\mu + \partial_\mu A_\alpha + \partial_\mu A_\alpha - \partial_\alpha A_\mu - \partial_\alpha A_\mu + \partial_\mu A_\alpha \\
&= 4\left(\partial_\mu A_\alpha - \partial_\alpha A_\mu\right) = 4F_{\mu\alpha}
\end{aligned}$$

while

$$\frac{\partial \mathcal{L}_{\text{guess}}}{\partial A^\alpha} \equiv 0$$

so equations (3) become:

$$\partial^\mu F_{\mu\alpha} = 0$$

which are Maxwell's equations in the absence of sources. We can fix up the Lagrangian by adding the interaction term $\frac{1}{c}J_\alpha A^\alpha$. Thus

$$\mathcal{L} = -\frac{1}{16\pi}F^{\alpha\beta}F_{\alpha\beta} - \frac{1}{c}J_\alpha A^\alpha \tag{4}$$

With this Lagrangian density:

$$\frac{\partial \mathcal{L}}{\partial A^\alpha} = -\frac{1}{c}J_\alpha$$

and Lagrange's equations become:

$$-\frac{1}{4\pi}\partial^\mu F_{\mu\alpha} + \frac{1}{c}J_\alpha = 0$$

or

$$\partial^\mu F_{\mu\alpha} = \frac{4\pi}{c} J_\alpha$$

which are the two Maxwell equations that include sources.

3 The Hamiltonian

Now we form the Hamiltonian. First let's look at the string. Using equation (2):

$$\begin{aligned} H &= \sum_n \frac{\partial \mathcal{L}}{\partial \hat{y}_n} \hat{y}_n - \mathcal{L} \\ &= \frac{1}{2} \sum_n 2\mu \hat{y}_n \hat{y}_n - \left(\mu \hat{y}_n^2 - T \frac{n^2\pi^2}{L^2} y_n^2 \right) \\ &= \frac{1}{2} \sum_n \mu \hat{y}_n^2 + T \frac{n^2\pi^2}{L^2} y_n^2 \\ &= \sum_n E_n \end{aligned}$$

where E_n is the total (kinetic plus potential) energy per mode.

By analogy, we get for the EM field system without sources:

$$\begin{aligned} T^{\alpha\beta} &= \frac{\partial \mathcal{L}}{\partial \left(\partial_\alpha A_\mu \right)} \partial^\beta A_\mu - g^{\alpha\beta} \mathcal{L} \\ &= -\frac{1}{4\pi} F^{\mu\alpha} \partial^\beta A_\mu - g^{\alpha\beta} \left(-\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} \right) \\ &= \frac{1}{4\pi} \left(-F^{\mu\alpha} \partial^\beta A_\mu + \frac{1}{4} g^{\alpha\beta} F^{\mu\nu} F_{\mu\nu} \right) \\ &= \frac{1}{4\pi} \left(F^{\alpha\mu} \partial^\beta A_\mu + \frac{1}{4} g^{\alpha\beta} F^{\mu\nu} F_{\mu\nu} \right) \end{aligned}$$

This tensor is not symmetric, because the first term contains only one half of the field tensor: $\partial^\beta A_\mu$ rather than F_μ^β. The conservation laws require that the energy tensor be symmetric, so we have to modify the result.

4 The energy-momentum tensor

Recall that the field energy (non-relativistic) is $\varepsilon = \frac{1}{8\pi}\left(E^2 + B^2\right)$ and the Poynting theorem may be written:

$$\frac{\partial}{\partial t} \frac{E^2 + B^2}{8\pi} + \vec{\nabla} \cdot \frac{c}{4\pi} \vec{E} \times \vec{B} + \vec{j} \cdot \vec{E} = 0 \tag{5}$$

We'd like to express this result in covariant form. We obviously need something quadratic in the fields. For example:

$$
\begin{aligned}
F_{\mu}^{\alpha}F^{\mu\beta} &= \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \\
&= \begin{pmatrix} E_x^2+E_y^2+E_z^2 & E_yB_z-E_zB_y & -E_xB_z+E_zB_x & E_xB_y-E_yB_x \\ E_yB_z-E_zB_y & -E_x^2+B_z^2+B_y^2 & -E_xE_y-B_yB_x & -E_xE_z-B_zB_x \\ -E_xB_z+E_zB_x & -E_xE_y-B_yB_x & -E_y^2+B_z^2+B_x^2 & -E_yE_z-B_zB_y \\ E_xB_y-E_yB_x & -E_xE_z-B_zB_x & -E_yE_z-B_zB_y & -E_z^2+B_y^2+B_x^2 \end{pmatrix} \\
&= \begin{pmatrix} E^2 & \left(\tilde{\mathbf{E}}\times\tilde{\mathbf{B}}\right)_x & \left(\tilde{\mathbf{E}}\times\tilde{\mathbf{B}}\right)_y & \left(\tilde{\mathbf{E}}\times\tilde{\mathbf{B}}\right)_z \\ \left(\tilde{\mathbf{E}}\times\tilde{\mathbf{B}}\right)_x & -E_x^2+B_z^2+B_y^2 & -E_xE_y-B_yB_x & -E_xE_z-B_zB_x \\ \left(\tilde{\mathbf{E}}\times\tilde{\mathbf{B}}\right)_y & -E_xE_y-B_yB_x & -E_y^2+B_z^2+B_x^2 & -E_yE_z-B_zB_y \\ \left(\tilde{\mathbf{E}}\times\tilde{\mathbf{B}}\right)_z & -E_xE_z-B_zB_x & -E_yE_z-B_zB_y & -E_z^2+B_y^2+B_x^2 \end{pmatrix}
\end{aligned}
$$

Now we'd like the $(0,0)$ component to be the energy ε. We can get that if we add the tensor $\frac{1}{4}g^{\alpha\beta}F^{\mu\nu}F_{\mu\nu} = \frac{1}{4}g^{\alpha\beta}2\left(B^2-E^2\right)$. Then

$$
\begin{aligned}
\Theta^{\alpha\beta} &= \frac{1}{4\pi}\left\{F_{\mu}^{\alpha}F^{\mu\beta} + \frac{1}{4}g^{\alpha\beta}F^{\mu\nu}F_{\mu\nu}\right\} \\
&= \frac{1}{4\pi}\left(\begin{matrix} E^2+\frac{B^2-E^2}{2} & \left(\vec{E}\times\vec{B}\right)_x & \left(\vec{E}\times\vec{B}\right)_y & \left(\vec{E}\times\vec{B}\right)_z \\ \left(\vec{E}\times\vec{B}\right)_x & -E_x^2+B_z^2+B_y^2-\frac{B^2-E^2}{2} & -E_xE_y-B_yB_x & -E_xE_z-B_zB_x \\ \left(\vec{E}\times\vec{B}\right)_y & -E_xE_y-B_yB_x & -E_y^2+B_z^2+B_x^2-\frac{B^2-E^2}{2} & -E_yE_z-B_zB_y \\ \left(\vec{E}\times\vec{B}\right)_z & -E_xE_z-B_zB_x & -E_yE_z-B_zB_y & -E_z^2+B_y^2+B_x^2-\frac{B^2}{} \end{matrix}\right. \\
&= \frac{1}{4\pi}\begin{pmatrix} \frac{B^2+E^2}{2} & \left(\vec{E}\times\vec{B}\right)_x & \left(\vec{E}\times\vec{B}\right)_y & \left(\vec{E}\times\vec{B}\right)_z \\ \left(\vec{E}\times\vec{B}\right)_x & \frac{B^2+E^2}{2}-E_x^2-B_x^2 & -E_xE_y-B_yB_x & -E_xE_z-B_zB_x \\ \left(\vec{E}\times\vec{B}\right)_y & -E_xE_y-B_yB_x & \frac{B^2+E^2}{2}-E_y^2-B_y^2 & -E_yE_z-B_zB_y \\ \left(\vec{E}\times\vec{B}\right)_z & -E_xE_z-B_zB_x & -E_yE_z-B_zB_y & \frac{B^2+E^2}{2}-E_z^2-B_z^2 \end{pmatrix}
\end{aligned}
$$

Then

$$\partial_\alpha\Theta^{\alpha 0} = \frac{\partial}{\partial ct}\frac{B^2+E^2}{8\pi} + \vec{\nabla}\cdot\frac{\vec{E}\times\vec{B}}{4\pi}$$

which is part of equation (5). On the right hand side we have -$\frac{1}{c}\vec{j}\cdot\vec{E} = \frac{1}{c}J_\alpha F^{\alpha 0}$. Thus we have:

$$\partial_\alpha\Theta^{\alpha 0} = \frac{1}{c}J_\alpha F^{\alpha 0}$$

and so we guess that the full set of conservation laws are given by:

$$\partial_\alpha \Theta^{\alpha\beta} = \frac{1}{c} J_\alpha F^{\alpha\beta}$$

I leave it to you to show that $\beta = i$ gives momentum conservation (Jackson equation 6.122).

5 Angular momentum

Cross products are not proper vectors. They are pseudo-vectors because they do not transform properly under reflections. Thus it is usually better to express quantities such as angular momentum of a particle ($\vec{L} = \vec{r} \times \vec{p}$) as antisymmetric tensors. For example the tensor

$$M_{ik} = \sum_{particles} (x_i p_k - x_k p_i)$$

has 3 independent components: the components of the vector $\vec{L}$.

Extending this idea, let's look at the tensor

$$M^{\alpha\beta} = \sum \left(x^\alpha p^\beta - x^\beta p^\alpha \right)$$

The 3×3 spacelike part is the tensor M_{ik} and thus represents the angular momentum of the system. In addition:

$$M^{i0} = \sum \left(x^i \frac{\varepsilon}{c} - ctp^i \right)$$

where ε is the energy of the particle.

Conservation of angular momentum for the system is expressed as M^{ij} =constant. Thus we conjecture that the full conservation law is $M^{\alpha\beta}$ =constant. (Or equivalently $\partial_\alpha M^{\alpha\beta} = 0$) This gives for the $(i, 0)$ component:

$$\sum \left(x^i \frac{\varepsilon}{c} - ctp^i \right) = \text{constant}$$

Now if we divide through by $\sum \varepsilon$, we get:

$$\frac{\sum x^i \varepsilon}{\sum \varepsilon} = \frac{\sum c^2 tp^i}{\sum \varepsilon} + \text{ constant}$$

The term on the left hand side is the position of the center of mass,

$$\vec{r}_{CM} = \frac{\sum \gamma m \vec{x}}{\sum \gamma m}$$

while the term on the right hand side is the CM velocity times t.

$$\vec{v}_{CM} = \frac{\sum \gamma m \vec{v}}{\sum \gamma m}$$

an eminently sensible result.

To get the equivalent result for the EM field we form the tensor

$$M^{\alpha\beta\gamma} = \Theta^{\alpha\beta} x^\gamma - \Theta^{\alpha\gamma} x^\beta$$

and then the conservation laws should be given by:

$$\partial_\alpha M^{\alpha\beta\gamma} = 0$$

Taking $\beta = 0$ gives the CM motion as above.

6 The Darwin Lagrangian

The analysis above is for source-free fields. We might attempt to add the free-particle Lagrangian to get a complete description of the particle-plus-field system, but this approach fails because of retardation effects. (The fields propagate at the speed of light.) We can calculate a complete Lagrangian in a single reference frame, inlcuding relativistic effects up to order $(v/c)^2$.

Let's start with a 2-particle system. Both particles produce fields and both can move under the influence of those fields. The interaction term for charge 1 interacting with the fields due to 2 is

$$\frac{q_1}{c} u_{1\alpha} A_2^\alpha = \frac{q_1}{c} (c\gamma, \gamma\vec{v}_1) \left(\phi_2, \vec{A}_2\right) = q_1 \gamma \left(\phi - \frac{\vec{v}_1}{c} \cdot \vec{A}\right) \tag{6}$$

If we now work in a single reference frame and use the coordinate time rather than proper time as our time variable, we should drop the factor γ. We want to evaluate this expression to second order in v/c. If we work in Coulomb gauge, the potential $\phi = q_2/r$ is exact. We only need $\tilde{\mathbf{A}}$ to first order since it appears in combination with v/c. This means we can ignore retardation effects. Then:

$$\vec{A}_2 \simeq \frac{1}{c} \int \frac{\vec{j}_t}{|\vec{x} - \vec{x}'|} dV'$$

where the transverse current is

$$\begin{aligned} \vec{j}_t &= \vec{j} - \vec{j}_l = q_2 \vec{v}_2 \delta(\vec{x} - \vec{x}_2) - \frac{1}{4\pi} \vec{\nabla} \int \frac{\vec{\nabla}' \cdot q_2 \vec{v}_2 \delta(\vec{x}' - \vec{x}_2)}{|\vec{x} - \vec{x}'|} dV' \\ &= q_2 \vec{v}_2 \delta(\vec{x} - \vec{x}_2) - \frac{q_2}{4\pi} \vec{\nabla} \frac{\vec{v}_2 \cdot (\vec{x} - \vec{x}_2)}{|\vec{x} - \vec{x}_2|^3} \end{aligned}$$

Therefore:

$$\vec{A} = \frac{q_2}{r} \frac{\vec{v}_2}{c} - \frac{q_2}{4\pi c} \int \frac{1}{|\vec{x} - \vec{x}'|} \vec{\nabla}' \frac{\vec{v}_2 \cdot (\vec{x}' - \vec{x}_2)}{|\vec{x}' - \vec{x}_2|^3} dV'$$

Let's look at the integral. First make a change of origin. Let $\vec{y} = \vec{x}' - \vec{x}_2$, and with

$\vec{r} = \vec{x} - \vec{x}_2$.,then

$$\begin{aligned}
\int \frac{1}{|\vec{x} - \vec{x}'|} \vec{\nabla}' \frac{\vec{v}_2 \cdot (\vec{x}' - \vec{x}_2)}{|\vec{x}' - \vec{x}_2|^3} dV' &= \int \frac{1}{|\vec{y} - \vec{r}|} \vec{\nabla}_y \frac{\vec{v}_2 \cdot \vec{y}}{y^3} d^3 y \\
&= \left. \frac{1}{|\vec{y} - \vec{r}|} \frac{\vec{v}_2 \cdot \vec{y}}{y^3} \right|_{S \text{ at } \infty} - \int \vec{\nabla}_y \frac{1}{|\vec{y} - \vec{r}|} \frac{\vec{v}_2 \cdot \vec{y}}{y^3} d^3 y \\
&= \vec{\nabla}_r \int \frac{1}{|\vec{y} - \vec{r}|} \frac{\vec{v}_2 \cdot \vec{y}}{y^3} d^3 y \\
&= \vec{\nabla}_r \int \sum_l \frac{r_<^l}{r_>^{l+1}} P_l(\mu) (\vec{v}_2 \cdot \hat{y})\, dy d\mu d\phi \\
&= \vec{\nabla}_r 2\pi \int_{-1}^{+1} d\mu \int_0^\infty dy \sum_l \frac{r_<^l}{r_>^{l+1}} P_l(\mu)\ \vec{v}_2 \cdot \left(\cos\theta \hat{\mathbf{r}} + \sin\theta \hat{\perp} \right)
\end{aligned}$$

where we have put the polar axis for $\vec{y}$ along $\vec{r}$.

The vector $\hat{\perp} = \hat{\mathbf{x}} \cos\phi + \hat{\mathbf{y}} \sin\phi$ and so the integrtaion over ϕ renders this term zero.

Next we make use of the orthogonality of the P_l, noting that cos$\theta = P_1$. Only $l = 1$ survives the integration over μ. We obtain:

$$\begin{aligned}
\text{integral} &= \vec{\nabla}_r 2\pi \int_0^\infty \frac{r_<}{r_>^2} \frac{2}{3} \vec{v}_2 \cdot \hat{r}\ dy \\
&= \frac{4\pi}{3} \vec{\nabla}_r \left(\int_0^r \vec{v}_2 \cdot \hat{r} \frac{y}{r^2} dy + \int_r^\infty \vec{v}_2 \cdot \hat{r} \frac{r}{y^2} dy \right) \\
&= \frac{4\pi}{3} \vec{\nabla}_r (\vec{v}_2 \cdot \hat{r}) \left(\frac{1}{2} + 1 \right) \\
&= 2\pi \vec{\nabla}_r (\vec{v}_2 \cdot \hat{r})
\end{aligned}$$

And thus

$$\begin{aligned}
\vec{A}_2 &= \frac{q_2}{r} \frac{\tilde{\mathbf{v}}_2}{c} - \frac{q_2}{4\pi c} 2\pi \vec{\nabla}_r \left(\vec{v}_2 \cdot \frac{\vec{r}}{r} \right) \\
&= \frac{q_2}{c} \left(\frac{\vec{v}_2}{r} - \frac{1}{2} \left[\frac{\vec{v}_2}{r} - \frac{\vec{v}_2 \cdot \vec{r}}{r^2} \hat{\mathbf{r}} \right] \right) \\
&= \frac{q_2}{2cr} \left(\vec{v}_2 + (\vec{v}_2 \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} \right)
\end{aligned}$$

Then the interaction term for 2 particles (equation 6 with the γ dropped) is:

$$\begin{aligned}
q_1 \left(\phi - \frac{\vec{v}_1}{c} \cdot \vec{A} \right) &= q_1 \left(\frac{q_2}{r} - \frac{\vec{v}_1}{c} \cdot \frac{q_2}{2cr} \left(\vec{v}_2 + (\vec{v}_2 \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} \right) \right) \\
&= \frac{q_1 q_2}{r} \left(1 - \frac{1}{2c^2} \left[\vec{v}_1 \cdot \vec{v}_2 + (\vec{v}_1 \cdot \hat{\mathbf{r}}) (\vec{v}_2 \cdot \hat{\mathbf{r}}) \right] \right)
\end{aligned}$$

Adding this term to the kinetic energy plus electric potential terms, we have the Darwin

Lagrangian for a collection of charged particles:

$$L_D = -\frac{1}{2}\sum_i m_i c^2 \sqrt{1-\frac{v_i^2}{c^2}} - \sum_{i>j} \frac{q_i q_j}{r_{ij}} + \frac{1}{2c^2}\sum_{i>j} \frac{q_i q_j}{r_{ij}} \left(\vec{v}_i \cdot \vec{v}_j + \left(\vec{v}_i \cdot \hat{\mathbf{r}}\right)\left(\vec{v}_j \cdot \hat{\mathbf{r}}\right)\right)$$

To be consistent, we should evaluate the first term to second order in v/c, i.e. $\left(1-\frac{v^2}{c^2}\right)^{1/2} \simeq 1 - \frac{1}{2}\frac{v^2}{c^2} + \left(\frac{1}{2}\left(-\frac{1}{2}\right)\frac{v^4}{2c^4}\right)$, so finally, dropping the constant leading term, we have

$$L_D = \frac{1}{2}\sum_i m_i v_i^2 + \frac{1}{8c^2}\sum_i m_i v_i^4 - \sum_{i>j} \frac{q_i q_j}{r_{ij}} + \frac{1}{2c^2}\sum_{i>j} \frac{q_i q_j}{r_{ij}} \left(\vec{v}_i \cdot \vec{v}_j + \left(\vec{v}_i \cdot \hat{\mathbf{r}}\right)\left(\vec{v}_j \cdot \hat{\mathbf{r}}\right)\right)$$

correct to second order in v/c.