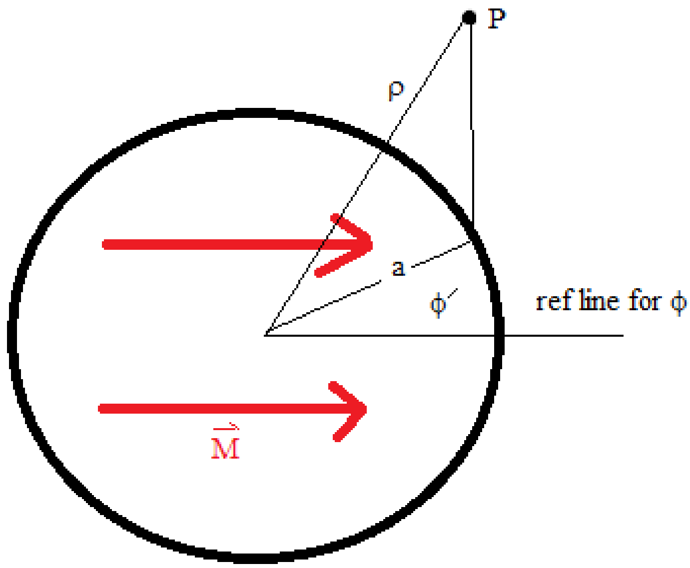
A disk magnet of radius a and height h, $h \ll a$, has uniform magnetizaton $\vec{M} = M\hat{x}$ throughout its interior. Find the magnetic fields $\vec{B}$ and $\vec{H}$.



As we have seen, there is an effective magnetic "charge density" $\vec{M} \cdot \hat{n}$ at the surface of the magnet. In this case we have

$$\vec{M} \cdot \hat{n} = M\hat{x} \cdot \hat{\rho} = M \cos \phi$$

The magnetic scalar potential is then given by (eqn 42, notes 1)

$$\begin{aligned} \Phi_m &= \frac{1}{4\pi} \int_0^h dz' \int_0^{2\pi} \frac{M \cos \phi'}{|\vec{x} - \vec{x}'|} a d\phi' \\ &= \frac{1}{4\pi} \int_0^h dz' \int_0^{2\pi} \frac{M \cos \phi'}{\sqrt{(z - z')^2 + \rho^2 + a^2 - 2\rho a \cos (\phi - \phi')}} a d\phi' \end{aligned}$$

The integral is nasty in the general case. Later in the semester we will develop tools that will allow us to find $\vec{H}$ everywhere. For now we will look at two special cases.

Outside the magnet at a great distance away, $z \gg h$, $\rho \gg a$, we have

$$\begin{aligned}
\Phi_m &= \frac{h}{4\pi}\int_0^{2\pi} \frac{M\cos\phi'}{\sqrt{z^2+\rho^2-2\rho a\cos\left(\phi-\phi'\right)}} a d\phi' \\
&= \frac{h}{4\pi R}\int_0^{2\pi} \frac{M\cos\phi'}{\sqrt{1-2\frac{\rho a}{R^2}\cos\left(\phi-\phi'\right)}} a d\phi' \quad \text{where } R^2 = z^2+\rho^2 \\
&= \frac{Mh}{4\pi R}\int_0^{2\pi} \cos\phi' \left\{1+\frac{\rho a}{R}\cos\left(\phi-\phi'\right)+\cdots\right\} a d\phi'
\end{aligned}$$

The first term integrates to zero. We expand the cosine in the second term, and make use of the orthogonality of the trig functions to get:

$$\begin{aligned}
\Phi_m &= \frac{Mh}{4\pi R}\int_0^{2\pi} \left[\cos\phi' \frac{\rho a}{R^2}\left(\cos\phi\cos\phi'+\sin\phi\sin\phi'\right)+\cdots\right] a d\phi' \\
&= \frac{Mh}{4\pi R^3}\pi a^2 x
\end{aligned}$$

With a magnetic moment $\vec{m}$ equal to $\vec{M}V = \pi a^2 h M\hat{x}$, the potential is

$$\Phi_m = \frac{\vec{m}\cdot\vec{r}}{4\pi R^3}$$

as expected for a dipole. (Compare with eqns 25 and 40 in Notes 1). In this region $\vec{B} = \mu_0\vec{H}$ because $\vec{M}$ is zero outside the magnet.

Inside the magnet and near the axis, $z - z' \ll a, \rho \ll a$ so we may expand a different way:

$$\begin{aligned}
\Phi_m &= \frac{1}{4\pi}\int_0^h dz' \int_0^{2\pi} \frac{M\cos\phi'}{\sqrt{a^2}}\left\{1-\frac{1}{2}\frac{\left(z-z'\right)^2+\rho^2-2\rho a\cos\left(\phi-\phi'\right)}{a^2}+\cdots\right\} a d\phi' dz' \\
&= \frac{hM}{4\pi}\int_0^{2\pi}\cos\phi'\left\{\rho\frac{\left(\cos\phi\cos\phi'+\sin\phi\sin\phi'\right)}{a}+\cdots\right\} d\phi' \\
&\simeq \frac{hM}{4a}x
\end{aligned}$$

The linear potential gives a uniform field:

$$\vec{H} = -\vec{\nabla}\Phi_m = -\frac{hM}{4a}\hat{x}$$

Note that $\vec{H}$ is opposite $\vec{M}$. The magnetic induction $\vec{B}$ is

$$\vec{B} = \mu_0\left(\vec{H}+\vec{M}\right) = \mu_0\vec{M}\left(1-\frac{h}{4a}\right)$$

and is in the same direction as $\vec{M}$.