

## 1 Relativistic E&M

Now we want to discover the set of tensors and the covariant relations that govern E&M.

We begin with the current 4-vector which is:

$$\mathbf{J} = (c\rho, \vec{j}) \quad (1)$$

Note that each component is dimensionally equivalent. The charge conservation law is

$$\frac{\partial J^\mu}{\partial x^\mu} = 0 = \frac{\partial c\rho}{\partial ct} + \vec{\nabla} \cdot \vec{j} \quad (2)$$

where the term on the left is the 4-divergence of the 4-vector  $\mathbf{J}$ .

Next we conjecture that the 4-potential is given by

$$\mathbf{A} = (\phi, \vec{A}) \quad (3)$$

We find that the Lorentz gauge condition is obtained by setting the 4-divergence to zero:

$$\frac{\partial A^\mu}{\partial x^\mu} = 0 = \frac{\partial \phi}{\partial ct} + \vec{\nabla} \cdot \vec{A} \quad (4)$$

We obtain the wave equation from the wave operator

$$\square^2 = \partial^\mu \partial_\mu = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2$$

Then

$$\begin{aligned} \square^2 \mathbf{A} &= \frac{4\pi}{c} \mathbf{J} \\ \partial^\mu \partial_\mu A^\alpha &= \frac{4\pi}{c} J^\alpha \end{aligned} \quad (5)$$

This gives us both of the relations we've seen before:

$$\begin{aligned} \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi &= 4\pi\rho \\ \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} &= \frac{4\pi}{c} \vec{j} \end{aligned}$$

The potential transforms with the Lorentz transformations according to the usual rule for vectors. Note particularly that the charge density is NOT a scalar!

Now for the fields. First note the relations between the fields and the potentials:

$$\vec{B} = \vec{\nabla} \times \vec{A}; \quad \vec{E} = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t} - \vec{\nabla} \phi$$

We know that curls are usually associated with antisymmetric tensors, so we define the field tensor

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha \quad (6)$$

This tensor is antisymmetric, so it has zeros along the diagonal. Recall that the gradient form has components

$$\partial_\alpha = \left( \frac{\partial}{\partial ct}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

so the corresponding vector has components

$$\partial^\alpha = \left( \frac{\partial}{\partial ct}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z} \right)$$

Thus

$$\begin{aligned} F^{10} &= \partial^1 A^0 - \partial^0 A^1 = -F^{01} \\ &= -\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial ct} = E_x \end{aligned}$$

Similarly we find  $F^{20} = E_y$  and  $F^{30} = E_z$ .

Now look at

$$\begin{aligned} F^{21} &= \partial^2 A^1 - \partial^1 A^2 = -F^{12} \\ &= -\frac{\partial A_x}{\partial y} + \frac{\partial A_y}{\partial x} = B_z \end{aligned}$$

Thus we have the components of the tensor:

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \quad (7)$$

Now we can find the transformed field components by transforming the tensor in the usual way:

$$\overline{F}^{\mu\nu} = \Lambda_\alpha^\mu \Lambda_\beta^\nu F^{\alpha\beta}$$

or, in matrix notation:

$$\begin{aligned}
\overline{\mathbf{F}} &= \Lambda F \Lambda^T \\
&= \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_x\gamma\beta & -E_x\gamma & -E_y & -E_z \\ E_x\gamma & -E_x\gamma\beta & -B_z & B_y \\ E_y\gamma - B_z\gamma\beta & -E_y\gamma\beta + B_z\gamma & 0 & -B_x \\ E_z\gamma + B_y\gamma\beta & -E_z\gamma\beta - B_y\gamma & B_x & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\gamma^2 E_x + \gamma^2 \beta^2 E_x & -E_y\gamma + B_z\gamma\beta & -E_z\gamma - B_y\gamma\beta \\ -\gamma^2 \beta^2 E_x + \gamma^2 E_x & 0 & E_y\gamma\beta - B_z\gamma & E_z\gamma\beta + B_y\gamma \\ E_y\gamma - B_z\gamma\beta & -E_y\gamma\beta + B_z\gamma & 0 & -B_x \\ E_z\gamma + B_y\gamma\beta & -E_z\gamma\beta - B_y\gamma & B_x & 0 \end{pmatrix}
\end{aligned}$$

But  $\gamma^2 (1 - \beta^2) = 1$ , so

$$\overline{\mathbf{F}} = \begin{pmatrix} 0 & E_x & -\gamma(E_y - \beta B_z) & -\gamma(E_z + \beta B_y) \\ -E_x & 0 & \gamma(E_y\beta - B_z) & \gamma(E_z\beta + B_y) \\ \gamma(E_y - \beta B_z) & -\gamma(E_y\beta - B_z) & 0 & -B_x \\ \gamma(E_z + \beta B_y) & -\gamma(E_z\beta + B_y) & B_x & 0 \end{pmatrix}$$

Thus we have the transformation rules:

The components of  $\vec{E}$  and  $\vec{B}$  parallel to the relative velocity are unchanged, and the perpendicular components transform as:

$$\vec{E}_\perp = \gamma (\vec{E}_\perp + \vec{\beta} \times \vec{B}) \quad (8)$$

and

$$\vec{B}_\perp = \gamma (\vec{B}_\perp - \vec{\beta} \times \vec{E}) \quad (9)$$

We'll come back to these results in a moment.

Next we want to write Maxwell's equations in covariant form. We can group the equations into the source-free equations

$$\begin{aligned}
\vec{\nabla} \cdot \vec{B} &= 0 \\
\vec{\nabla} \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} &= 0
\end{aligned}$$

and the equations with sources

$$\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= 4\pi\rho \\
\vec{\nabla} \times \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} &= \frac{4\pi\vec{j}}{c}
\end{aligned}$$

The second pair are obtained from the covariant relation

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta \quad (10)$$

For example, with  $\beta = 0$  we have

$$\vec{\nabla} \cdot \vec{E} = \frac{4\pi}{c} \rho$$

and with  $\beta = 1$

$$\begin{aligned} \partial_\alpha F^{\alpha 1} &= \frac{4\pi}{c} J^1 \\ \partial_0 F^{01} + \partial_2 F^{21} + \partial_3 F^{31} &= \frac{4\pi}{c} J^1 \\ \frac{\partial}{\partial t} (-E_x) + \frac{\partial}{\partial y} (B_z) + \frac{\partial}{\partial z} (-B_y) &= \frac{4\pi}{c} J_x \end{aligned}$$

This equation is the  $x$ -component of

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

The other components follow similarly.

Two obtain the source-free equations we first define the dual tensor

$$\mathcal{F}^{\alpha\beta} = \frac{1}{2} \varepsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} \quad (11)$$

where  $\varepsilon^{\alpha\beta\gamma\delta}$  is defined similarly to  $\varepsilon_{ijk}$ . ( $\varepsilon^{\alpha\beta\gamma\delta} = 1$  if  $\alpha\beta\gamma\delta =$  an even permutation of 0123, and so on.) First compute the covariant components

$$\begin{aligned} F_{\alpha\beta} &= g_{\alpha\gamma} g_{\beta\delta} F^{\gamma\delta} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \end{aligned} \quad (12)$$

and then we have, for example

$$\mathcal{F}^{01} = \frac{1}{2} \varepsilon^{01\gamma\delta} F_{\gamma\delta} = \frac{1}{2} (F_{23} - F_{32}) = F_{23} = -B_x$$

and

$$\begin{aligned}\mathcal{F}^{12} &= \frac{1}{2}\varepsilon^{12\gamma\delta}F_{\gamma\delta} = \frac{1}{2}\varepsilon^{1203}F_{03} + \frac{1}{2}\varepsilon^{1230}F_{30} \\ \frac{1}{2}(F_{03} - F_{30}) &= F_{23} = -E_z\end{aligned}$$

To get 1203 from 0123 We have to do two interchanges: First interchange 2 and 0 to get 1023 then interchange 1 and 0 to get 0123. Thus this is an even permutation and  $\varepsilon^{1203} = +1$ . Thus

$$\mathcal{F}^{12} = \frac{1}{2}(F_{03} - F_{30}) = E_z$$

Thus

$$\mathcal{F} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix} \quad (13)$$

Then the remaining two Maxwell equations may be written in covariant form as

$$\partial_\alpha \mathcal{F}^{\alpha\beta} = 0 \quad (14)$$

Finally we note the Lorentz force may be written in covariant form as

$$F^\alpha = \frac{q}{c} F^{\alpha\beta} u_\beta$$

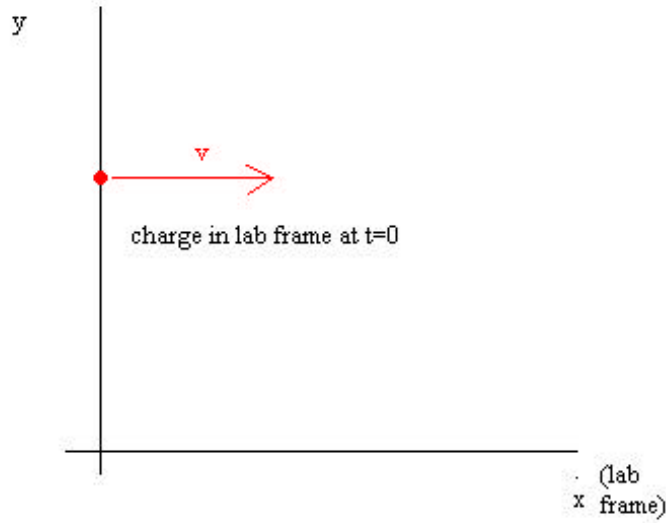
For example

$$\begin{aligned}F_x &= \frac{q}{c}(F^{1\beta}u_\beta) = \frac{q}{c}\gamma[E_x c + (-B_z)(-u_y) + B_y(-u_z)] \\ &= q\gamma\left[E_x + \left(\frac{\vec{u}}{c} \times \vec{B}\right)_x\right]\end{aligned}$$

This is the non-relativistic result in the limit  $\beta \rightarrow 0$ ,  $\gamma \rightarrow 1$ .

## 2 Fields due to a moving point charge

Let a charge  $q$  move at constant velocity  $\vec{v}$  in the lab frame, and let us set up coordinates as shown in the diagram.



In its own rest frame, the field due to a point charge obeys the usual Coulomb law:

$$\vec{E} = \frac{q}{r^2} \hat{r}, \quad \vec{B} = 0.$$

Thus the field at the origin in the lab frame is

$$\bar{E}_x = -\frac{qv\bar{t}}{(\bar{y}^2 + (v\bar{t})^2)^{3/2}}, \quad \bar{E}_y = -\frac{q\bar{y}}{(\bar{y}^2 + (v\bar{t})^2)^{3/2}}, \quad \bar{E}_z = 0$$

Now we transform to the lab frame using equations (8) and (9) with velocity in the negative- $x$ -direction:

$$\begin{aligned} E_x &= \bar{E}_x = -\frac{qv\bar{t}}{(\bar{y}^2 + (v\bar{t})^2)^{3/2}} \\ E_y &= \gamma\bar{E}_y = -\frac{\gamma q\bar{y}}{(\bar{y}^2 + (v\bar{t})^2)^{3/2}} \\ E_z &= 0 \end{aligned}$$

and

$$\begin{aligned}
 B_x &= \gamma \left( -\vec{\beta} \times \vec{E} \right)_x = 0 \\
 B_y &= \gamma \left( -\vec{\beta} \times \vec{E} \right)_y = -\gamma\beta\bar{E}_z = 0 \\
 B_z &= \gamma \left( -\vec{\beta} \times \vec{E} \right)_z = \gamma\beta\bar{E}_y = -\frac{\gamma\beta q\bar{y}}{\left(\bar{y}^2 + (v\bar{t})^2\right)^{3/2}}
 \end{aligned}$$

Now we still need to transform the coordinates to the lab frame:

$$\bar{x} = \gamma(x - vt); \quad \bar{y} = y \quad \text{and} \quad c\bar{t} = \gamma(ct - \beta x)$$

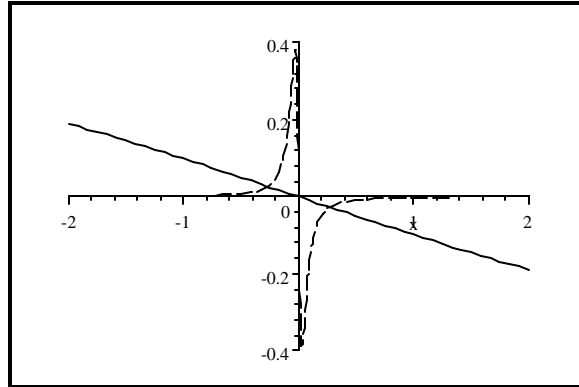
Since our observation point is at the origin,  $x = 0$ . Thus

$$\begin{aligned}
 E_x &= -\frac{q\beta\gamma ct}{\left(y^2 + (\beta\gamma ct)^2\right)^{3/2}} \\
 E_y &= -\frac{\gamma q\bar{y}}{\left(y^2 + (\beta\gamma ct)^2\right)^{3/2}} \\
 B_z &= -\frac{\gamma\beta q\bar{y}}{\left(y^2 + (\beta\gamma ct)^2\right)^{3/2}} = \beta E_y
 \end{aligned}$$

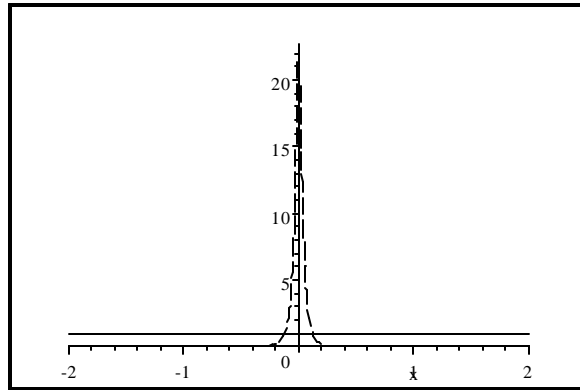
First check that these results are correct in the non-relativistic limit ( $\beta \rightarrow 0$ ,  $\gamma \rightarrow 1$ ). The interesting result is the relativistic limit as  $\gamma$  becomes very large. The fields become impulsive, (large for a very small time interval). The fields fall rapidly to zero for

$$|t| > \frac{y}{\beta\gamma c}$$

which is very small when  $\gamma$  is large. In addition, the magnitude of the fields become large for  $t \approx 0$ . The diagram shows the field components as functions of time for  $\beta = 0.1$  ( $\gamma = 1.005$ ) and  $\beta = 0.999$  ( $\gamma = 22.366$ ).



$E_x y^2 / q$  versus  $ct/y$ . Solid:  $\beta = 0.1$ . Dashed,  $\beta = 0.999$



$E_y y^2 / q$  versus  $ct/y$ . Solid:  $\beta = 0.1$ . Dashed,  $\beta = 0.999$

### 3 Invariants

We can form several invariants from the field tensors  $F^{\alpha\beta}$  and  $\mathcal{F}^{\alpha\beta}$ . First we evaluate

$$F^{\alpha\beta}F_{\alpha\beta} = B^2 - E^2$$

Thus if we have a pure electric field in one frame,  $E > B$  in all other frames. The fields maintain their dominantly electric character, and similarly for magnetic fields.

A second invariant is

$$F_{\alpha\beta}\mathcal{F}^{\alpha\beta} = -\vec{E} \cdot \vec{B}$$

Since this product is invariant, if either of  $\vec{E}$  or  $\vec{B}$  is zero in one frame, the fields in any other frame are perpendicular.

Check that these results hold true for the point charge fields we calculated above.

A special case is the EM wave, for which both invariants are zero: ( $\vec{E} \perp \vec{B}$  and  $E = B$ ). These properties are therefore true in all frames.