Physics 705

1 Tensors and relativity

1.1 History

Physical laws should not depend on the reference frame used to describe them. This idea dates back to Galileo, who recognized projectile motion as free fall in a moving reference frame. E&M presents some problems in Galilean relativity. When we push a magnet into a coil, we get an emf $\oint \vec{E} \cdot d\vec{l}$ and a current flows. But if we hold the magnet fixed and move the coil, both electric and magnetic fields contribute to the emf: $\oint \left(\vec{E} + \vec{v} \times \vec{B}\right) \cdot d\vec{l}$. These relations appear to violate Galilean relativity. Einstein noticed that if we could move at c, the electric and magnetic fields in an EM wave in our moving frame would not satisfy Maxwell's equations. His solution was to retain the idea that physical laws are independent of reference frame, and to retain Maxwell's equations, but to change our notions of space and time.

1.2 Mathematics of invariance

Our first task is to express physical laws in a coordinate-independent way. We do this using the mathematics of tensors.

Scalars are quantities that are independent of reference frame. They are also called invariants or tensors of rank zero. Examples are the mass and charge of a particle.

Vectors (tensors of rank one) are geometrical quantities, often described as *arrows*, that are also described by their components in a given reference frame. When transforming to a new reference frame, all vector components obey the same **transformation law.** For example, if we allow the set of transformations to be rotations of a Cartesian coordinate system, then the transformation law is

$$\overline{v}^i = A^{ij} v^j$$

where the components

$$A^{ij} = \cos\theta_{ij}$$

and θ_{ij} is the angle between the ith new axis and the jth old axis. (Lea section 1.1.2)

Physical laws maintain their form in all reference frames related by these transformations. For example:

$$\vec{F} = m\vec{a}, \text{ or, in index notation, } F^i = ma^i$$

Transform to a new frame:

$$\begin{aligned} \overline{F}^i &= A^{ij} F^j = A^{ij} m a^j \\ &= m A^{ij} a^j = m \overline{a}^j \end{aligned}$$

and so the law is true in the bar frame too. (Remember that m is a scalar so $m = \overline{m}$.

The dot, inner, or scalar product of two vectors is a scalar. Using the summation convention:

$$\overline{v}^i \overline{v}^i = A^{ij} v^j A^{ik} v^k = A^{ij} A^{ik} v^j v^k$$

(When using index notation, no index may ever appear in a product more than twice. Nlotice how I used k in the second transformation, to avoid repeating the j.) But since the transpose of a rotation matrix is its inverse:

$$\begin{aligned} A^{ij} A^{ik} v^j v^k &= \left(A^T\right)^{ji} A^{ik} v^j v^k \\ \overline{v}^i \overline{v}^i &= \delta^{jk} v^j v^k = v^j v^j \end{aligned}$$

and so the product is invariant.

The current in a magnetized plasma is related to the electric field through the conductivity tensor:

$$J^i = \sigma^{ik} E^k$$

where, with $z-$axis along $\vec{B}$:

$$\sigma^{ik} = i \frac{Ne^2}{\omega m} \frac{1}{1 - \omega_c^2/\omega^2} \begin{pmatrix} 1 & -i\omega_c/\omega & 0 \\ i\omega_c/\omega & 1 & 0 \\ 0 & 0 & 1 - \omega_c^2/\omega^2 \end{pmatrix}$$

But what if the $z-$axis is at an angle θ to $\vec{B}$? We need to rotate our coordinate system. To see how this rank-2 tensor transforms, we recall that the physical law must remain invariant in form:

$$\overline{J}^i = \overline{\sigma}^{ik} \overline{E}^k$$

Let's transform the vectors:

$$\overline{J}^i = A^{ik} J^k = A^{ik} \sigma^{km} E^m$$

But

$$E^m = \left(A^T\right)^{mn} \overline{E}^n$$

and so

$$\overline{J}^i = A^{ik} \sigma^{km} \left(A^T\right)^{mn} \overline{E}^n = \overline{\sigma}^{in} \overline{E}^n$$

Since this must be true no matter what the components of J and E, we have the transformation law for the tensor:

$$\overline{\sigma}^{in} = A^{ik} \sigma^{km} \left(A^T\right)^{mn} = A^{ik} A^{nm} \sigma^{km} \tag{1}$$

From this we extrapolate to the general rule:

a rank-n tensor transforms through the application of n transformation matrices.

A rank n tensor has N^n components, where N is the dimension of the space. The tensor components are labelled with n indices, and we need one transformation matrix for each.

Let's perform the transformation. With a rank-2 tensor we can do the math with matrices, using the leftmost version of equation (1). Let

$$K = i\frac{Ne^2}{\omega m}\frac{1}{1-\omega_c^2/\omega^2}$$

Then:

$$\begin{aligned}
\overline{\overline{\sigma}} &= \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} K \begin{pmatrix} 1 & -i\omega_c/\omega & 0 \\ i\omega_c/\omega & 1 & 0 \\ 0 & 0 & 1-\omega_c^2/\omega^2 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \\
&= K \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -i\frac{\omega_c}{\omega} & \sin\theta \\ i\frac{\omega_c}{\omega}\cos\theta & 1 & i\frac{\omega_c}{\omega}\sin\theta \\ -\left(1-\frac{\omega_c^2}{\omega^2}\right)\sin\theta & 0 & \left(1-\frac{\omega_c^2}{\omega^2}\right)\cos\theta \end{pmatrix} \\
&= K \begin{pmatrix} \cos^2\theta + \left(\sin^2\theta\right)\left(1-\frac{\omega_c^2}{\omega^2}\right) & -i\frac{\omega_c}{\omega}\cos\theta & \cos\theta\sin\theta - (\sin\theta)\left(1-\frac{\omega_c^2}{\omega^2}\right)\cos\theta \\ i\frac{\omega_c}{\omega}\cos\theta & 1 & i\frac{\omega_c}{\omega}\sin\theta \\ \cos\theta\sin\theta - (\sin\theta)\left(1-\frac{\omega_c^2}{\omega^2}\right)\cos\theta & -i\frac{\omega_c}{\omega}\sin\theta & \sin^2\theta + \left(1-\frac{\omega_c^2}{\omega^2}\right)\cos^2\theta \end{pmatrix} \\
&= \begin{pmatrix} 1-\left(\sin^2\theta\right)\frac{\omega_c^2}{\omega^2} & -i\frac{\omega_c}{\omega}\cos\theta & \cos\theta\sin\theta\left(\frac{\omega_c^2}{\omega^2}\right) \\ i\frac{\omega_c}{\omega}\cos\theta & 1 & i\frac{\omega_c}{\omega}\sin\theta \\ (\cos\theta\sin\theta)\frac{\omega_c^2}{\omega^2} & -i\frac{\omega_c}{\omega}\sin\theta & 1-\left(\cos^2\theta\right)\frac{\omega_c^2}{\omega^2} \end{pmatrix}
\end{aligned}$$

The tensor acts as a mapping to map one vector in our space ($\vec{E}$) to another ($\vec{J}$).

1.3 Various bits of tensor math

Contraction occurs when we sum over an index, as in the dot product or in computing $\vec{J}$ in the example above. These products are inner products. We can also form an outer product such as

$$T^{ik} = v^i u^k$$

It is easy to show that such products transform as tensors by applying the appropriate transformation law to each element of the product.

The quotient rule (Lea section A.2)

> If a tensor b is the inner or outer product of c and d, and c is a tensor with arbitrary components, then the set of components d is also a tensor.

Suppose

$$a^i = b^{ik} c^k$$

where a is a vector and b is a tensor. Then c is also a vector. Let's transform both sides.

$$\overline{a}^j = A^{ji} a^i = A^{ji} b^{ik} c^k \tag{2a}$$

But from the definition of a in the bar frame:

$$\overline{a}^j = \overline{b}^{jk} \overline{c}^k \tag{3}$$

Thus, writing $\overline{b}$ in equation (3) in terms of b, we have

$$\overline{a}^j = A^{ji} A^{km} b^{im} \overline{c}^k \tag{4}$$

Multiply equations (2a) and (4) on the right by A^{nj} and set the results equal:

$$\delta^{ni} b^{ik} c^k = \delta^{ni} A^{km} b^{im} \overline{c}^k$$

Relabelling dummy indices on the right hand side, $k \to p$, $m \to k$, we have:

$$b^{nk} c^k = b^{nm} A^{km} \overline{c}^k = b^{nk} A^{pk} \overline{c}^p$$

Relabel again, $p \to m$, and rearrange to get:

$$b^{nk} \left(c^k - A^{mk} \overline{c}^m \right) = 0$$

Since the components of b^{nk} are arbitrary, we may conclude that

$$c^k = A^{mk} \overline{c}^m = \left(A^{-1} \right)^{km} \overline{c}^m$$

or equivalently

$$\overline{c}^m = A^{mk} c^k$$

and thus c is a vector.

1.4 General tensor calculus

Lea section A.4

2 Relativity

2.1 Invariance of c

Einstein's big idea was to recognize that Maxwell's equations give c as the speed of light in every reference frame: c is *invariant.* To make this so, we must make the arena in which physics is done be a four-dimensional space-time. Events in space time are described by the coordinates

$$x^\mu = (ct, x, y, z)$$

.(It is conventional to let the Greek indices run over the four values $0, 1, 2, 3$.) The line element is

$$ds^2 = c^2 dt^2 - dr^2$$

For two events along a light ray, $dr = cdt$ and the line element is *null:*

$$ds^2 = 0 \ \text{ along a light ray.}$$

This statement is invariant because the line element is a scalar. The metric is:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

where

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$$

The interval between two events may be *timelike* ($ds^2 > 0$) or *spacelike* ($ds^2 < 0$) or *null.* We may also write

$$ds^2 = c^2 d\tau^2$$

where $d\tau$ is the proper time between the two events. If the two events occur at the same place in some reference frame, then $dr = 0$ and

$$d\tau = dt$$

in that reference frame.

> The proper time between two events is measured by an observer who sees both events happen at the same place.

All other observers measure a greater time interval between the events:

$$dt' = \sqrt{d\tau^2 + dr'^2/c^2}$$

Now if the prime frame moves at constant velocity $v\hat{x}$ with respect to the unprime frame, then

$$dx' = vdt'$$

and

$$dt'^2\left(1 - v^2/c^2\right) = d\tau^2$$

or

$$dt' = \gamma d\tau \tag{5}$$

where

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \text{ and } \beta = \frac{v}{c}$$

This is time dilation.

2.2 The Lorentz transformation

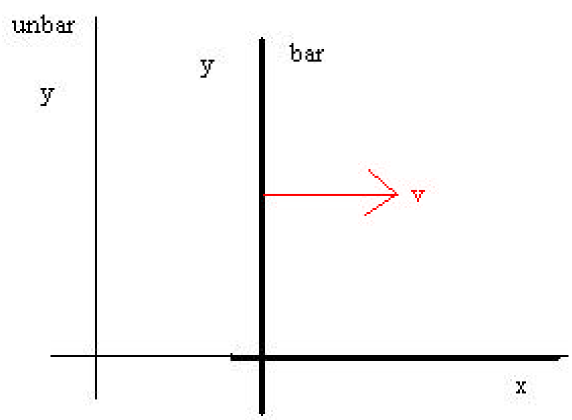
We construct the Lorentz transformation from a few simple physical principles.

(1) A particle moving uniformly in one frame moves uniformly in all inertial frames. (Newton's second law) Thus the transformation must be linear in the coordinates

$$\overline{x}^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + \xi^{\nu}$$

The constant vector ξ^{ν} is a change of origin which is not significant physically, so let's drop it.

To simplify the derivation, we put the $x-$axis along the direction of the relative velocity of the two frames.



The spatial origin of the bar frame then moves to the right at speed v in the unbar frame.

$$\overline{x} = \overline{y} = \overline{z} = 0,\ \overline{t} = T' \text{ corresponds to } x = vT,\ t = T,\ y = z = 0$$

Thus we expect

$$\begin{aligned} \overline{x} &= A\left(x + Bct\right) \\ c\overline{t} &= E\left(ct + Fx\right) \end{aligned}$$

with

$$0 = A\left(vT + BcT\right) \Rightarrow B = -\beta$$

and

$$T' = E\left(cT + FvT\right)$$

(2) The second requirement is invariance of the line element:

$$ds^2 = E^2\left(cdt + Fdx\right)^2 - A^2\left(dx - \beta cdt\right)^2 = c^2dt^2 - dx^2$$

Expanding the middle expression and equating coefficients, we have

$$\begin{array}{ll} dt^2 \text{ term:} & E^2 - A^2\beta^2 = 1 \\ \text{cross term:} & E^2F + A^2\beta = 0 \\ dx^2 \text{ term:} & E^2F^2 - A^2 = -1 \end{array}$$

We can solve these three equations for E, A and F.

$$\begin{aligned} E^2 &= 1 + A^2\beta^2 \\ F &= \frac{-A^2\beta}{E^2} = \frac{-A^2\beta}{1 + A^2\beta^2} \end{aligned}$$

and finally

$$\begin{aligned} A^2 - 1 &= E^2F^2 = \left(1 + A^2\beta^2\right)\left(\frac{-A^2\beta}{1 + A^2\beta^2}\right)^2 \\ &= \frac{A^4\beta^2}{1 + A^2\beta^2} \end{aligned}$$

So

$$\begin{aligned} A^4\beta^2 + A^2\left(1 - \beta^2\right) - 1 &= A^4\beta^2 \\ A^2 &= \frac{1}{1 - \beta^2} = \gamma^2 \end{aligned}$$

Then

$$E^2 = 1 + \gamma^2\beta^2 = \gamma^2$$

and

$$F = \frac{-\beta\gamma^2}{1 + \gamma^2\beta^2} = -\beta$$

(3) The final physical principle we invoke is that we expect time to run forward in both frames, so that E must be positive. We can make a similar argument for A. Thus the Lorentz transformation is:

$$\begin{aligned} \overline{x} &= \gamma\left(x - \beta ct\right) \\ c\overline{t} &= \gamma\left(ct - \beta x\right) \end{aligned}$$

$$\bar{y} = y \quad \text{and } \bar{z} = z$$

To see why the last two are true, think of a relativistic train on rails. The train must remain on the rails when viewed from either the train frame or the rail frame, so distances perpendicular to $\vec{v}$ cannot be affected by the transformation. Note that we get back the Galilean transformation as $\beta \to 0$.

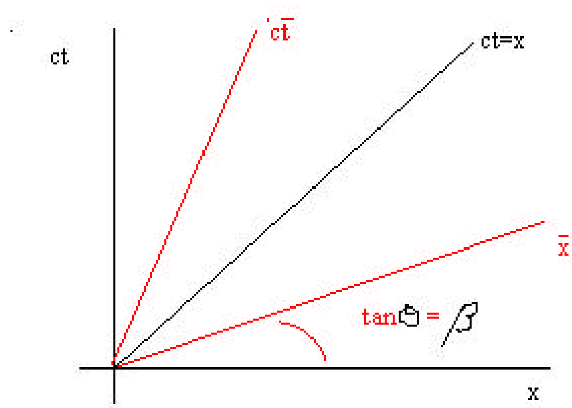
Now we write the transformation matrix

$$\Lambda^{\alpha}{}_{\beta} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

O course we must also include the rotation matrices, which appear as the space-space components. For example, for rotation about the $z-$axis, we have:

$$\Lambda_{\rm rot} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In a space-time diagram, the Lorentz transformation rotates both axes toward the diagonal (the light cone).



Beware: the scale on the bar axes is not the same as on the unbar axes! This diagram clearly shows that events that are simultaneous in the unbar frame are not simulatneous in bar, and vice-versa.

2.3 Scenario for physics

First we define the space through the coordinates and the metric. Next we define the allowable transformation laws (Lorentz transformations + rotations). These then define the vectors and tensors that we shall use. We then choose those vectors and tensors that define relevant physical quantities, and the set of equations that describe the physical processes that occur.

We start with a differential displacement:

$$d\overleftrightarrow{\mathbf{x}} = (cdt, d\vec{r})$$

To construct the veocity we differentiate with respect to the scalar proper time. Then the velocity is

$$\frac{d\overleftrightarrow{\mathbf{x}}}{d\tau} = \left(c\frac{dt}{d\tau}, \frac{d\vec{r}}{d\tau}\right) = \gamma\left(c, \vec{v}\right)$$

and its invariant length squared is

$$\overleftrightarrow{\mathbf{v}} \cdot \overleftrightarrow{\mathbf{v}} = v^{\alpha} g_{\alpha\beta} v^{\beta} = \gamma^2\left(c^2 - v^2\right) = c^2$$

A nice result!

Momentum follows in the obvious way:

$$\overleftrightarrow{\mathbf{p}} = m\overleftrightarrow{\mathbf{v}} = \gamma m\left(c, \vec{v}\right)$$

and acceleration is

$$\overleftrightarrow{\mathbf{a}} = \frac{d\overleftrightarrow{\mathbf{v}}}{d\tau} = \gamma^2\left(0, \vec{a}\right) + \frac{d\gamma}{d\tau}\left(c, \vec{v}\right)$$

where $\vec{a}$ is the 3-acceleration and

$$\begin{aligned} \frac{d\gamma}{d\tau} &= \frac{d}{d\tau}\frac{1}{\sqrt{1-\beta^2}} = -\frac{\gamma^3}{2}\left(-2\vec{\beta}\cdot\frac{d\vec{\beta}}{d\tau}\right) \\ &= \gamma^3\vec{\beta}\cdot\overrightarrow{\dot{\beta}} = \gamma^4\vec{\beta}\cdot\frac{\overrightarrow{a}}{c} \end{aligned}$$

Then,

$$\overleftrightarrow{\mathbf{a}} = \gamma^2\left(\gamma^2\left[\vec{\beta}\cdot\vec{a}\right], \quad \vec{a} + \gamma^2\left[\vec{\beta}\cdot\vec{a}\right]\vec{\beta}\right)$$

In the particle's rest frame ($\beta = 0, \gamma = 1$) the four acceleration equals the three-acceleration. The invariant product

$$\begin{aligned} \overleftrightarrow{\mathbf{a}}\cdot\overleftrightarrow{\mathbf{a}} &= \gamma^4\left(\gamma^4\left[\vec{\beta}\cdot\vec{a}\right]^2 - \left\{a^2 + 2\gamma^2\left[\vec{\beta}\cdot\vec{a}\right]^2 + \gamma^4\beta^2\left[\vec{\beta}\cdot\vec{a}\right]^2\right\}\right) \\ &= \gamma^4\left(\gamma^2\left[\vec{\beta}\cdot\vec{a}\right]^2 - a^2 - 2\gamma^2\left[\vec{\beta}\cdot\vec{a}\right]^2\right) \\ &= -\gamma^4\left(\gamma^2\left[\vec{\beta}\cdot\vec{a}\right]^2 + a^2\right) = -\gamma^4\left(a_\perp^2 + \gamma^2 a_\parallel^2\right) \end{aligned}$$

If $\overleftrightarrow{\mathbf{a}} = 0$ in one frame then it is zero in all frames.

2.3.1 Velocity transformation

Let's transform the velocity vector. Suppose a particle has velocity $\vec{\overline{u}}$ in a bar frame moving at velocity $\vec{v} = v\hat{x}$ with respect to unbar. Then we have:

$$u^\alpha = \Lambda^\alpha{}_\beta \overline{u}^\beta$$

where the transformation matrix transforms from bar to unbar:

$$\Lambda^\alpha{}_\beta = \Lambda^\alpha{}_\beta = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus, using $\Gamma = 1/\sqrt{1 - u^2/c^2}$

$$\begin{aligned} \begin{pmatrix} \Gamma c \\ \Gamma u_x \\ \Gamma u_y \\ \Gamma u_z \end{pmatrix} &= \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \overline{\Gamma c} \\ \overline{\Gamma u_x} \\ \overline{\Gamma u_y} \\ \overline{\Gamma u_z} \end{pmatrix} \\ &= \begin{pmatrix} \gamma\overline{\Gamma}c + \gamma\beta\overline{\Gamma u_x} \\ \gamma\beta\overline{\Gamma}c + \gamma\overline{\Gamma u_x} \\ \overline{\Gamma u_y} \\ \overline{\Gamma u_z} \end{pmatrix} \end{aligned}$$

Thus

$$\Gamma = \gamma\overline{\Gamma} + \gamma\beta\frac{\overline{\Gamma u_x}}{c} = \gamma\overline{\Gamma}\left(1 + \beta\frac{\overline{u_x}}{c}\right) = \gamma\frac{1 + \beta\overline{u}_x/c}{\sqrt{1 - \overline{u}^2/c^2}}$$

Then

$$\begin{aligned} u_x &= \frac{\gamma\overline{\Gamma}}{\Gamma}\left(\beta c + \overline{u_x}\right) \\ &= \frac{\beta c + \overline{u_x}}{1 + \beta\overline{u}_x/c} \end{aligned}$$

Note again that we get back the Galilean transformation as $\beta \to 0$.

The perpendicular components are:

$$u_y = \frac{\overline{\Gamma u_y}}{\Gamma} = \frac{\overline{u_y}}{\gamma\left(1 + \beta\overline{u}_x/c\right)}$$

and similarly for u_z.

2.3.2 Work and energy

The work done by a force $\vec{F}$ on a particle of mass m in a single frame is

$$W = \vec{F} \cdot \vec{u}$$

so we look at the four-vector dot product

$$\overleftrightarrow{\mathbf{F}} \cdot \overleftrightarrow{\mathbf{u}} = m \frac{d\overleftrightarrow{\mathbf{u}}}{d\tau} \cdot \overleftrightarrow{\mathbf{u}} = \frac{m}{2} \frac{d}{d\tau} \left(\overleftrightarrow{\mathbf{u}} \cdot \overleftrightarrow{\mathbf{u}} \right) = \frac{m}{2} \frac{d}{d\tau} c^2 = 0$$

Alternatively:

$$\begin{aligned} \overleftrightarrow{\mathbf{F}} \cdot \overleftrightarrow{\mathbf{u}} &= m \frac{d\overleftrightarrow{\mathbf{u}}}{d\tau} \cdot \overleftrightarrow{\mathbf{u}} = \gamma \left(\frac{d}{d\tau} (\gamma m c), \frac{d}{d\tau} \vec{p} \right) \cdot (c, \vec{u}) \\ &= \gamma \left(\frac{d}{d\tau} (\gamma m c^2) - \vec{F} \cdot \vec{u} \right) = 0 \end{aligned}$$

Thus the work energy theorem is recovered in the form

$$\frac{d}{d\tau} (\mathcal{E}) = \frac{d}{d\tau} \left(\gamma m c^2 \right) = \vec{F} \cdot \vec{u} = W$$

where we identify $\gamma m c^2$ as the particle's energy.

2.3.3 Doppler shift

The phase of a wave is the scalar

$$\phi = \vec{k} \cdot \vec{x} - \omega t$$

suggesting that the wave vector components are

$$\overleftrightarrow{\mathbf{k}} = \left(\frac{\omega}{c}, \vec{k} \right)$$

and then

$$\phi = -\overleftrightarrow{\mathbf{k}} \cdot \overleftrightarrow{\mathbf{x}}$$

Applying the Lorentz transformation we obtain the Doppler shift formulae:

$$\overline{\omega} = \gamma \left(\omega - \vec{v} \cdot \vec{k} \right)$$

$$\overline{k_x} = \gamma \left(k_x - \beta \omega / c \right)$$

and

$$\overline{k_y} = k_y$$

Notice that there is a Doppler shift even if $\vec{v} \cdot \vec{k} = 0$. This result is initially surprising, but we should realize that this effect is due to time dilation. Astrophysicists call this the "transverse Doppler effect".

2.4 Collision problems

The momentum 4-vector is

$$\overleftrightarrow{\mathbf{p}} = m\overleftrightarrow{\mathbf{v}} = \gamma m\,(c, \vec{v}) = (\mathcal{E}/c, \vec{p})$$

and the dot product $\overleftrightarrow{\mathbf{p}} \cdot \overleftrightarrow{\mathbf{p}} = (mc)^2$ is *invariant.* That means we can evaluate it in any reference frame and the results are the same. We also have the usual conservation laws for momentum and energy, which may now be written as a single conservation law for the 4-vector momentum in a given reference frame:

$$\overleftrightarrow{\mathbf{p}}_{\text{before}} = \overleftrightarrow{\mathbf{p}}_{\text{after}}$$

All these relations can be used to solve problems.

2.5 More on Lorentz transformations

The set of transformations of coordinates in spacetime is represented by the Lorentz Group. (see Lea optional topic). We can investigate some of the properties of the group elements. We have already found the matrix that represents a "boost" by β along the $x-$axis.

$$\Lambda^{\mu}_{\nu} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6}$$

Now we want to find the other elements of the group.

First of all, we know that the magnitude of the position vector must be invariant under the transformation $x' = \Lambda x$ (This is called preservation of the norm.)

$$x \cdot g \cdot x = x' \cdot g \cdot x' = x \cdot \Lambda^T \cdot g \cdot \Lambda \cdot x$$

and thus

$$\Lambda^T \cdot g \cdot \Lambda = g \tag{7}$$

(In index notation

$$\overline{x}^{\mu}\overline{x}_{\mu} = x^{\nu}x_{\nu}$$

or

$$\Lambda^{\mu}_{\nu}x^{\nu}g_{\mu\alpha}\Lambda^{\alpha}_{\beta}x^{\beta} = x^{\nu}x_{\nu} = x^{\nu}g_{\nu\beta}x^{\beta}$$

This result is true for arbitrary components x^{μ}, and so

$$\Lambda^{\mu}_{\nu}g_{\mu\alpha}\Lambda^{\alpha}_{\beta} = g_{\nu\beta}$$

)

Taking the determinant of this relation, we get:

$$\det(\Lambda)^2 \det(g) = \det(g)$$

so that

$$\det(\Lambda) = \pm 1$$

We already know that in 3-space, the det = -1 represents reflection, whereas det=+1 represents rotation. Similarly here, we usually restrict attention to the "proper" transformations with $\det(\Lambda) = +1$.

Now let's investigate equation (7) further. Since g is symmetric, so is $\Lambda g \Lambda^T$. But this condition is automatically satisfied. J says there are only 10 linearly independent elements "because of symmetry under transposition" but Λ itself is not symmetric! The boost matrices are symmetric but the rotation matrices are not. They have antisymmetric off-diagonal elements. But the group element Λ does not have 16 independent elements all the same. How to prove it?? We work with the generators of the group (see Lea).

The group elements Λ may be written in terms of the generators as

$$\Lambda = e^L$$

where the exponential of a matrix is defined using the series expansion of the exponential:

$$e^L = 1 + L + \frac{L \cdot L}{2} + \cdots$$

Now since $\det(\Lambda) = 1$, then we must have

$$\det\left(e^L\right) = 1 = e^{tr(L)} \Rightarrow Tr(L) = 0$$

Starting from equation (7), multiplying on the left by $g = g^{-1}$, we get:

$$\begin{aligned} g \cdot \Lambda^T \cdot g \cdot \Lambda &= g \cdot g = 1 \\ g \cdot \Lambda^T \cdot g &= \Lambda^{-1} \end{aligned}$$

or, in terms of the generators:

$$\begin{aligned} g e^{L^T} g &= g\left(1 + L^T + \frac{L^T \cdot L^T}{2} + \cdots\right) g \\ &= gg + gL^T g + \frac{gL^T L^T g}{2} + \cdots \\ &= 1 + gL^T g + \frac{gL^T g g L^T g}{2} + \cdots \\ &= \exp\left(gL^T g\right) = e^{-L} \end{aligned}$$

since $gg = 1$. Thus:

$$gL^T g = -L \tag{8}$$

and thus

$$L^T g = -gL = (gL)^T$$

so the product gL is antisymmetric. Now L has elements:

$$\begin{pmatrix} L_{00} & L_{01} & L_{02} & L_{03} \\ L_{10} & L_{11} & L_{12} & L_{13} \\ L_{20} & L_{21} & L_{22} & L_{23} \\ L_{30} & L_{31} & L_{32} & L_{33} \end{pmatrix}$$

and so gL has elements:

$$\begin{pmatrix} L_{00} & L_{01} & L_{02} & L_{03} \\ -L_{10} & -L_{11} & -L_{12} & -L_{13} \\ -L_{20} & -L_{21} & -L_{22} & -L_{23} \\ -L_{30} & -L_{31} & -L_{32} & -L_{33} \end{pmatrix}$$

And since this matrix is antisymmetric, then $L_{10} = L_{01}$, $L_{20} = L_{02}$, $L_{12} = -L_{21,}$ etc. Thus L is traceless, as we predicted, and has only 6 independent elements:

$$L = \begin{pmatrix} 0 & L_{01} & L_{02} & L_{03} \\ L_{10} & 0 & L_{12} & L_{13} \\ L_{02} & -L_{12} & 0 & L_{23} \\ L_{03} & -L_{13} & -L_{23} & 0 \end{pmatrix}$$

The elements L_{0i} represents *boosts* – velocity transformations, while L_{ij} represent rotations.

Now define the basis matrices
S_1 : all elements zero except $L_{23} = -1$
S_2 : all elements zero except $L_{13} = +1$
S_3 : all elements zero except $L_{12} = -1$
K_1 : all elements zero except $L_{01} = +1$
K_2 : all elements zero except $L_{02} = +1$
K_3 : all elements zero except $L_{03} = +1$

The squares of these matrices are all diagonal. For example:

$$S_1 S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and

$$K_1 K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

etc. Also

$$S_1^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = -S_1$$

Thus

$$
\begin{aligned}
e^{-S_1} &= 1 - S_1 + \frac{1}{2} S_1 \cdot S_1 - \frac{1}{3!} S_1^3 + \cdots \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - S_1^2 - S_1 + \frac{S_1^2}{2} + \frac{S_1}{3!} + \frac{\left(-S_1^2\right)}{4!} + \cdots \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - S_1 \sin(1) - S_1^2 \cos(1) \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos 1 & \sin 1 \\ 0 & 0 & -\sin 1 & \cos 1 \end{pmatrix} = \text{ rotation through 1 radian about the } x-\text{axis}
\end{aligned}
$$

Thus the matrix S_1 generates rotations about the $x-$axis. The matrix θS_1 generates a rotation through angle θ about the $x-$axis. Similarly, the other S matrices generate rotations about the $y-$ and $z-$axes.

Similarly we can show that the K matrices generate boosts. $K_1^3 = +K_1$,and so

$$
\begin{aligned}
e^{-\xi K_1} &= 1 - \xi K_1 + \frac{(\xi K_1)^2}{2} - \frac{1}{3!} K_1^3 + \cdots \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + K_1^2 - \xi K_1 + \frac{(\xi K_1)^2}{2} - \xi^3 \frac{K_1}{3!} \cdots \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - K_1 \sinh \xi + K_1^2 \cosh \xi \\
&= \begin{pmatrix} \cosh \xi & -\sinh \xi & 0 & 0 \\ -\sinh \xi & \cosh \xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

Compare this with equation (6). The two matrices are identical if we identify $\beta = \tanh \xi$, or $\xi = \tanh^{-1} \beta$ is the *rapidity.* The most general generator may be written

$$L = -\vec{\omega} \cdot \vec{S} - \vec{\xi} \cdot \vec{K}$$

which generates the transformation $\Lambda = \exp\left(-\vec{\omega} \cdot \vec{S} - \vec{\xi} \cdot \vec{K}\right)$ Thus each matrix Λ has 6 independent parameters: the components of $\vec{\omega}$ and $\vec{\xi}$.

Now the matrices K_i do not commute. In fact:

$$[K_i, K_j] = K_i K_j - K_j K_i = -\varepsilon_{ijk} S_k$$

For example

$$K_1 K_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_2 K_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So

$$K_1 K_2 - K_2 K_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -S_3$$

Now let's see what happens if we do one boost after another:

$$\begin{aligned} \Lambda_1 \Lambda_2 &= \exp\left(-\vec{\xi}_1 \cdot \vec{K}\right) \exp\left(-\vec{\xi}_2 \cdot \vec{K}\right) \\ &= \left(1 - \vec{\xi}_1 \cdot \vec{K} + \frac{\left(\vec{\xi}_1 \cdot \vec{K}\right)^2}{2} + \cdots\right) \left(1 - \vec{\xi}_2 \cdot \vec{K} + \frac{\left(\vec{\xi}_2 \cdot \vec{K}\right)^2}{2} + \cdots\right) \\ &= 1 - \vec{\xi}_1 \cdot \vec{K} - \vec{\xi}_2 \cdot \vec{K} + \left(\vec{\xi}_1 \cdot \vec{K}\right)\left(\vec{\xi}_2 \cdot \vec{K}\right) + \cdots \end{aligned}$$

As an example, let $\vec{\xi}_1 = \xi_1 \hat{x}$ and $\vec{\xi}_2 = \xi_2 \hat{y}$. Then

$$\begin{aligned} \Lambda_1 \Lambda_2 &= 1 - \begin{pmatrix} 0 & \xi_1 & 0 & 0 \\ \xi_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & \xi_2 & 0 \\ 0 & 0 & 0 & 0 \\ \xi_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \xi_1 & 0 & 0 \\ \xi_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \xi_2 & 0 \\ 0 & 0 & 0 & 0 \\ \xi_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \cdots \\ &= 1 - \begin{pmatrix} 0 & \xi_1 & \xi_2 & 0 \\ \xi_1 & 0 & 0 & 0 \\ \xi_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \xi_1 \xi_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \cdots \end{aligned}$$

which is not a pure boost! The boosts do not commute; in fact

$$\Lambda_1 \Lambda_2 - \Lambda_2 \Lambda_1 = -\xi_1 \xi_2 S_3$$

a rotation!

This can give rise to some interesting kinematic effects.

Suppose a system (e.g. an electron) moves with a velocity $\vec{v}(t)$ in the lab frame and a vector associated with that system (e.g. its spin) has a time derivative in the rest frame of the system. What is the observed rate of change in the lab frame?

First we investigate the coordinate system moving with the system. At time t :

$$x' = \Lambda\left(\vec{\beta}\right) x$$

where x are the lab frame coordinates. At time $t + \delta t$

$$x'' = \Lambda\left(\vec{\beta} + \delta\vec{\beta}\right) x$$

Thus

$$x'' = \Lambda\left(\vec{\beta} + \delta\vec{\beta}\right) \Lambda^{-1}\left(\vec{\beta}\right) x'$$

is the relation between the rest frame coords at times t and $t + \delta t$. We may choose our coordinates so that $\vec{\beta}$ lies along the $x-$axis and $\delta\vec{\beta}$ is in the $x-y-$plane. Then $\vec{\beta} + \delta\vec{\beta}$ has components $(\beta + \delta\beta_1, \delta\beta_2, 0)$, and

$$\gamma + \delta\gamma = \frac{1}{\sqrt{1 - \left(\vec{\beta} + \delta\vec{\beta}\right)^2}} = \frac{1}{\sqrt{1 - \beta^2 - 2\vec{\beta}\cdot\delta\vec{\beta}}} = \gamma\frac{1}{\sqrt{1 - \frac{2\vec{\beta}\cdot\delta\vec{\beta}}{1-\beta^2}}} = \gamma\left(1 + \gamma^2\vec{\beta}\cdot\delta\vec{\beta}\right)$$

So

$$\delta\gamma = \gamma^3\vec{\beta}\cdot\delta\vec{\beta} = \gamma^3\beta\delta\beta_1$$

and by Jackson (11.98), to first order in $\delta\vec{\beta}$,

$$\begin{aligned}
\Lambda\left(\vec{\beta} + \delta\vec{\beta}\right) &= \begin{pmatrix} \gamma + \delta\gamma & -(\gamma + \delta\gamma)(\beta + \delta\beta_1) & -\gamma\delta\beta_2 & 0 \\ -(\gamma + \delta\gamma)(\beta + \delta\beta_1) & \gamma + \delta\gamma & (\gamma + \delta\gamma - 1)\frac{\delta\beta_2}{\beta} & 0 \\ -(\gamma + \delta\gamma)\delta\beta_2 & (\gamma + \delta\gamma - 1)\frac{\delta\beta_2}{\beta} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \gamma + \gamma^3\beta\delta\beta_1 & -\left(\gamma + \gamma^3\beta\delta\beta_1\right)(\beta + \delta\beta_1) & -\gamma\delta\beta_2 & 0 \\ -\left(\gamma + \gamma^3\beta\delta\beta_1\right)(\beta + \delta\beta_1) & \gamma + \gamma^3\beta\delta\beta_1 & (\gamma - 1)\frac{\delta\beta_2}{\beta} & 0 \\ -\gamma\delta\beta_2 & (\gamma - 1)\frac{\delta\beta_2}{\beta} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \gamma + \gamma^3\beta\delta\beta_1 & -\gamma\beta - \gamma\delta\beta_1\left(1 + \gamma^2\beta^2\right) & -\gamma\delta\beta_2 & 0 \\ -\gamma\beta - \gamma\delta\beta_1\left(1 + \gamma^2\beta^2\right) & \gamma + \gamma^3\beta\delta\beta_1 & (\gamma - 1)\frac{\delta\beta_2}{\beta} & 0 \\ -\gamma\delta\beta_2 & (\gamma - 1)\frac{\delta\beta_2}{\beta} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

But

$$\left(1 + \gamma^2\beta^2\right) = 1 + \frac{\beta^2}{1 - \beta^2} = \frac{1}{1 - \beta^2} = \gamma^2$$

Then the transformation matrix is:

$$
\begin{aligned}
\Lambda_T &= \Lambda\left(\vec{\beta}+\delta\vec{\beta}\right)\Lambda^{-1}\left(\vec{\beta}\right) \\
&= \begin{pmatrix} \gamma+\gamma^3\beta\delta\beta_1 & -\gamma\beta-\gamma^3\delta\beta_1 & -\gamma\delta\beta_2 & 0 \\ -\gamma\beta-\gamma^3\delta\beta_1 & \gamma+\gamma^3\beta\delta\beta_1 & (\gamma-1)\frac{\delta\beta_2}{\beta} & 0 \\ -\gamma\delta\beta_2 & (\gamma-1)\frac{\delta\beta_2}{\beta} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \gamma^2-\gamma^2\beta^2 & \gamma^4\beta^2\delta\beta_1-\gamma^4\delta\beta_1 & -\gamma\delta\beta_2 & 0 \\ \gamma^4\beta^2\delta\beta_1-\gamma^4\delta\beta_1 & \gamma^2-\gamma^2\beta^2 & (\gamma-1)\frac{\delta\beta_2}{\beta} & 0 \\ -\gamma\delta\beta_2 & -\gamma\delta\beta_2\frac{\gamma\beta^2-\gamma+1}{\beta} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & -\gamma^2\delta\beta_1 & -\gamma\delta\beta_2 & 0 \\ -\gamma^2\delta\beta_1 & 1 & (\gamma-1)\frac{\delta\beta_2}{\beta} & 0 \\ -\gamma\delta\beta_2 & -(\gamma-1)\frac{\delta\beta_2}{\beta} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

Thus:

$$
\begin{aligned}
\Lambda_T &= 1-\frac{\gamma-1}{\beta}\delta\beta_2 S_3-\gamma^2\delta\beta_1 K_1-\gamma\delta\beta_2 K_2 \\
&= 1-\frac{\gamma-1}{\beta^2}\left(\tilde{\boldsymbol{\beta}}\times\boldsymbol{\delta}\tilde{\boldsymbol{\beta}}\right)\cdot\tilde{\mathbf{S}}-\left(\gamma^2\delta\tilde{\boldsymbol{\beta}}_{\parallel}+\gamma\delta\tilde{\boldsymbol{\beta}}_{\perp}\right)\cdot\tilde{\mathbf{K}}
\end{aligned}
$$

Now if we define the infinitesimal velocity $\triangle\tilde{\boldsymbol{\beta}}\equiv\gamma^2\delta\tilde{\boldsymbol{\beta}}_{\parallel}+\gamma\delta\tilde{\boldsymbol{\beta}}_{\perp}$, then

$$\Lambda\left(\triangle\tilde{\boldsymbol{\beta}}\right)=\exp\left(-\triangle\tilde{\boldsymbol{\beta}}\cdot\tilde{\mathbf{K}}\right)=1-\triangle\tilde{\boldsymbol{\beta}}\cdot\mathbf{K}$$

and define a rotation angle

$$\triangle\vec{\ }=\frac{\gamma-1}{\beta^2}\left(\vec{\beta}\times\delta\vec{\beta}\right)$$

and the corresponding rotation matrix

$$R\left(\triangle\vec{\ }\right)=\exp\left(-\triangle\vec{\ }\cdot\vec{S}\right)=1-\triangle\vec{\ }\cdot\vec{S}$$

Then we may write the transformation from x' to x'' as

$$\Lambda_T=\Lambda\left(\triangle\vec{\beta}\right)R\left(\triangle\vec{\ }\right)=R\left(\triangle\vec{\ }\right)\Lambda\left(\triangle\vec{\beta}\right)$$

That is, the transformation is equivalent to a boost plus a rotation.

The rotation is troublesome, because rotating frames are non-inertial. We can account for the rotation of the frame by including an additional term in the time derivative:

$$\left.\frac{d\vec{G}}{dt}\right|_{\text{non-inertial}}=\left.\frac{d\vec{G}}{dt}\right|_{\text{inertial}}+\vec{\omega}\times\vec{G}$$

where $\vec{G}$ is a vector and $\vec{\omega}$ is the angular velocity of the frame. In this case:

$$\vec{\omega} = \lim_{\delta t \to 0} \frac{\triangle \vec{\ }}{\delta t} = \frac{\gamma - 1}{\beta^2} \vec{\beta} \times \frac{d\vec{\beta}}{dt}$$

or, using $\beta^2 = 1 - 1/\gamma^2 = \left(\gamma^2 - 1\right)/\gamma^2$,we may write this in Jackson's form

$$\vec{\omega} = \frac{\gamma^2}{\gamma + 1} \vec{\beta} \times \frac{d\vec{\beta}}{dt}$$

This effect is purely kinematic. It gives rise to the phenomenon of Thomas precession. The phenomenon occurs only when the system has an acceleration component perpendicular to its instantaneous velocity.