Jackson Notes 3 2012

1 Method of images

The method of images is a method that allows us to solve certain potential problems as well as obtaining a Green's function for certain spaces. Recall that the Green's function satisfies the equation

$$\nabla^2 G\left(\vec{x}', \vec{x}\right) = -4\pi\delta\left(\vec{x} - \vec{x}'\right) \tag{1}$$

subject to the boundary conditions

$$G_D = 0 \text{ on S} \tag{2}$$

or

$$\frac{\partial G}{\partial n} = C \text{ on S} \tag{3}$$

and thus the solution is of the form

$$G\left(\vec{x}', \vec{x}\right) = \frac{1}{\left|\vec{x} - \vec{x}'\right|} + \psi\left(\vec{x}', \vec{x}\right) \tag{4}$$

where

$$\nabla^2 \psi\left(\vec{x}', \vec{x}\right) = 0 \text{ in } V \tag{5}$$

Thus the Green's function is $4\pi\varepsilon_0$ times the potential at $\vec{x}$ due to a unit point charge at $\vec{x}'$ in the volume V plus an additional term, with no sources in V, that fixes up the boundary conditions, that is, a term due to sources outside of V. Thus we can make progress in finding the Dirichlet Green's function by finding the potential due to a point charge in V with grounded boundaries.

1.1 Plane boundary

Suppose the volume of interest is the half-space $z > 0$. A point charge q is placed at a distance d from the $x - y-$plane, which is a conducting boundary. What is the potential for $z > 0$?

First, the conducting plane must be at a constant potential, which we may take to be zero. (If the potential is V_0, we can just add V_0 to our solution at the end.) Then we may place an *image charge* $-q$ at distance d from the boundary, but on the opposite side. Then the potential due to these two charges everywhere on the boundary $z = 0$ is zero. Since the image charge we added is *outside* our volume, it does not contribute to the value of $\nabla^2\Phi$ in V. Thus, putting the $z-$axis through q, the solution is

$$\Phi\left(\vec{x}\right) = \frac{1}{4\pi\varepsilon_0}\left\{\frac{q}{\left|\vec{x} - d\hat{z}\right|} - \frac{q}{\left|\vec{x} + d\hat{z}\right|}\right\}$$

Properties of this solution are explored in Jackson problem 2.1. Note that the image charge represents the charge density drawn onto the plane through the ground wire.

To obtain the Green's function for the half-space, we simply set $q = 1$ and multiply by $4\pi\varepsilon_0$. Then

$$G\left(\vec{x}, \vec{x}'\right) = \frac{1}{\left|\vec{x} - \vec{x}'\right|} - \frac{1}{\left|\vec{x} - \vec{x}''\right|} \tag{6}$$

where

$$\vec{x}'' = (x', y', -z')$$

is the position vector of the image point. This expression shows clearly that the physial dimensions of G are [1/length].

This Green's function (6) is explicitly in the form (4), but it is not very convenient to use. For example, suppose the problem of interest has potential V_0 within a circle of radius a on the plane $z = 0$, with the rest of the plane grounded. Then we would need first to compute (notes 2.5 equation 6)

$$\left.\frac{\partial G}{\partial n'}\right|_{z'=0} = -\left.\frac{\partial G}{\partial z'}\right|_{z'=0} = -\left.\frac{2z}{|\vec{x} - \vec{x}'|^3}\right|_{z'=0}$$

(note that the outward normal on the plane at $z = 0$ is $-\hat{z}$) and then evaluate

$$\begin{aligned}\Phi(\tilde{\mathbf{x}}) &= \frac{V_0}{4\pi}\int_{\text{circle}} \left.\frac{2z}{|\vec{x} - \vec{x}'|^3}\right|_{z'=0} dx'dy' \\ &= \frac{V_0 z}{2\pi}\int_{\text{circle}} \frac{1}{\left(r^2 + r'^2 - 2rr'\cos\left(\phi - \phi'\right) + z^2\right)^{3/2}} r'dr'd\phi'\end{aligned}$$

where r, ϕ are polar coordinates in the $x - y$ plane. This is ugly. (We encountered a similar integral in the bar magnet example, but that was only a 1/2 power. Later on we will identify some methods for doing this integral, but we will also find more convenient expressions for G.)

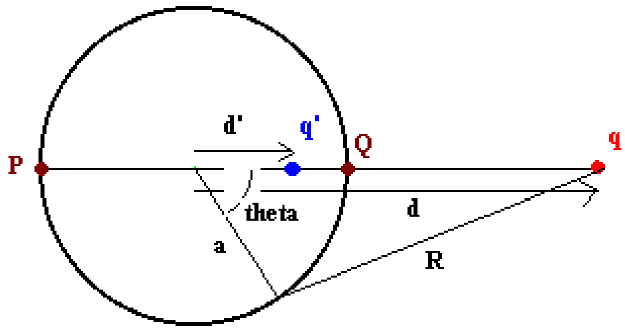
1.2 Images in a sphere

Now let the conducting boundary be a sphere of radius a and suppose we have a point charge q at a distance d from the center of the sphere, where $d > a$. We want to find the potential outside the sphere.

Learning from our experience above, we conjecture that we can place an image charge *inside* the sphere (and thus outside our volume V) and form the potential in V as the sum of the potential due to the two charges. Let the image charge have magnitude q' and be at $r = d'$. Then we have

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left\{\frac{q}{\left|\vec{x} - \vec{d}\right|} + \frac{q'}{\left|\vec{x} - \vec{d}'\right|}\right\}$$

The boundary condition is that $\Phi(\vec{x}) = 0$ everywhere on the surface of the sphere $(r = a)$. The system has azimuthal symmetry about the line from the center of the sphere to the charge q. Rotate the system about this line and nothing changes. Thus the image charge q' must lie on this line at a distance d' from the center. Then we have two unknowns in our potential: q' and d', and we need only pick two points on the sphere to solve for the two unknowns. The most convenient two points lie on the ends of the diameter through the image charge, as shown (P and Q in the diagram).



$$\Phi_P = \frac{1}{4\pi\varepsilon_0}\left\{\frac{q}{d+a} + \frac{q'}{d'+a}\right\} = 0 \tag{7}$$

and

$$\Phi_Q = \frac{1}{4\pi\varepsilon_0}\left\{\frac{q}{d-a} + \frac{q'}{a-d'}\right\} = 0 \tag{8}$$

Then from equation (7)

$$q\,(d'+a) + q'\,(d+a) = 0$$

and from (8)

$$q\,(a-d') + q'\,(d-a) = 0$$

Adding these two relations eliminates d' and gives

$$2qa + 2q'd = 0 \Rightarrow q' = -q\frac{a}{d} \tag{9}$$

This result has the nice property that the image charge is negative if q is positive, as we found in the planar case. Once again the image charge represents the charge drawn onto the surface of the sphere through the ground wire.

Now we subtract the two relations to obtain an expression for d' :

$$\begin{aligned} 2qd' + 2q'a &= 0 \\ d' &= -a\frac{q'}{q} = -a\left(-\frac{a}{d}\right) = \frac{a^2}{d} \end{aligned} \tag{10}$$

The image charge is inside the sphere if $d > a$, as we need. Conversely, if $d < a$, the image charge is outside the sphere. (You are asked to confirm this result in Problem 2.2.)

We can perform two checks on this result. First let's find the potential at an arbitrary point on the sphere.

$$\begin{aligned} 4\pi\varepsilon_0\Phi(a,\theta) &= \frac{q}{R}+\frac{q'}{R'} \\ &= \frac{q}{\sqrt{d^2+a^2-2ad\cos\theta}}-\frac{qa/d}{\sqrt{(d')^2+a^2-2ad'\cos\theta}} \\ &= \frac{q}{d}\left\{\frac{1}{\sqrt{1+\frac{a^2}{d^2}-2\frac{a}{d}\cos\theta}}-\frac{a}{\sqrt{(a^2/d)^2+a^2-2(a^3/d)\cos\theta}}\right\}=0 \end{aligned}$$

for all θ, as expected.

Second, let's check that we get back the result from §1.1 as $a\to\infty$ (plane boundary). We have to be a bit careful here, because if we immediately let $a\to\infty$, the point from which we are measuring our distances moves off infinitely far to the left, and we will learn nothing. So first we write our results in terms of distance from the surface of the sphere. The charge q is a distance $d-a=h$ from the surface, and then

$$q'=-q\frac{a}{a+h}=-\frac{q}{1+h/a}\to -q \text{ as } a\to\infty$$

as required. The distance of the image from the surface is

$$h'=a-d'=a-\frac{a^2}{a+h}=\frac{ah}{a+h}=\frac{h}{1+h/a}\to h \text{ as } a\to\infty$$

and this is the second required result.

Finally we can write the Dirichlet Green's function for the region outside a sphere of radius a by setting $q=1$ and multiplying by $4\pi\varepsilon_0$:

$$G_D(\vec{x},\vec{x}')=\frac{1}{|\vec{x}-\vec{x}'|}-\frac{a}{r'}\frac{1}{|\vec{x}-\vec{x}''|}$$

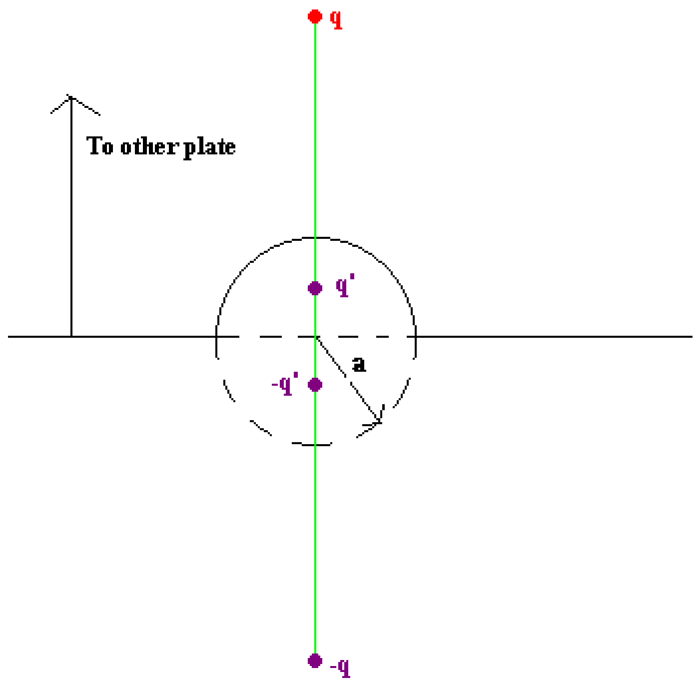
where $\vec{x}''$ is the position vector of the image point. Again this is pretty ugly.

1.3 Images in a cylinder

The basic ideas and methods are the same as we have used in the plane and sphere cases. See Jackson problem 2.11.

1.4 Use of images to solve problems

Jackson P 2.10 asks us to compute the potential inside a parallel-plate capacitor with a small hemispherical boss on one plate.



The clue is in part (c) of the problem: We *model* the system by putting a point charge q at a very large distance d from the plane. Then we can use the image system shown to model the capacitor, since the image system puts potential zero on the lower plate. (q and $-q$, q' and $-q'$ form pairs that make the potential on the plane zero; q and q', $-q$ and $-q'$ form pairs that make the potential on the sphere zero.) Then we can show that as $d \to \infty$ we obtain a uniform field at large distance from the lower plate, and we set that uniform field equal to the given value of E_0, thus determining the necessary charge q.

Setup: From §1.2, the image charge $q' = -\frac{a}{d}q$ and its distance from the plane is $d' = \frac{a^2}{d}$. The potential due to the four charges (one real charge and three images) is

$$\Phi(r,\theta) = kq \left\{ \begin{array}{c} 1/\sqrt{r^2+d^2-2rd\cos\theta} - 1/\sqrt{r^2+d^2+2rd\cos\theta} \\ -\left(\frac{a}{d}\right)/\sqrt{r^2+\frac{a^4}{d^2}-2r\frac{a^2}{d}\cos\theta} + \left(\frac{a}{d}\right)/\sqrt{r^2+\frac{a^4}{d^2}+2r\frac{a^2}{d}\cos\theta} \end{array} \right\} \quad (11)$$

To see why this works, look at the potential for $d \gg r \gg a$. We drop terms in $(a/d)^2$ to get

$$\Phi(r,\theta) \simeq \frac{kq}{d} \left\{ \begin{array}{c} 1/\sqrt{\frac{r^2}{d^2}+1-2\frac{r}{d}\cos\theta} - 1/\sqrt{\frac{r^2}{d^2}+1+2\frac{r}{d}\cos\theta} \\ -\left(\frac{a}{r}\right)/\sqrt{1-2\frac{a^2}{dr}\cos\theta} + \left(\frac{a}{r}\right)/\sqrt{1+2\frac{a^2}{rd}\cos\theta} \end{array} \right\}$$

Next expand the square roots, dropping terms in $(r/d)^2$ and $(a/r)\times(a/d)$.

$$\begin{aligned} \Phi(r,\theta) &\simeq \frac{kq}{d}\left\{1+\frac{r}{d}\cos\theta - \left(1-\frac{r}{d}\cos\theta\right) - \frac{a}{r} + \frac{a}{r}\right\} \\ &= \frac{kq}{d}2\frac{r}{d}\cos\theta = \frac{2kq}{d^2}z \text{ to first order in small quantities} \end{aligned}$$

This corresponds to the given uniform field provided that $E_0 = 2kq/d^2$, or, if we choose

$$q = E_0 d^2/2k. \tag{12}$$

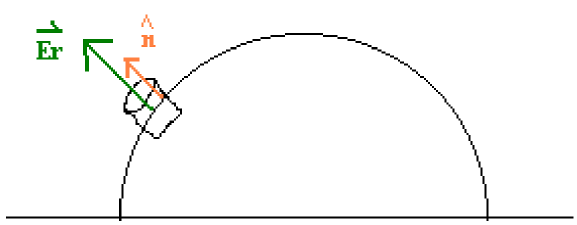
Solve: **(a)** To find the surface charge densities, begin with the field components. From (11), we have

$$\begin{aligned}\frac{\partial\Phi}{\partial r} &= kq\left\{-\frac{r-d\cos\theta}{\left(r^2+d^2-2rd\cos\theta\right)^{3/2}}+\frac{r+d\cos\theta}{\left(r^2+d^2+2rd\cos\theta\right)^{3/2}}\right.\\ &\left.\quad+\frac{a\left(r-a^2\cos\theta/d\right)}{d\left(r^2+\frac{a^4}{d^2}-2r\frac{a^2}{d}\cos\theta\right)^{3/2}}-\frac{a\left(r+a^2\cos\theta/d\right)}{d\left(r^2+\frac{a^4}{d^2}+2r\frac{a^2}{d}\cos\theta\right)^{3/2}}\right\}\end{aligned}$$

and at $r = a$

$$\begin{aligned}E_r &= kq\left\{\frac{a-d\cos\theta}{\left(a^2+d^2-2ad\cos\theta\right)^{3/2}}-\frac{a+d\cos\theta}{\left(a^2+d^2+2ad\cos\theta\right)^{3/2}}\right.\\ &\left.\quad-\frac{a\left(a-a^2\cos\theta/d\right)}{d\left(a^2+\frac{a^4}{d^2}-2a\frac{a^2}{d}\cos\theta\right)^{3/2}}+\frac{a\left(a+a^2\cos\theta/d\right)}{d\left(a^2+\frac{a^4}{d^2}+2a\frac{a^2}{d}\cos\theta\right)^{3/2}}\right\}\\ &= kq\left\{\frac{a-d\cos\theta}{\left(a^2+d^2-2ad\cos\theta\right)^{3/2}}-\frac{a+d\cos\theta}{\left(a^2+d^2+2ad\cos\theta\right)^{3/2}}\right.\\ &\left.\quad-\frac{da^2\left(d-a\cos\theta\right)}{a^3\left(d^2+a^2-2ad\cos\theta\right)^{3/2}}+\frac{da^2\left(d+a\cos\theta\right)}{a^3\left(d^2+a^2+2ad\cos\theta\right)^{3/2}}\right\}\\ &= kq\left[\frac{a-d\cos\theta-\frac{d}{a}\left(d-a\cos\theta\right)}{\left(a^2+d^2-2ad\cos\theta\right)^{3/2}}-\frac{a+d\cos\theta-\frac{d}{a}\left(d+a\cos\theta\right)}{\left(a^2+d^2+2ad\cos\theta\right)^{3/2}}\right]\\ &= kqa\left(1-\frac{d^2}{a^2}\right)\left[\frac{1}{\left(a^2+d^2-2ad\cos\theta\right)^{3/2}}-\frac{1}{\left(a^2+d^2+2ad\cos\theta\right)^{3/2}}\right]\end{aligned}$$

and thus the charge density is (notes 1 eqn 5 with $\vec{E} = 0$ inside the boss):



$$\sigma(\theta) = \varepsilon_0 E_r = \varepsilon_0 kqa\left(1-\frac{d^2}{a^2}\right)\left[\frac{1}{\left(a^2+d^2-2ad\cos\theta\right)^{3/2}}-\frac{1}{\left(a^2+d^2+2ad\cos\theta\right)^{3/2}}\right]$$

Analysis: Notice that σ is zero in the corners at $\theta = \pi/2$, .as expected. Also since

$1 \geq \cos\theta \geq 0$ on the boss and $d > a$, the chrage density is negative on the boss if q is positive, as expected.

Solve: The total charge on the boss is

$$
\begin{aligned}
Q_{\text{boss}} &= 2\pi \int_0^{\pi/2} \sigma(\theta)\, a^2 \sin\theta\, d\theta \\
&= \frac{2\pi}{4\pi} q a^3 \left(1 - \frac{d^2}{a^2}\right) \int_0^1 \left(\frac{1}{(a^2 + d^2 - 2a\mu d)^{3/2}} - \frac{1}{(a^2 + d^2 + 2a\mu d)^{3/2}} \right) d\mu \\
&= -\frac{qa^3}{2} \left(1 - \frac{d^2}{a^2}\right) \frac{1}{2ad} \left(\frac{-2}{(a^2 + d^2 - 2da\mu)^{1/2}} - \frac{2}{(a^2 + d^2 + 2da\mu)^{1/2}} \right) \Bigg|_0^1
\end{aligned}
$$

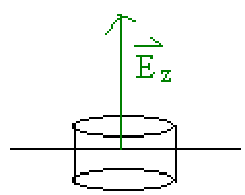
$$
\begin{aligned}
Q_{\text{boss}} &= -\frac{q}{2} \left(1 - \frac{d^2}{a^2}\right) \frac{a^2}{d} \left(\frac{-1}{(a^2 + d^2 - 2ad)^{1/2}} + \frac{2}{\sqrt{a^2 + d^2}} - \frac{1}{(a^2 + d^2 + 2ad)^{1/2}} \right) \\
&= -\frac{q}{2} \left(a^2 - d^2\right) \frac{1}{d} \left(\frac{-1}{d - a} - \frac{1}{a + d} + \frac{2}{\sqrt{a^2 + d^2}} \right)
\end{aligned}
$$

$$
Q_{\text{boss}} = -q \left(1 - \frac{d^2 - a^2}{d\sqrt{a^2 + d^2}}\right)
$$

which is Jackson's result.

Analysis: As $d \to \infty$ for fixed q, $Q_{\text{boss}} \to 0$. . Is this what you would have expected? The induced charge also goes to zero as $a \to 0$, .as the boss disappears in this case.

Solve: The charge density on the plane is



$$
\sigma(\rho) = \varepsilon_0 E_z = -\varepsilon_0 \left. \frac{\partial \Phi}{\partial z} \right|_{z=0}
$$

Here it is more convenient to express the potential in terms of z :

$$
\begin{aligned}
\Phi(\rho, z) &= kq \left\{ \frac{1}{\sqrt{z^2 + \rho^2 + d^2 - 2zd}} - \frac{1}{\sqrt{z^2 + \rho^2 + d^2 + 2zd}} \right. \\
&\qquad \left. - \frac{a}{d\sqrt{z^2 + \rho^2 + \frac{a^4}{d^2} - 2z\frac{a^2}{d}}} + \frac{a}{d\sqrt{z^2 + \rho^2 + \frac{a^4}{d^2} + 2z\frac{a^2}{d}}} \right\} \quad (13)
\end{aligned}
$$

Thus

$$\begin{aligned}\sigma(\rho) &= -\varepsilon_0 kq\left[\frac{-(z-d)}{(z^2+\rho^2+d^2-2zd)^{3/2}}-\frac{-(z+d)}{(z^2+\rho^2+d^2+2zd)^{3/2}}\right.\\ &\quad\left.\left.+\frac{a\left(z-a^2/d\right)}{d\left(z^2+\rho^2+\frac{a^4}{d^2}-2z\frac{a^2}{d}\right)^{3/2}}-\frac{a\left(z+a^2/d\right)}{d\left(z^2+\rho^2+\frac{a^4}{d^2}+2z\frac{a^2}{d}\right)^{3/2}}\right]\right|_{z=0}\\ &= -\frac{qd}{2\pi}\left(\frac{1}{(\rho^2+d^2)^{3/2}}-\frac{a^3}{d^3\left(\rho^2+\frac{a^4}{d^2}\right)^{3/2}}\right)\end{aligned}$$

where the second term is negligible if $d \gg a$.

The total charge on the plane is:

$$\begin{aligned}Q_{\text{plane}} &= -\frac{qd}{2\pi}2\pi\int_a^\infty\left(\frac{1}{(\rho^2+d^2)^{3/2}}-\frac{a^3}{d^3\left(\rho^2+\frac{a^4}{d^2}\right)^{3/2}}\right)\rho d\rho\\ &= -\frac{qd}{2}\left.\left(\frac{-2}{(\rho^2+d^2)^{1/2}}-\frac{a^3(-2)}{d^3\left(\rho^2+\frac{a^4}{d^2}\right)^{1/2}}\right)\right|_a^\infty\\ &= -qd\left(\frac{1}{(a^2+d^2)^{1/2}}-\frac{a^3}{d^3\left(a^2+\frac{a^4}{d^2}\right)^{1/2}}\right)\\ &= -\frac{q}{d}\frac{d^2-a^2}{\sqrt{d^2+a^2}}\end{aligned}$$

Analysis: $Q_{\text{plane}} \to -q$ as $a \to 0$ (flat plate) or $d \to \infty$. The total induced charge is:

$$\begin{aligned}Q_{\text{boss}}+Q_{\text{plane}} &= -q\left(1-\frac{d^2-a^2}{d\sqrt{a^2+d^2}}\right)-q\frac{1}{d}\frac{d^2-a^2}{\sqrt{d^2+a^2}}\\ &= -q\end{aligned}$$

as expected.

Solve: Now let's put in the value for q that gets us to the capacitor-plus-boss system (eqn 12): $q = E_0 d^2/2k$. Then, for $d \gg a$, the charge on the boss is:

$$\begin{aligned}Q_{\text{boss}} &= -q\left(1-\frac{1-a^2/d^2}{\sqrt{a^2/d^2+1}}\right)\simeq -\frac{E_0}{2k}d^2\left[1-\left(1-\frac{a^2}{d^2}\right)\left(1-\frac{1}{2}\frac{a^2}{d^2}\right)+\mathcal{O}\left(\frac{a}{d}\right)^4\right]\\ &= -\frac{E_0}{2k}d^2\left(\frac{3}{2}\frac{a^2}{d^2}\right)=-\frac{3}{4}\frac{E_0}{k}a^2=-3\pi\varepsilon_0 E_0 a^2\end{aligned}$$

Analysis: This is Jackson's answer in (b). Notice that d disappears in the limit $d \to \infty$, as required.

Solve: The charge densities are:

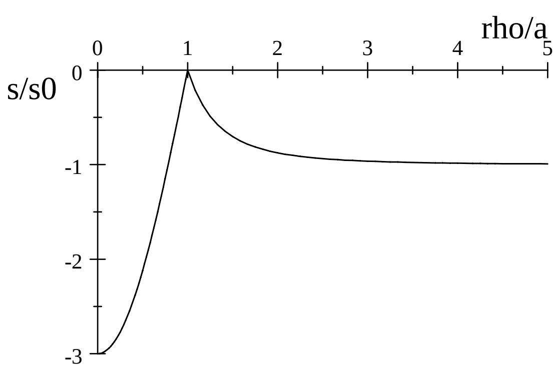
$$\begin{aligned}
\sigma(\theta) &= \varepsilon_0 E_0 \frac{a}{2}\left(1-\frac{d^2}{a^2}\right)\left(\frac{d^2}{(a^2+d^2-2ad\cos\theta)^{3/2}}-\frac{d^2}{(a^2+d^2+2ad\cos\theta)^{3/2}}\right)\\
&= \varepsilon_0 E_0 \frac{a}{2d}\left(1-\frac{d^2}{a^2}\right)\left(\frac{1}{(a^2/d^2+1-2a/d\cos\theta)^{3/2}}-\frac{1}{(a^2/d^2+1+2a/d\cos\theta)^{3/2}}\right)\\
&\simeq \varepsilon_0 E_0 \frac{a}{2d}\left(-\frac{d^2}{a^2}\right)\left(1+3\frac{a}{d}\cos\theta-\left(1-3\frac{a}{d}\cos\theta\right)\right)\\
&= \varepsilon_0 E_0 \frac{a}{2d}\left(-\frac{d^2}{a^2}\right)\left(6\frac{a}{d}\cos\theta\right) = -3\varepsilon_0 E_0\cos\theta
\end{aligned}$$

on the boss, and on the plane

$$\begin{aligned}
\sigma(\rho) &= -\frac{E_0 d^3}{4\pi k}\left(\frac{1}{(\rho^2+d^2)^{3/2}}-\frac{a^3}{d^3\left(\rho^2+\frac{a^4}{d^2}\right)^{3/2}}\right)\\
&= -\frac{E_0}{4\pi k}\left(1-\frac{3\rho^2}{2d^2}-\frac{a^3}{\rho^3\left(1+\frac{a^4}{d^2\rho^2}\right)^{3/2}}\right)\\
&= -\frac{E_0}{4\pi k}\left(1-\frac{3\rho^2}{2d^2}-\frac{a^3}{\rho^3}\left(1-\frac{3}{2}\frac{a^4}{d^2\rho^2}\right)\right)\\
&\to -\varepsilon_0 E_0\left(1-\frac{a^3}{\rho^3}\right) \text{ as } d\to\infty
\end{aligned}$$

Analysis: Note that the charge density is zero where the boss meets the plate ($\rho = a$, $\theta = \pi/2$), as expected near a sharp "hole" in the conductor. Also $\sigma \to -\varepsilon_0 E_0$ as $\rho \to \infty$, the expected result for a flat plate.

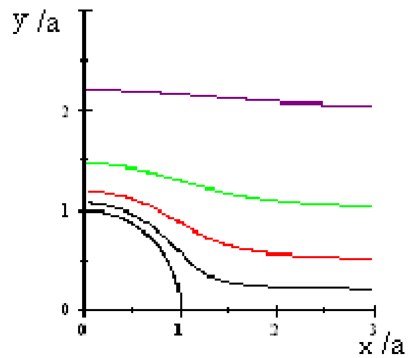
The plot shows $\sigma/\varepsilon_0 E_0$ versus ρ/a for $\rho > a$ and versus $2\theta/\pi$ for $0 < \theta < \pi/2$



Solve: The potential (13) is

$$\begin{aligned}
\Phi(\rho,z) &= \frac{kq}{d}\left\{\frac{1}{\sqrt{1+\left(\frac{z}{d}\right)^2+\left(\frac{\rho}{d}\right)^2-2\frac{z}{d}}}-\frac{1}{\sqrt{1+\left(\frac{z}{d}\right)^2+\left(\frac{\rho}{d}\right)^2+2\frac{z}{d}}}\right.\\
&\qquad\left.-\frac{a}{d\sqrt{\left(\frac{z}{d}\right)^2+\left(\frac{\rho}{d}\right)^2+\frac{a^4}{d^4}-2\frac{z}{d}\frac{a^2}{d^2}}}+\frac{a}{d\sqrt{\left(\frac{z}{d}\right)^2+\left(\frac{\rho}{d}\right)^2+\frac{a^4}{d^4}+2\frac{z}{d}\frac{a^2}{d^2}}}\right\}\\
&\simeq \frac{E_0 d}{2}\left\{1+\frac{z}{d}-\left[1-\frac{z}{d}\right]\right\}-\frac{E_0}{2}\frac{a}{\sqrt{z^2+\rho^2}}\left\{-2\frac{z}{d}\frac{a^2}{z^2+\rho^2}\right\}\\
&\simeq E_0 z\left(1-\frac{a^3}{\left(z^2+\rho^2\right)^{3/2}}\right)\text{ as } d\to\infty
\end{aligned}$$

Analysis: This function is plotted below. All distances are scaled by a.



Values of $\Phi/E_0 a$ are: black 0.2, red 0.5, green 1, purple 2

2 Expansion in orthogonal functions

To obtain a more useful form of the Green's function, we'll want to expand in orthogonal functions that are (relatively) easy to integrate. We begin with some basics and we'll see how to obtain solutions for the potential in boundary value problems with no charges inside the volume. Then we can move on to compute the Green's function.

2.1 Sturm-Liouville theory

See Lea Chapter 8 sections 1 and 2.

2.2 General method for finding the potential

1. Choose coordinates so that the boundaries of the region correspond to constant-coordinate surfaces. Then write the defining equation

$$\nabla^2\Phi = 0$$

in terms of the chosen coordinates, called u, v, and w, separate variables, and solve to determine the eigenfunctions. You'll get equations something like:

$$\frac{D_u U}{U} = -(\alpha + \beta); \quad \frac{D_v V}{V} = \alpha; \quad \frac{D_w W}{W} = \beta$$

D_u is a differential operator of the form

$$f(u) \frac{d^2}{du^2} + g(u) \frac{d}{du} + h(u)$$

and similarly for D_v and D_w.

2Note which boundary has a *non-zero* value of potential. This boundary should be defined by the condition $u =$ constant for one of the coordinates, u. You will want to choose your separation constants so that the sets of eigenfunctions in the other two coordinates v and w are orthogonal functions.

3Then use the boundaries on which $\Phi = 0$ to determine your eigenfunctions V and W, and eigenvalues, (which I'll call $\alpha = \frac{D_v V}{V}$ and $\beta = \frac{D_w W}{W}$). If the boundaries are at a finite distance the eigenvalues will be countable and can be labelled with an integer: α_m. Otherwise they will form a continuous set.

4The last eigenfunction is determined from the differential equation in the final coordinate, u:

$$\gamma = \frac{D_u U}{U} = -\alpha - \beta$$

At this point you should have a general solution of the form:

$$\Phi(u, v, w) = \sum_{m,n} A_{mn} U_{mn} (u : -\alpha_m - \beta_n) V_m (v : \alpha_m) W_n (w : \beta_n)$$

where the coefficients A_{mn} are still undetermined.

5Now use the final, non-zero, boundary condition together with orthogonality of the eigenfunctions V_m and W_n to find the coefficients A_{mn}

2.3 Rectangular coordinates

See Lea §8.2 and Griffiths Ch 3 §3, Jackson §2.9, or check here:
http://www.physics.sfsu.edu/~lea/courses/ugrad/360notes7.PDF

2.3.1 Rectangular 2-D problems with non-zero potential on two sides.

Since the general method allows for non-zero potential on only one side, we have to solve these problems by superposition. Suppose we have a rectangular region measuring $a \times b$ with potential V on the sides at $x = 0$ and $y = b$, the sides at $x = a$ and $y = 0$ being grounded. We solve two problems, each with three sides grounded, one having potential V at $x = 0$ and one having potential V at $y = b$. Then we add the results. The solution is

$$\Phi = \Phi_1 + \Phi_2$$

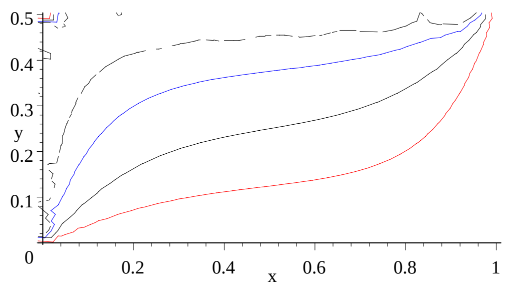
where (Lea Example 8.1)

$$\Phi_2 = \frac{4V}{\pi} \sum_{m=0}^{\infty} \frac{1}{2m+1} \sin\left[\frac{(2m+1)\pi x}{a}\right] \frac{\sinh\frac{(2m+1)\pi y}{a}}{\sinh\frac{(2m+1)\pi b}{a}}$$

and

$$\Phi_1 = \frac{4V}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin\left[\frac{(2n+1)\pi y}{b}\right] \frac{\sinh\frac{(2n+1)\pi(a-x)}{b}}{\sinh\frac{(2n+1)\pi a}{b}}$$

With $a = 2b$ the potential looks like this (the axes are x/a and y/a):



Blue 0.75, black, 0.5, red 0.25, dashed 0.9

2.3.2 Continuous set of eigenvalues

If one dimension of our rectangle becomes infinite, then the set of eigenvalues is no longer countable but becomes a continuous set. Suppose we let $b \to \infty$ in the example above. With $\Phi(x, y) = X(x) Y(y)$, Laplace's equation takes the form

$$\frac{X''}{X} = k^2 = -\frac{Y''}{Y}$$

. The solutions to the differential equation that satisfy the boundary conditions $\Phi = 0$ at $y = 0$ and at $x = a$ but Φ non-zero at $x = 0$ are of the form

$$\sin ky \sinh k(a - x)$$

But we no longer have an upper boundary in y to determine the values of k. (Effectively, Φ_1 in the previous problem has gone to zero because of the b in the denominator of the argument, leaving Φ_2 with an undetermined k.). However, since the eigenfunction is an even function of k, we need only positive values of k. Thus the solution is

$$\Phi(x, y) = \int_0^{\infty} A(k) \sin ky \sinh[k(a - x)] \, dk$$

To find the function $A(k)$, we use the known potential $V(y)$ at $x = 0$:

$$\Phi(0, y) = V(y) = \int_0^{\infty} A(k) \sin ky \sinh ka \, dk$$

We now make use of the orthogonality of the functions $\sin ky$ by multiplying both sides by $\sin k'y$ and integrating over y :

$$\int_0^\infty V(y) \sin k'y \, dy = \int_0^\infty \int_0^\infty A(k) \sin ky \sinh ka \, dk \; \sin k'y \, dy$$

On the RHS, we have

$$\begin{aligned}
\int_0^\infty \sin ky \sin k'y \, dy &= \frac{1}{2} \int_0^\infty \left[\cos(k-k')y - \cos(k+k')y\right] \, dy \\
&= \frac{1}{4} \int_0^\infty \left\{ e^{i\left(k-k'\right)y} + e^{-i\left(k-k'\right)y} - e^{i\left(k+k'\right)y} - e^{-i\left(k+k'\right)y} \right\} \, dy \\
&= \frac{1}{4} \int_{-\infty}^\infty \left\{ \exp\left[i(k-k')y\right] - \exp\left[i(k+k')y\right] \right\} \, dy \\
&= \frac{\pi}{2} \left[\delta(k-k') - \delta(k+k')\right]
\end{aligned}$$

where we used Lea eqn 6.16. Thus

$$\int_0^\infty V(y) \sin k'y \, dy = \frac{\pi}{2} \int_0^\infty A(k) \sinh ka \left[\delta(k-k') - \delta(k+k')\right] \, dk$$

Since k and k' are both positive, only the first delta function contributes, and this integral reduces to

$$\int_0^\infty V(y) \sin k'y \, dy = \frac{\pi}{2} A(k') \sinh k'a$$

and thus, dropping the primes,

$$A(k) = \frac{2}{\pi} \int_0^\infty V(y) \frac{\sin ky}{\sinh ka} \, dy$$

For example, if $V(y) = V_0 e^{-y/c}$, then

$$\begin{aligned}
A(k) &= \frac{2}{\pi} \int_0^\infty V_0 e^{-y/c} \frac{\sin ky}{\sinh ka} \, dy \\
&= \frac{1}{\pi i} \frac{V_0}{\sinh ka} \left[\frac{e^{(ik-1/c)y}}{ik - \frac{1}{c}} - \frac{e^{(-ik-1/c)y}}{-ik - \frac{1}{c}} \right]_0^\infty \\
&= \frac{1}{\pi i} \frac{V_0}{\sinh ka} \left[\frac{-1}{ik - \frac{1}{c}} - \frac{1}{ik + \frac{1}{c}} \right] \\
&= \frac{1}{\pi i} \frac{V_0}{\sinh ka} \left[\frac{2ic^2 k}{k^2 c^2 + 1} \right]
\end{aligned}$$

and thus

$$\Phi(x,y) = \frac{V_0}{\pi} \int_0^\infty \left(\frac{2c^2 k^2}{k^2 c^2 + 1} \right) \frac{\sin ky}{k} \frac{\sinh\left[k(a-x)\right]}{\sinh ka} \, dk$$

Analysis: We can see right away that our result is dimensionally correct. Also, since $x \le a$, $k(a-x) \le ka$ and thus

$$\frac{\sinh\left[k(a-x)\right]}{\sinh ka} \le 1$$

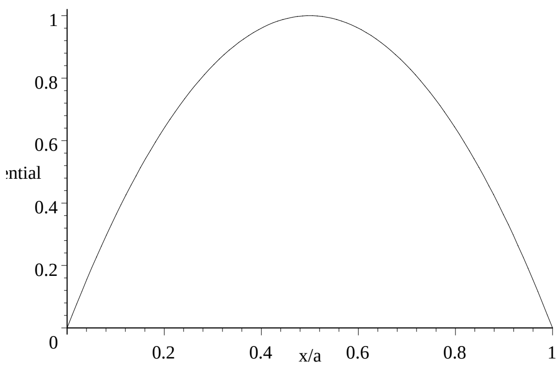
for all k, so the integral converges. I couldn't do this integral analytically, so let's compute some values in the case $c = a$:

$\Phi\left(\frac{a}{2},\frac{a}{2}\right)/V_0 = \frac{2}{\pi}0.314\,45 = 0.20019$
$\Phi\left(\frac{a}{2},a\right)/V_0 = \frac{2}{\pi}0.281\,18 = 0.179$

$\Phi\left(\frac{a}{2},2a\right)/V_0 = \frac{2}{\pi}0.119\,04 = 7.578\,3\times 10^{-2}$
The potential decreases rapidly for $y > a$

2.3.3 A 3-D problem

Find the potential inside a cubical box of side a with grounded walls, except for the side at $z = a$ which has potential $V_0\left[1-\left(\frac{2x}{a}-1\right)^2\right]$. This potential increases from zero at the walls to V_0 at $x = a/2$,as shown in the diagram.



There is no charge inside the box, so the potential satisfies Laplace's equation:

$$\nabla^2\Phi = 0$$
$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2} = 0$$

Try separating variables:

$$\Phi = XYZ$$

Then

$$X''YZ + XY''Z + XYZ'' = 0$$

or

$$\frac{X''}{X}+\frac{Y''}{Y}+\frac{Z''}{Z} = 0 \quad (14)$$

For equation 14 to hold for *all* values of x, y, z we must have:

$$\frac{X''}{X} = k_1,\ \frac{Y''}{Y} = k_2,\ \frac{Z''}{Z} = k_3 \text{ and } k_1+k_2+k_3 = 0$$

The potential is zero at $x = 0$ and at $x = a$, so we need k_1 to be negative, $k_1 = -\alpha^2$, which gives the solutions $X = \sin\alpha x$ and $X = \cos\alpha x$, which have multiple zeros. We choose the sine function to make $X(0) = 0$, and then choose $\alpha = n\pi/a$ to make $X(a) = 0$. A similar reasoning gets $Y = \sin(m\pi y/a)$. Then

$$k_3 = \left(n^2+m^2\right)\frac{\pi^2}{a^2}$$

and to make $Z(0) = 0$ the appropriate solution for Z is the $\sinh$.

$$\Phi(x,y,z) = \sum_{n,m=1}^{\infty} A_{nm} \sin n\pi\frac{x}{a} \sin m\pi\frac{y}{a} \sinh\sqrt{(n^2+m^2)}\frac{\pi z}{a}$$

Finally we evaluate the potential at $z = a$:

$$V_0\left[1-\left(\frac{2x}{a}-1\right)^2\right] = \sum_{n,m=1}^{\infty} A_{nm} \sin n\pi\frac{x}{a} \sin m\pi\frac{y}{a} \sinh\sqrt{(n^2+m^2)}\pi$$

We make use of the orhogonality of the sines by multiplying both sides by $\sin n'\pi x/a \sin m'\pi y/a$ and integrating over x and y :

$$\int_0^a\int_0^a V_0\left[1-\left(\frac{2x}{a}-1\right)^2\right]\sin\left(n'\pi\frac{x}{a}\right)\sin\left(m'\pi\frac{y}{a}\right)dxdy = A_{n'm'}\frac{a^2}{4}\sinh\sqrt{(n')^2+(m')^2}\pi$$

Drop the primes, and let $u = 2x/a - 1$, to get

$$\int_{-1}^{1} V\left(1-u^2\right)\sin n\pi\left(\frac{u+1}{2}\right)\frac{a}{2}du\;\frac{-\cos m\pi y/a}{m\pi/a}\Big|_0^a = A_{nm}\frac{a^2}{4}\sinh\sqrt{(n^2+m^2)}\pi$$

$$\frac{a}{2}\int_{-1}^{1} V_0\left(1-u^2\right)\left(\sin\frac{n\pi u}{2}\cos\frac{n\pi}{2}+\cos\frac{n\pi u}{2}\sin\frac{n\pi}{2}\right)du\frac{1-(-1)^m}{m\pi/a} = A_{nm}\frac{a^2}{4}\sinh\sqrt{(n^2+m^2)}\pi \quad (15)$$

The result is zero unless m is odd. When we integrate over u the sine term gives zero because we have an odd integrand over an even range. For the cosine term, we do integation by parts:

$$\begin{aligned}
\int_{-1}^{1}\left(1-u^2\right)\cos\frac{n\pi u}{2}du &= \frac{2}{n\pi}\sin\frac{n\pi u}{2}\Big|_{-1}^{+1} - 2\int_0^1 u^2\cos\frac{n\pi u}{2}du \\
&= \frac{4}{n\pi}\sin\frac{n\pi}{2} - 2\left[u^2\frac{2}{n\pi}\sin\frac{n\pi u}{2}\Big|_0^{+1} - 2\frac{2}{n\pi}\int_0^1 u\sin\frac{n\pi u}{2}du\right]
\end{aligned}$$

$$\begin{aligned}
\int_{-1}^{1}\left(1-u^2\right)\cos\frac{n\pi u}{2}du &= \frac{4}{n\pi}\sin\frac{n\pi}{2} \\
&\quad -2\left[\frac{2}{n\pi}\sin\frac{n\pi}{2} + 2u\left(\frac{2}{n\pi}\right)^2\cos\frac{n\pi u}{2}\Big|_0^{+1} - 2\left(\frac{2}{n\pi}\right)^2\int_0^1\cos\frac{n\pi u}{2}du\right] \\
&= -2\left[\frac{2^3}{(n\pi)^2}\cos\frac{n\pi}{2} - 2\left(\frac{2}{n\pi}\right)^3\sin\frac{n\pi u}{2}\Big|_0^{+1}\right] \\
&= -\frac{16}{(n\pi)^2}\cos\frac{n\pi}{2} + \frac{32}{(n\pi)^3}\sin\frac{n\pi}{2}
\end{aligned}$$

and so the left hand side of (15) is

$$\begin{aligned}
& V_0 \frac{a^2}{m\pi}\left(\frac{32}{(n\pi)^3}\sin^2 n\frac{\pi}{2} - \frac{16}{(n\pi)^2}\sin\frac{n\pi}{2}\cos\frac{n\pi}{2}\right) \\
= \;& V_0 \frac{a^2}{m\pi}\left(\frac{16}{(n\pi)^3}(1-\cos n\pi) - \frac{8}{(n\pi)^2}\sin n\pi\right) \\
= \;& V_0 \frac{a^2}{m\pi}\frac{16}{(n\pi)^3}\left(1-(-1)^n\right) \\
= \;& V_0 \frac{32a^2}{mn^3\pi^4} \text{ for } m \text{ and } n \text{ odd}
\end{aligned}$$

Thus for m odd and n odd

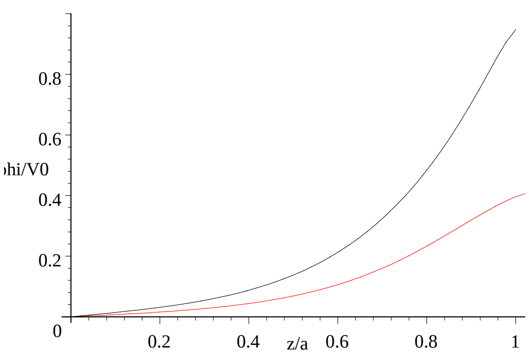
$$A_{nm} = V_0 \frac{128}{mn^3\pi^4}\frac{1}{\sinh\sqrt{n^2+m^2}\pi}$$

So

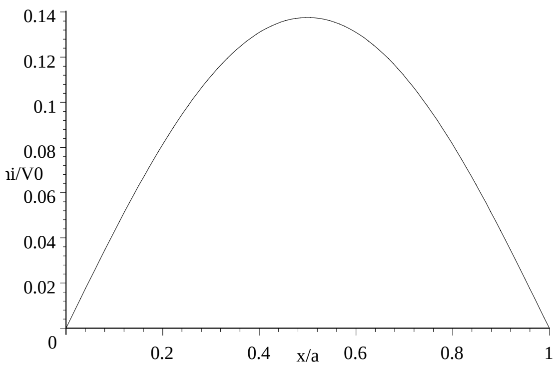
$$\Phi(x,y,z) = \frac{128}{\pi^4}V_0 \sum_{m=1,\text{odd}}^{\infty}\sum_{n=1,\text{odd}}^{\infty}\frac{1}{mn^3}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{a}\frac{\sinh\sqrt{n^2+m^2}\frac{\pi z}{a}}{\sinh\sqrt{n^2+m^2}\pi}$$

Analysis: The result is dimensionally correct, and converges nicely.

The plot shows the series up to $m = n = 5$. The black line is the potential as a function of z for $x = y = a/2$, and the red line for $x = y = a/4$. The potential increases faster near the center of the box, as expected, since the potential on the top side is maximum at $x = a/2$. .



At $z = a/2, y = a/2$ the potential versus x/a looks like this:



3 Conformal mapping

Recall that both the real and imaginary parts of an analytic function satisfy Laplace's equation in two dimensions. We may use this fact to solve 2-d potential problems.

3.0.4 Potential in a wedge.

Suppose the region of interest is defined by the angular wedge $0 \leq \theta \leq \alpha$. Further suppose the potential is zero on the conducting boundaries at $\theta = 0$ and $\theta = \alpha$. Then the analytic function

$$f(z) = z^{\pi/\alpha} = r^{\pi/\alpha} e^{i\theta\pi/\alpha} = r^{\pi/\alpha}\left(\cos\frac{\pi}{\alpha}\theta + i\sin\frac{\pi}{\alpha}\theta\right)$$

has imaginary part

$$v(r,\theta) = r^{\pi/\alpha}\sin\frac{\pi}{\alpha}\theta$$

that satisfies the boundary conditions. If $\alpha = \pi/m$ for some integer m, then $f(z)$ is analytic everywhere. If α is an arbitrary real number, then $f(z)$ may have a branch point at the origin, but we may choose the branch cut so that $f(z)$ is still analytic in our region everywhere except AT the origin. Thus $v(r,\theta)$ also satisfies Laplace's equation in the volume, and so it is a solution of the correct form. In fact the function $f(z) = z^{n\pi/\alpha}$ for any n has the same nice properties. Thus the potential in a wedge-shaped region with opening angle α and conducting boundaries at potential V_0 is described by the complex potential

$$\Phi(z) = A + iV_0 + \Sigma_n a_n z^{n\pi/\alpha}$$

The imaginary part is

$$V_0 + \sum_{n=-\infty}^{+\infty} a_n r^{n\pi/\alpha}\sin\frac{n\pi}{\alpha}\theta$$

which is the potential in the region. The coefficients a_n must be chosen to satisfy any remaining boundary conditions in r. Compare with Jackson 2.72 which he derives using separation of variables in plane polar coordinates.

The sum is over positive n if the origin is included within our region. The solution near the origin is dominated by the first $(n = 1)$ term. Then the field near the origin has components

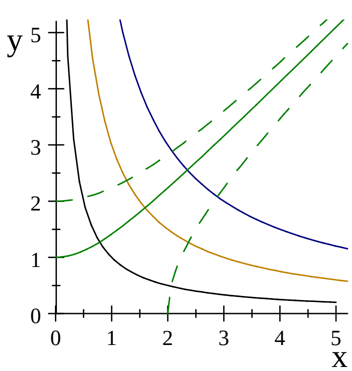
$$\vec{E} = a_1 \frac{\pi}{\alpha} r^{\pi/\alpha - 1} \left[\sin\left(\frac{\pi}{\alpha}\theta\right) \hat{r} + \cos\left(\frac{\pi}{\alpha}\theta\right) \hat{\theta} \right]$$

Thus $\left|\vec{E}\right| \to 0$ as $r \to 0$ if $\pi/\alpha > 1$ but $\left|\vec{E}\right| \to \infty$ as $r \to 0$ if $\pi/\alpha < 1$. The field is small in a hole ($\alpha < \pi$) but very large near a spike ($\alpha > \pi$).

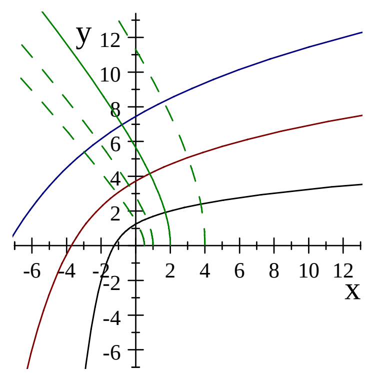
The diagram shows the cases of $\alpha = \pi/2$ and $3\pi/2$. For $\alpha = \pi/2$, the equipotential surfaces are

$$r^2 \sin 2\theta = \text{constant} = 2xy$$

Thus the equipotentials are hyperbolae.



equipotentials and field lines for $\alpha = \pi/2$



$\alpha = 3\pi/2$

We may use conformal mapping to find the potential in a 2-d boundary value problem with more complex boundaries. See Lea Ch 2 section 2.8. We choose the mapping to simplify the boundary shape, solve the simpler problem, and then map back.

4 Finite element analysis

The big idea: we want a procedure for numerically solving a potential problem for a source $g(\vec{x})$ in a region R with specified boundary conditions.

Let's look at a Dirichlet problem in 2 dimensions.

We start with two general relations. First, if $\nabla^2 \psi = -g$ in R, then

$$\int_R \phi \left(\nabla^2 \psi + g\right) dx dy = 0$$

for *any* function ϕ. The next relation we need is Green's identity from Chapter 1 (Notes 2.5 eqn 1).

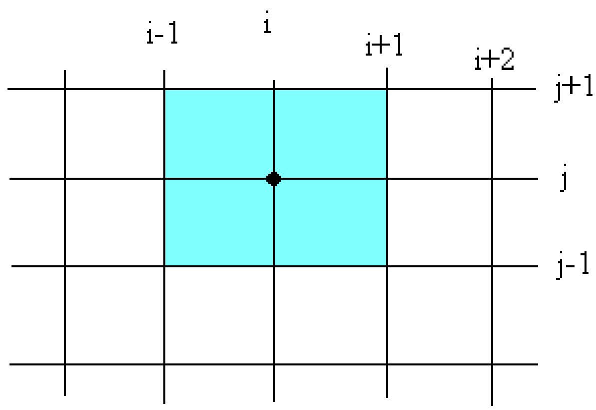
$$\int \left(\phi \nabla^2 \psi + \vec{\nabla}\phi \cdot \vec{\nabla}\psi\right) dV = \int_S \phi \frac{\partial \psi}{\partial n} dA = 0$$

where we have chosen the function ϕ to be zero on the boundary. (Note: in a two dimensional problem the "volume" is an area and the bounding "surface" S is actually a curve.) Now inserting our differential equation for ψ, we get:

$$\int_R \left(\vec{\nabla}\phi \cdot \vec{\nabla}\psi - g\phi\right) dx dy = 0 \qquad (16)$$

The next step is to set up a grid over the region R. We expand the desired solution ψ in a set of functions ϕ_{ij} each of which is zero except on a small region around the grid point (i, j) . For example, let:

$$\phi_{ij} = \begin{cases} \left(1 - \frac{|x-x_i|}{h}\right)\left(1 - \frac{|y-y_j|}{h}\right) & \text{for} \quad |x - x_i| < h, |y - y_j| < h \\ 0 & \text{otherwise} \end{cases}$$



and then let

$$\psi(x, y) = \sum_{k,l} \psi_{kl}\phi_{kl}(x, y)$$

Effectively, the function ϕ_{kl} smears the potential value ψ_{kl} over one grid square, converting a set of numerical values ψ_{kl} into a continuous function.$\psi(x, y)$. Now stuff this assumed form into the integral 16, and take the function $\phi \equiv \phi_{ij}$.

$$\int_R \sum_{k,l} \psi_{kl}\vec{\nabla}\phi_{kl} \cdot \vec{\nabla}\phi_{ij} dxdy = \int_R \phi_{ij} g(x, y)\, dxdy \qquad (17)$$

The integral on the right is non zero only on a square region surrounding (x_i, y_j) , and because the function ϕ_{ij} is "peaky" we can approximate the integral as:

$$\int_R \phi_{ij} g(x, y)\, dxdy \simeq g(x_i, y_j) \int_{\text{square}} \phi_{ij} dxdy$$

which may be evaluated as follows. Let $u = x - x_i$ and $v = y - y_j$. Then:

$$\begin{aligned} \int_{\text{square}} \phi_{ij} dxdy &= \left(\int_0^h \left(1 - \frac{u}{h}\right) du + \int_{-h}^0 \left(1 + \frac{u}{h}\right) du\right)\left(\int_0^h \left(1 - \frac{v}{h}\right) dv + \int_{-h}^0 \left(1 + \frac{v}{h}\right) dv\right) \\ &= \left(\frac{h}{2} + \frac{h}{2}\right)\left(\frac{h}{2} + \frac{h}{2}\right) = h^2 \end{aligned}$$

Thus (17) becomes:

$$\sum_{k,l} \psi_{kl} \int_R \vec{\nabla}\phi_{kl} \cdot \vec{\nabla}\phi_{ij} dxdy = h^2 g(x_i, y_j) \qquad (18)$$

where the sum on the left is over all the grid points. However, the integrand is non-zero only on a small rectangular region $2h$ by $2h$ surrounding the grid point (i, j). First note that inside this square region,

$$\vec{\nabla}\phi_{ij} = \mp\frac{1}{h}\left(1 \mp \frac{v}{h}\right)\hat{\mathbf{x}} + \left(1 \mp \frac{u}{h}\right)\left(\mp\frac{1}{h}\right)\hat{\mathbf{y}}$$

where in the first factor of each term we take the upper sign on the right half of our square $(0 < u < h)$ and the bottom sign on the left $(-h < u < 0)$; in the second factor we take the top sign on the top of the box $(0 < v < h)$ and the bottom sign on the bottom $(-h < v < 0)$.

Thus if $k = i$ and $l = j$:

$$\begin{aligned}
\int_{box} \vec{\nabla}\phi_{ij} \cdot \vec{\nabla}\phi_{i,j} dxdy &= \frac{1}{h^2}\int_0^h \int_0^h \left[\left(1 - \frac{v}{h}\right)^2 + \left(1 - \frac{u}{h}\right)^2\right] dudv \\
&\quad + \frac{1}{h^2}\int_0^h \int_{-h}^0 \left[\left(1 + \frac{v}{h}\right)^2 + \left(1 - \frac{u}{h}\right)^2\right] dudv \\
&\quad + \frac{1}{h^2}\int_{-h}^0 \int_0^h \left[\left(1 - \frac{v}{h}\right)^2 + \left(1 + \frac{u}{h}\right)^2\right] dudv \\
&\quad + \frac{1}{h^2}\int_{-h}^0 \int_{-h}^0 \left[\left(1 + \frac{v}{h}\right)^2 + \left(1 + \frac{u}{h}\right)^2\right] dudv \\
&= 4\int_0^1 \int_0^1 \left(\alpha^2 + \beta^2\right) d\alpha d\beta \\
&= \frac{4}{3}\left(\alpha^3\beta + \alpha\beta^3\right)\Big|_0^1\Big|_0^1 = \frac{8}{3}
\end{aligned}$$

where in the four terms we took $\alpha = 1 - \frac{v}{h}$ (terms1 and 3), $\alpha = 1 + \frac{v}{h}$ (terms 2 and 4), $\beta = 1 - \frac{u}{h}$ (terms1 and 2) or $\beta = 1 + \frac{u}{h}$ (terms 3 and 4).

Also if $k = i + 1$ and $l = j$:

$$\begin{aligned}
\phi_{i+1,j} &= \left(1 - \frac{|x - x_i - h|}{h}\right)\left(1 - \frac{|y - y_j|}{h}\right) \\
&= \left(1 - \frac{|u - h|}{h}\right)\left(1 - \frac{|v|}{h}\right) \text{ for } |u - h| < h \text{ and } |v| < h
\end{aligned}$$

and zero otherwise. The overlap region is on the right side of our box $(0 < u < h)$ where

$$\vec{\nabla}\phi_{i+1,j} = \frac{1}{h}\left(1 \mp \frac{v}{h}\right)\hat{x} + \frac{u}{h}\left(\mp\frac{1}{h}\right)\hat{y}$$

giving

$$\begin{aligned}
\int_{box} \vec{\nabla}\phi_{ij} \cdot \vec{\nabla}\phi_{i+1,j} dxdy &= \frac{1}{h^2} \int_0^h \int_0^h \left(-\left(1 - \frac{v}{h}\right)^2 + \left(1 - \frac{u}{h}\right) \frac{u}{h} \right) dudv \\
&\quad + \frac{1}{h^2} \int_{-h}^0 \int_0^h \left(-\left(1 + \frac{v}{h}\right)^2 + \left(1 - \frac{u}{h}\right) \frac{u}{h} \right) dudv \\
&= \int_0^1 \int_0^1 \left(-\alpha^2 + \beta\left(1 - \beta\right)\right) d\alpha d\beta \\
&= 2 \left(-\frac{\alpha^3 \beta}{3} + \frac{\alpha\beta^2}{2} - \frac{\alpha\beta^3}{3} \right) \Big|_0^1 \Big|_0^1 \\
&= 2\left(\frac{1}{2} - \frac{2}{3}\right) = -\frac{1}{3}
\end{aligned}$$

We get the same result in all the overlap regions. Thus equation (18) may be written as a matrix equation:

$$\mathbb{K}\Psi = \mathbb{G}$$

where the matrix $\mathbb{K}$ is described as "sparse" – it has only a few non-vanishing elements and they are all near the diagonal, like this:

$$\mathbb{K} = \frac{1}{3} \begin{pmatrix} 8 & -1 & -1 & 0 & 0 \\ -1 & 8 & -1 & -1 & 0 \\ -1 & -1 & 8 & -1 & -1 \\ 0 & -1 & -1 & 8 & -1 \\ 0 & 0 & -1 & -1 & 8 \end{pmatrix}$$

Matrices of this type are relatively easy to invert numerically. The column vectors Ψ and $\mathbb{G}$ contain the values of the potential and the source at the grid points. and we have reduced the potential problem to a matrix inversion.

$$\Psi = \mathbb{K}^{-1}\mathbb{G}$$

For regions with odd shapes, the square grid we used above does not fit very well. Triangles of arbitrary size and shape can be fit onto a region of almost any shape. So now we'll modify the method above to use triangles instead of squares. By varying the sizes and shapes of the triangles we can also get better resolution where things change more rapidly.

The basic triangular element has vertices at points that we label 1,2 and 3 with coordinates (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) . We approximate the potential solution $\psi(x, y)$ in this triangle by a Taylor-series-type expansion of the form:

$$\psi(x, y) = A + Bx + Cy$$

The 3 values of the potential at the 3 vertices give enough information to evaluate the 3 coefficients A, B and C. To make the numerical computation more efficient, it is convenient to define three "shape functions" $N_i(x, y) = a_i + b_i x + c_i y$ that have the properties:

$$N_i(x_i, y_i) = 1$$

and

$$N_i(x_j, y_j) = 0, \; i \neq j$$

These functions are the analogue of the ϕ_{ij} that we used with the square grid. They have the effect of smearing the potential at node i over the whole triangle. Then, for example, for N_1 we have:

$$\begin{aligned} a_1 + b_1 x_1 + c_1 y_1 &= 1 \\ a_1 + b_1 x_2 + c_1 y_2 &= 0 \\ a_1 + b_1 x_3 + c_1 y_3 &= 0 \end{aligned}$$

This set of equations has a nontrivial solution for the coefficients a, b and c if the determinant:

$$D = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \neq 0$$

The determinant is:

$$\begin{aligned} D &= x_2 y_3 - x_3 y_2 + x_3 y_1 - x_1 y_3 + x_1 y_2 - x_2 y_1 \\ &= (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1) \end{aligned}$$

This is related to the area of the triangle. To see how, remember that we can write the area using the cross product:

$$A = \frac{1}{2} \left| \vec{\ell}_1 \times \vec{\ell}_2 \right|$$

where $\vec{\ell}_1$ and $\vec{\ell}_2$ are vectors along the sides of the triangle. In terms of the coordinates:

$$\vec{\ell}_1 = (x_2 - x_1)\,\hat{x} + (y_2 - y_1)\,\hat{y}$$

and

$$\vec{\ell}_2 = (x_3 - x_1)\,\hat{x} + (y_3 - y_1)\,\hat{y}$$

Then

$$2A = (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)$$

and so

$$D = 2A$$

and so is never zero. Then the solution for the coefficients is:

$$\begin{aligned} a_1 &= \frac{1}{2A}(x_2 y_3 - x_3 y_2) \\ b_1 &= \frac{1}{2A}(y_2 - y_3) \\ c_1 &= -\frac{1}{2A}(x_2 - x_3) \end{aligned}$$

and similarly for the others.

Now the procedure runs pretty much as before. In equation (16) we take:

$$\psi = \sum_j \phi_j N_j$$

where the sum is over all the vertices of all the triangles. Next take $\phi = N_i$ for one vertex of one triangle. Then relation (16) becomes:

$$\begin{aligned} \sum_{j=1}^{3} \phi_j \int_{\text{triangle}} \vec{\nabla} N_i \cdot \vec{\nabla} N_j \, dxdy &= \int g N_i \, dxdy \simeq g(\overline{x}, \overline{y}) \int N_i \, dxdy \\ &= g(\overline{x}, \overline{y})\, A\, (a_i + b_i \overline{x} + c_i \overline{y}) \end{aligned} \tag{19}$$

where $\overline{x}$, $\overline{y}$ are the coordinates of the center of the triangle. Now using our solutions for the coefficients, we have:

$$\begin{aligned} a_1 + b_1\overline{x} + c_1\overline{y} &= \frac{1}{2A}\left[(x_2y_3 - x_3y_2) + (y_2 - y_3)\frac{(x_1+x_2+x_3)}{3} - (x_2 - x_3)\frac{(y_1+y_2+y_3)}{3}\right] \\ &= \frac{1}{6A}\left[x_2y_3 - x_3y_2 + x_1y_2 - x_1y_3 - x_2y_1 + x_3y_1\right] = \frac{D}{6A} = \frac{1}{3} \end{aligned}$$

On the left hand side, we use the result that

$$\vec{\nabla}N_i = b_i\hat{\mathbf{x}} + c_i\hat{\mathbf{y}}$$

and is constant over the area of the triangle, so (19) becomes

$$\sum_{j=1}^{3}\phi_j\left(b_ib_j + c_ic_j\right)A = g\left(\overline{x},\overline{y}\right)\frac{A}{3}$$

To combine the triangles, we define

$$k_{ij} = \left(b_ib_j + c_ic_j\right)A \tag{20}$$

If $i = j$, k_{ii} refers to a single node. For each internal node i we sum over all the triangles connected to that node.

$$K_{ii} = \sum_{\text{triangles}} k_{ii}$$

The elements k_{ij}, $i \neq j$, are associated with two nodes, *i.e.* with the side of the triangle connecting the nodes. So we sum over all the triangles with a side along ij (usually two).

$$K_{ij} = \sum_{\text{triangles}} k_{ij} \qquad i < j \leq N$$

If a node is on the boundary, the value of the potential there will be known. These terms are moved to the right hand side and serve as source terms. So we have:

$$G_i = \frac{1}{3}\sum_{\text{triangles}} A_tg_t - \sum_{j=N+1}^{N_0} K_{ij}\phi_j$$

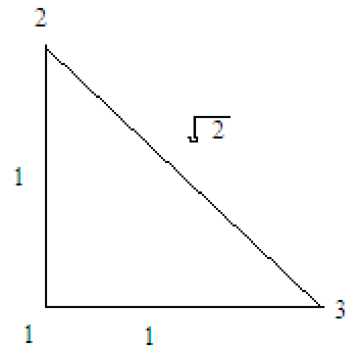
where the numbers $N+1$ to N_0 label the nodes on the boundary.

Then combining the results for all the triangles, we have the matrix equation:

$$\mathbb{K}\Phi = \mathbb{G}$$

where again $\mathbb{K}$ is a sparse matrix.

Example of how to get the k_{ij}. Consider an isoceles right triangle of sides 1,1 and $\sqrt{2}$. Put the origin at the right angle, and label the vertices 1 (the origin), 2 (on the y-axis) and 3 (on the x-axis). Then the area of the triangle is 1/2, and $2A = 1$



.

$$a_1 = x_2 y_3 - x_3 y_2 = 0 - 1 = -1$$
$$b_1 = y_2 - y_3 = 1$$
$$c_1 = -(x_2 - x_3) = -(-1) = 1$$
$$b_2 = y_3 - y_1 = 0$$
$$c_2 = -(x_3 - x_1) = -1$$

Then from (20) we get

$$k_{11} = \frac{1}{2}\left(b_1^2 + c_1^2\right) = \frac{1}{2}(1+1) = 1$$
$$k_{12} = \frac{1}{2}(b_1 b_2 + c_1 c_2) = \frac{1}{2}(0 + 1(-1)) = -\frac{1}{2}$$

and so on. (See Jackson Fig 2.16). These coefficients depend on the shape of the triangle but not on its orientation or size.