

## Fast Light and Causality

By Evan Bierman

It has become possible to control the velocity of light to such a high precision that it is now possible to make certain parts of light exceed the speed of light,  $c$ , slow light down, and stop it completely. However, Einstein's theory of special relativity, SR, is not invalidated in light of these new findings. To understand further, the terminology needs to be fully explained.

Light has both properties of waves and particles. The light particle is called a photon which is a chargeless massless elementary particle with spin equal to 1. Therefore it is a boson and is defined to always travel at speed  $c$ . However, light also has properties of waves and some of those properties may travel faster than  $c$  without invalidating SR. Light can be thought of as electromagnetic waves and can be represented by the spacial and time varying electric field ( $E$ ). For most cases the magnetic field ( $B$ ) is ignored because the magnitude of  $B$  is a factor  $c$  smaller than  $E$ .

Group velocity,  $v_g$ , is the velocity at which the entire wave packet propagates and can be thought of as the bulk motion of the waves. This term is sometimes confused with signal velocity,  $v_s$ , which is similar but there are a few key differences. The signal velocity is defined as measuring the deviation from a zero point. Brillouin and Sommerfeld proved that this cannot be faster than  $c$ , however the group velocity can be faster than light. The group velocity is defined from a set of phases in the group wave packet. These are very similar definitions but distinctly different in their mathematical definitions and their real world applications.

Phase velocity  $v_p$  is defined to be the velocity at which points of constant phase move through the medium.  $v_p$  has always been predicted to be able to travel faster than  $c$ , however,  $v_g$  is not supposed to from common thought. In regions where the  $v_g$  has exceeded  $c$ , classically the meaning of  $v_g$  lost its validity and these cases were ignored. However, it has been shown through modern experiments that  $v_g$  can retain its definition and exceed  $c$ , thereby seemingly causing a contradiction with Einstein. Through careful examination, one notes that this is not in conflict with causality and this is a valid theoretical and experimental result.

Causality means to follow Einstein's postulate that nothing can move faster than  $c$ , and more generally, that an event may not occur before the initiation of the event. In terms of space-time diagrams, the events must always be contained within the causal parts of the light cone and none may exist elsewhere in the spacetime diagram.

There is also the front velocity,  $v_f$ , energy velocity,  $v_e$ , and information velocity,  $v_i$ . None of these can exceed the speed of light and are causal at all times.  $v_f$  is the point of discontinuity at zero and always travels at  $c$ . In terms of Fourier analysis, the front is composed of the higher frequency components with smaller amplitudes. This is what Brillouin and Sommerfeld called a forerunner. This is contrast to the group region which is composed of smaller frequency components with larger amplitudes. Energy velocity is a more complicated topic to give a complete description for, but nevertheless cannot exceed  $c$ .  $v_i$  does not have a clear definition in mathematical terms but it is the underlying theme in causality and will never exceed  $c$ .

It has been shown experimentally that  $v_g$  does indeed exceed  $c$ , in a meaningful way, and

without any wave distortion. When talking about exceeding  $c$ , this refers specifically to  $v_g$ . Fast light or slow light is always mentioned in terms of wave propagation through a medium, and usually, compared to some reference light source traveling in air or vacuum. For most experiments, air propagation is used without any significant loss of precision. Slow light refers to  $v_g < c$  and fast light is  $v_g > c$ . There is also the possibility of negative group velocity, where the light exits the medium before the signal enters the medium.

To show these characteristics, use the mathematical expression for electromagnetic waves which can then be used to derive the common relationships of waves.

$$c \text{ is latin for celer, which means fast. } c \equiv 2.99792458 * 10^8 \frac{\text{m}}{\text{s}} \quad (1)$$

$$E = \tilde{E}_0 e^{i(k\vec{r} - \omega t + \phi)}; \text{ This is the general form which can be complex} \quad (2)$$

$$\text{The phase of the wave is } \Phi = (k\vec{r} - \omega t + \phi) \quad (3)$$

$$\text{where } k = \frac{2\pi}{\lambda} \text{ is the wave number} \quad (4)$$

$$\omega = 2\pi f \text{ is the angular frequency which is constant in all materials} \quad (5)$$

$\vec{r}$  is the distance from the origin, and  $\phi$  is an initial phase offset, which for this case is set to zero.  $\tilde{E}_0$  may be real or complex and contains the amplitude information and polarization information.  $k$  may also be expressed in any medium as  $nk_0$ . Where  $n$  is the index of refraction of the material and contains information about the material such that the speed within the material, and the materials response to electric fields.  $n$  is in general a function of  $\omega$ , the frequency of light traveling through the medium.

$$\text{frequency (f or } \omega) \times \text{Wavelength}(\lambda, \frac{1}{k}) = \text{speed of light in the medium (v)} \quad (6)$$

$$\text{travel time } dt = \frac{dr}{v} \quad (7)$$

To find the  $v_p$ , just track a point of constant phase. In a given time  $dt$ , the wave will travel  $dr$

$$\text{set } \Phi = 0 = k dr - \omega dt \Rightarrow v_p = \frac{dr}{dt} = \frac{\omega}{k} = \frac{c}{n(\omega)} \quad (8)$$

Setting the phase to be zero, a general expression for the phase velocity in a medium has been derived. The medium's specific properties have not been stated, but they will be mentioned later.

By Fourier analysis it is always possible to break down a wave packet into an infinite summation of different frequencies. For the wave packet to stay constant with no distortion, each frequency must travel together at the same speed and must add in phase at all points to be able to reproduce the wave packet at any time or point in space. The problem with group velocity is that most mediums cause distortion of the wave such that there is a spreading of Fourier wave frequen-

cies and causes the wave packet to spread. Each frequency is affected differently by the medium and causes this dispersion of the wave packet. To define the group velocity means that the phase must not change due to its frequencies changing to a first order approximation. Therefore

$$\frac{d\Phi}{d\omega} = 0 \quad (9)$$

$$\Phi = k r - \omega t = \frac{n(\omega) \omega}{c} r - \omega t = \frac{\omega}{c} (n(\omega) r - c t) \quad (10)$$

$$0 = \frac{d\Phi}{d\omega} = \frac{d}{d\omega} \left( \frac{n(\omega) \omega}{c} r - \omega t \right) = \frac{\omega}{c} \frac{dn}{d\omega} r + \frac{n(\omega)}{c} r - t \quad (11)$$

$$t = \frac{r}{v_g} \text{ and then solving for } v_g \quad (12)$$

$$0 = \frac{\omega \frac{dn}{d\omega} r + n(\omega) r}{c} - \frac{r}{v_g}$$

$$\Rightarrow v_g = \frac{c}{\omega \frac{dn}{d\omega} + n(\omega)} \quad (13)$$

Alternatively the definition of the wave number is that  $k = \frac{n(\omega) \omega}{c}$

$$\frac{dk}{d\omega} = \frac{\frac{dn}{d\omega} \omega + n(\omega)}{c} \Rightarrow \frac{d\omega}{dk} = \frac{c}{\omega \frac{dn}{d\omega} + n(\omega)} \quad (14)$$

$$\Rightarrow v_g = \frac{d\omega}{dk} \quad (15)$$

From the basic relation that  $v = \frac{c}{n_g}$  the group index can be defined as

$$n_g = n(\omega) + \frac{dn}{d\omega} \omega \quad \text{where } n_p = n(\omega) \quad (16)$$

The group index differs from the phase index by a factor of the phase and the derivative of the phase index. Slow light is when  $\frac{dn}{d\omega}$  is large and positive (large normal dispersion), and fast light is when  $\frac{dn}{d\omega}$  is large and negative (large anomalous dispersion). To take advantage of the change in index in terms of frequency, the response of the system, and more specifically the electrons and polarization, must be able to keep up. To determine this, model light traveling through a medium as a damped forced harmonic oscillator system where the general equation is the driving force equals the natural acceleration plus the "spring" force plus the dampening term. The damping term is important and without this term the oscillator would run away. In terms of light through a medium, this damping term comes from reradiation of energy mostly and also

from collisions with surrounding atoms. The spring terms is a combination of the "spring constant" and inertial mass.

$$F_D = \ddot{r} + \omega_0^2 r + 2 \beta \dot{r}$$

$$\omega_0^2 = \frac{k_e}{m_e}$$

$F_D = q E = q_e E_0 \text{Cos}[\omega t]$ ; where the cosine term is an equivalent way of expressing a light ray's oscillations. For this paper the damping term can be considered a constant. Clearly the solution to this problem is a frequency dependent amplitude times the driving oscillations.

$$\frac{q_e E_0}{m} \text{Cos}[\omega t] = \ddot{r} + \omega_0^2 r + 2 \beta \dot{r} \quad (17)$$

This differential equation can be solved by adding a complementary solution and particular solution together. The commentary solution is just the solution to the problem letting the term on the left equal 0. The particular solution reproduces the right hand side.

$$r_c(t) = e^{-\beta t} [A_1 e^{t\sqrt{\beta^2 - \omega_0^2}} + A_2 e^{-t\sqrt{\beta^2 - \omega_0^2}}] \quad (18)$$

$$r_p(t) = \frac{\frac{q_e}{m_e}}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4 \omega^2 \beta^2}} E_0 \text{Cos}\left[\omega t - \text{ArcTan}\left[\frac{2 \omega \beta}{\omega_0^2 - \omega^2}\right]\right] = D E(t) \quad (19)$$

$$D = \frac{\frac{q_e}{m_e}}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4 \omega^2 \beta^2}} \text{ and the } E(t) \text{ is the oscillating electric field}$$

However, the complementary solution dies out because of the damping factor and the particular solution is left. The polarization is an important property of the material and will help determine the index. Also the polarization and permeability of the material are dependent on each other.

$$P = N q_e r \text{ where } N \text{ is the total number of contributing electrons} \quad (20)$$

$$\epsilon = \epsilon_0 + \frac{P(t)}{E(t)} \quad (21)$$

$$n^2 = \frac{\epsilon}{\epsilon_0} = 1 + \frac{P(t)}{\epsilon_0 E(t)} = 1 + \frac{D}{\epsilon_0} = 1 + \frac{q_e}{\epsilon_0 m_e} \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4 \omega^2 \beta^2}} \quad (22)$$

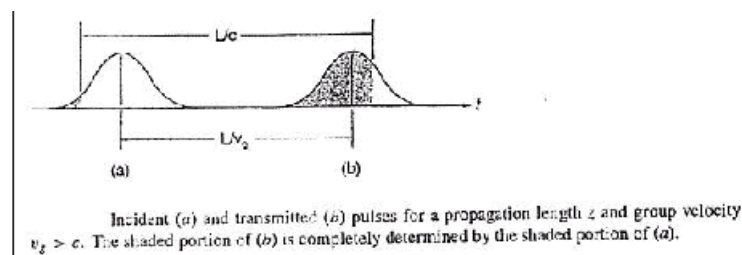
In general this expression can be expanded to involve all different possible characteristics

$$n^2 = 1 + \frac{N q_e^2}{\epsilon_0 m_e} \left[ \sum_j \frac{f_j}{\sqrt{(\omega_{0j}^2 - \omega^2)^2 + 4 \omega^2 \beta^2}} - \frac{f_e}{\omega^2 - i \beta \omega} \right] \quad (23)$$

$$\text{Plasma frequency } \omega_p = \sqrt{\frac{Nq_e^2}{\epsilon_0 m_e}} \Rightarrow n^2(\omega) = 1 - \left(\frac{\omega_p}{\omega}\right)^2; \quad (24)$$

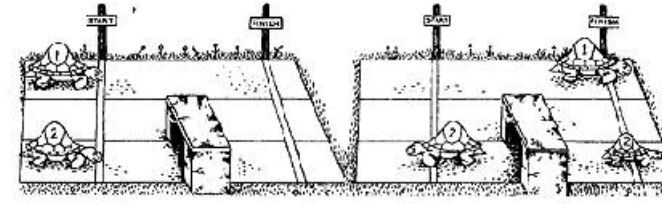
$N$  is the number of responding particles.  $f_i$  represents the strength of the different oscillator states for bound electrons in a substance or rather it represents transition probabilities. This is like a Boltzmann distribution of states where each state has a different probability of being occupied, but the sum over all states is normalized.  $\omega_{0j}$  are the characteristic absorbing and emitting frequencies which are based on quantum mechanical properties and can have an infinite amount of states.  $f_e$  represents the conducting electrons which have no natural frequency. Just looking at the free electron response the plasma frequency can be defined, see equation 24. It shows that when absorption is small, the amplitude is real, and the materials are transparent. This is assuming large frequencies, ignoring damping, and therefore most substances won't have any affect on x-rays and gamma rays. For frequencies below the plasma frequency the amplitude is imaginary which causes the damping of the field in the material, the absorption is small, and the material is opaque. A substance which absorbs strongly at a certain frequency actively reflects that particular frequency instead of absorbing. This, for example, causes such things as gold transmitting green light but absorbing gold light.

Looking at the equation, at large frequencies, the index can be less than one causing the phase velocity to be greater than  $c$ . This doesn't break causality because the phase cannot carry information with it. A single phase or frequency cannot carry information because only when there is a difference of a single quantity can there be information. Group velocity can also be greater than  $c$  if  $\frac{dn}{d\omega}$  is negative from the equation above. Group velocity can even be negative if  $\frac{dn}{d\omega}$  is negative enough. However, this also does not break causality because of non-locality of the wave packet. For example, if the group velocity is measured as the peak of the wave packet, the transmitted peak does not have to come from the peak of the incident wave. The information to form the peak of the transmitted wave can come from the tail of the incident ray. Thereby seeming to violate special relativity, but instead it is the reformation of the wave packet.



**Figure 1**

Another example of how group velocity can be greater than  $c$  follows from Figure 2. (Chaio, Milonni) This is comparing two light beams, with one having to travel through a certain medium, to two turtles racing, and one having to go through a barrier.



**Figure 2**

The probability that a turtle can pass through the barrier is small. Therefore, any turtle, or signal peak, that does pass through the barrier will be smaller. The shape of the turtle, or group, will be the same, except now it is smaller. Since the shape is the same, just smaller, the peak is closer to front, or shell to the nose. Therefore, the shell of the turtle, or group signal, passes through the finish line before the turtle that didn't go through the barrier. The wave fronts, or turtle noses, always travel at  $c$  and therefore the noses have always stayed causal. In this manner group velocity is allowed to travel faster than  $c$ , however, the faster it goes, or the more reshaping of the pulse, the more noise that is picked up. This comes from the fact that the bigger the barrier is, the smaller the transmitted turtle. In Figure 1, this is represented by using a smaller and smaller piece of the tail to make the peak. At a certain point, the quantum mechanical noises become larger than the signal if  $c$  is exceeded by too much. Signals can travel faster than light in a vacuum, as in the turtle case above, but causality is not broken because the noses always travel at  $c$ . In reality, the front is not traveling faster than  $c$ , and the nose to nose time is always  $c$ . The shell to shell time can be larger than  $c$  and this can be used in application in real life where the group velocity determines some quantity. For example, this let one quantum tunneling group to send Mozart's symphony across a barrier at speed  $4.7c$ .

The noses stayed causal and this makes sense from the Sommerfeld forerunner. Again, think of the front as the superposition of infinitely high frequencies with small amplitudes, the speed of the front must travel at  $c$ . Substituting in infinity into the equation above for the index of refraction produces a value of 1. This means that the speed is  $c$  for infinite frequencies in any medium. This also makes sense thinking about a wave propagating through a medium with electrons. As a wave passes the electron, there is some inertia to the electron which delays the response time. By the time the wave goes by and the electron reacts, the front of the wave has moved on leaving the electron to only interact with other parts of the wave.

This topic is one of the most fundamental and active research topics today. Using quantum optics physicists are exploring the quantum nature of the world. Einstein's view of a deterministic and local universe is being disproved by each new experiment. We are learning that we live in a probabilistic and non-local universe where quantum mechanics rules. However, just like light has both properties of waves and particles, our universe has both properties of deterministic and probabilistic, local and non-local. For example, the EPR paradox has been proven to be true and Bell's local variables have never been found. However, to make use of the EPR paradox, the information about each event would still have to be transmitted to each other and compared, thereby preserving SR and causality. This example is the same as if a laser is shined on the moon from earth and then the laser is moved. The laser point on the moon would go faster than the

speed of light but two consecutive points on the moon could not affect each other. One point on the moon could not cause a change that could propagate to the next point. The laser beam is moving causally here on Earth but the effects of which are moving non-causally somewhere else.

Optics mixed with relativity and Quantum Mechanics is clearly one of the most interesting fields of modern physics. The experiments on light, particles, group velocity, and the wave nature of reality are revealing the underlying fabric of our universe. We are investigating the nature of this fabric in the hopes of understanding who we are and why we are here. This is the heart of the study of physics. This is why physics is the truly fundamental science. These studies could one day cause revolutions unheard of in today's society. Just imagine if we could show a computer to Galileo or a car to Socrates. Physics continues to amaze us all.

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