Chapter 7. The Wave Equation

The vector spaces that we have described so far were finite dimensional. Describing position in the space we live in requires an infinite number of position vectors, but they can all be represented as linear combinations of three carefully chosen linearly independent position vectors. There are other analogous situations in which a complicated problem can be simplified by focusing attention on a set of basis vectors. In addition to providing mathematical convenience, these basis vectors often turn out to be interesting objects in their own right.

The pressure field inside a closed cylindrical tube (an "organ pipe") is a function of time and three space coordinates which can vary in almost an arbitrary way. Yet there are certain simple patterns which form a basis for describing all the others, and which are recognizably important and interesting. In this case they are the fundamental resonance of the organ pipe and its overtones. They are the "notes" which we use to make music, identified by our ears as basis vectors of its response space. A similar situation occurs with the vibrating strings of guitars and violas (or with the wires strung between telephone poles), where arbitrary waves moving on the string can be represented as a superposition of functions corresponding again to musical notes.

Another analogous situation occurs with quantum-mechanical electron waves near a positively charged nucleus. An arbitrarily complicated wave function can be described as a linear superposition of a series of "states," each of which corresponds (in a certain sense) to one of the circular Bohr electron orbits.

We will choose the stretched string to examine in mathematical detail. It is the easiest of these systems to describe, with a reasonable set of simplifying assumptions. The mathematical variety that it offers is a rich preview of mathematical physics in general.

A. Qualitative properties of waves on a string

Many interesting and intellectually useful experiments can be carried out by exciting waves on a stretched string, wire or
cord and observing their propagation. Snapping a clothesline or hitting a telephone line with a rock produces a pulse which travels along the line as shown in figure 7-1, keeping its shape. When it reaches the point where the line is attached, it reverses direction and heads back to the place where it started, upside down. The pulse maintains its shape as it travels, only changing its form when it reaches the end of the string.

A string which has vibrated for a long time tends to "relax" into simpler motions which are adapted to the length of the string. Figure 7-2 shows two such motions. The upper motion has a sort of sinusoidal shape at any given moment, with a point which never moves (a node) at either end of the string. If the string is touched with a finger at the midpoint, it then vibrates as sketched in the lower diagram, with nodes at the ends and also at the touched point. It will be noticed that the musical note made by the string in the second mode of vibration is an octave higher than the note from the first mode.

Another interesting motion can be observed by stretching the string into the shape of a pulse like that shown in figure 7-1, and then releasing it, so that the string is initially motionless. The pulse is observed to divide into two pulses, going in opposite directions, which run back and forth along the string. But after a time, the string "relaxes" into a motion like that in the upper part of figure 7-2.

How can a pulse propagate without its shape changing? Why does it reverse the direction of its displacement after reflection? Why does the guitar string vibrate with a sinusoidal displacement? Why does the mode with an additional displacement node vibrate with twice the frequency? Why does a traveling pulse relax into a stationary resonant mode? We will try to build up a mathematical description of waves in a string which predicts all these properties.

B. The wave equation.

The propagation of a wave disturbance through a medium is described in terms of forces exerted on an element of mass of the medium by its surroundings, and the resulting acceleration of the mass element. In a string, any small length of the string experiences two forces, due to the rest of the string pulling it to the side, in the two different directions. Figure 7-3 shows an element of the string of length $dx$, at a point where the string is curving downwards. The forces pulling on each end of the string are also shown, and it is clear that there is a net downwards force, due to the string tension. We will write down Newton's law for this string element, and show that it leads to the
partial differential equation known as the wave equation. But first, we will discuss a set of approximations which makes this equation simple.

First, we will make the small-angle approximation, assuming here that the angle which the string makes with the horizontal is small ($\theta \ll 1$). In this approximation, $\cos \theta \approx 1$ and $\sin \theta \approx \theta$.

Secondly, we will assume that the tension $T$ is constant throughout the string. Two of these assumptions ($\cos \theta = 1$ and $T$ constant) result in a net force of zero in the longitudinal ($x$) direction, so that the motion of the string is purely transverse. We will ignore a possible small longitudinal motion and assume that it would have only a small effect on the transverse motion which we are interested in.

In the transverse ($y$) direction, the forces do not cancel. The transverse component of the force on the left-hand side of the segment of string has magnitude $T \sin \theta$. We will relate $\sin \theta$ to the slope of the string, according to the relation from analytic geometry:

$$\text{slope} = \frac{dy}{dx} = \frac{\text{rise}}{\text{run}} = \tan \theta \approx \sin \theta.$$  (7-1)

The last, approximate, equality is due to the small-angle approximation. Thus the transverse force has magnitude approximately equal to $T \frac{dy}{dx}$, and Newton's second law for the transverse motion gives:

$$F_y = (dm)\frac{\partial y}{\partial x}$$

$$T \frac{\partial y}{\partial x} - T \frac{\partial^2 y}{\partial x^2} = (\mu dx)\frac{\partial^2 y}{\partial t^2}.$$  (7-2)

Here we have used the linear mass density $\mu$ (mass per unit length) to calculate $dm = \mu dx$.

**Partial derivatives.**

In the equation above we replaced the slope $\frac{dy}{dx}$ by $\frac{\partial y}{\partial x}$, a **partial derivative**. We need to explain briefly the difference between these two types of derivatives.

The displacement of the string is a function of two variables, the position $x$ along the string, and the time $t$. There are two physically interesting ways to take the derivative of such a function. The rate of change of $y$ with respect to $x$, at a fixed instant of time, is the slope of the string at that time, and the second derivative is related to the curvature of the string. Similarly, the rate of change of $y$ with respect to time at a fixed position gives the velocity of the string at that point, and the second time derivative gives the acceleration. These derivatives of a function with respect to one variable, while holding all other variables constant, are referred to as partial derivatives.

The partial derivatives of an arbitrary function $g(u,v)$ of two independent variables $u$ and $v$ are defined as follows:
(\frac{\partial g}{\partial u})_v \equiv \lim_{\delta u \to 0} \frac{g(u + \delta u, v) - g(u, v)}{\delta u} \tag{7-3}

(\frac{\partial g}{\partial v})_u \equiv \lim_{\delta v \to 0} \frac{g(u, v + \delta v) - g(u, v)}{\delta v}

The subscript indicates the variable which is held constant while the other is varied, and it can often be left off without ambiguity. Second partial derivatives can be defined, too:

\begin{align*}
\left( \frac{\partial^2 g}{\partial u^2} \right)_v & \equiv \lim_{\delta u \to 0} \frac{\frac{\partial g}{\partial u} (u + \delta u, v) - \frac{\partial g}{\partial u} (u, v)}{\delta u} \\
\left( \frac{\partial^2 g}{\partial v^2} \right)_u & \equiv \lim_{\delta v \to 0} \frac{\frac{\partial g}{\partial v} (u, v + \delta v) - \frac{\partial g}{\partial v} (u, v)}{\delta v} \tag{7-4}
\end{align*}

There is a fourth partial second-order partial derivative which we have omitted - but, for mathematically well behaved functions, it can be shown that

\begin{equation}
\frac{\partial^2 g}{\partial u \partial v} = \frac{\partial^2 g}{\partial v \partial u} \tag{7-5}
\end{equation}

The definitions give above are the formal definitions. Often in practice, however, taking a partial derivative simply means taking the derivative with respect to the stated variable, treating all other variables as constants.

Example. Calculate all the first and second partial derivatives of the function \( g(u, v) = u^3 v + \sin u \).

Solution: Evaluate the six partial derivatives. We will note in particular whether or not the two cross partial derivatives come out to be equal, as they should.

\begin{align*}
\frac{\partial g}{\partial u} & = \frac{\partial}{\partial u} (u^3 v + \sin u) = 3u^2 v + \cos u \quad (a) \\
\frac{\partial g}{\partial v} & = \frac{\partial}{\partial v} (u^3 v + \sin u) = u^3 \quad (b) \\
\frac{\partial^2 g}{\partial u^2} & = \frac{\partial}{\partial u} \left( \frac{\partial g}{\partial u} \right) = \frac{\partial}{\partial u} (3u^2 v + \cos u) = 6uv - \sin u \quad (c) \\
\frac{\partial^2 g}{\partial v^2} & = \frac{\partial}{\partial v} \left( \frac{\partial g}{\partial v} \right) = \frac{\partial}{\partial v} (u^3) = 0 \quad (d) \\
\frac{\partial^2 g}{\partial u \partial v} & = \frac{\partial}{\partial u} \left( \frac{\partial g}{\partial v} \right) = \frac{\partial}{\partial u} (3u^2 v + \cos u) = 3u^2 \quad (e) \\
\frac{\partial^2 g}{\partial v \partial u} & = \frac{\partial}{\partial v} \left( \frac{\partial g}{\partial u} \right) = \frac{\partial}{\partial v} (3u^2 v + \cos u) = 3u^2 \quad (f)
\end{align*}
Check the two cross partial derivatives. Sure enough, \( \frac{\partial^2 g}{\partial u \partial v} = \frac{\partial^2 g}{\partial v \partial u} \).

**Wave velocity.**

Now we will work out the partial differential equation resulting from Newton's second law. Starting with equation 7-2, we get

\[
\frac{\partial y}{\partial x} \bigg|_{x+dx} - \frac{\partial y}{\partial x} \bigg|_{x} = \frac{\mu \partial^2 y}{dx} \frac{\partial t^2}{T}
\]

But we recognize the left-hand side as just the definition of the second partial derivative with respect to \( x \), provided we let \( dx \) be arbitrarily small:

\[
\lim_{dx \to 0} \left( \frac{dy}{dx_{x+dx}} - \frac{dy}{dx_{x}} \right) = \frac{\partial^2 y}{\partial x^2}
\]

And so we get the differential equation for the string, also known as the wave equation:

\[
\frac{\partial^2 y}{\partial x^2} = \frac{\mu \partial^2 y}{T \partial t^2}
\]

It is easy to see that the dimensions of the two constants must be such that a characteristic velocity \( \nu \) can be defined as follows:

\[
\nu = \sqrt{\frac{T}{\mu}}
\]

In terms of this characteristic velocity, the differential equation becomes

\[
\frac{\partial^2 y}{\partial x^2} = \frac{1}{\nu^2} \frac{\partial^2 y}{\partial t^2}
\]

**The Wave Equation**

Now we must find the solutions to this partial differential equation, and find out what it is that moves at speed \( \nu \).

### C. Sinusoidal solutions.

Since we associate the sinusoidal shape with waves, let's just try the function

\[
y(x,t) = A \sin(kx - \omega t)
\]

At a fixed time (say, \( t = 0 \), this function (shown in figure 7-4) describes a
sinusoidal wave, \( \sin kx \), which repeats itself after the argument increases by \( 2\pi \). The distance over which it repeats itself after the argument increases by \( 2\pi \) is called the wavelength \( \lambda \). It is thus related to \( k \) as follows:

\[
k\lambda = 2\pi \quad \Rightarrow \quad k = \frac{2\pi}{\lambda} \quad (7-13)
\]

Similarly, if we observe the motion at \( x = 0 \) as a function of time, it repeats itself after a time interval \( T \), as the argument increases by \( 2\pi \). This means

\[
\omega T = 2\pi \quad \Rightarrow \quad \omega = \frac{2\pi}{T} \quad (7.14)
\]

Here \( \omega \) is the angular frequency, in radians per second, related to the frequency \( f \) (in cycles per second, or Hz) by \( \omega = 2\pi f \). This leads to the set of relations for sinusoidal waves,

\[
\begin{align*}
k &= \frac{2\pi}{\lambda} \\
\omega &= \frac{2\pi}{T} \\
f &= \frac{\omega}{2\pi} = \frac{1}{T}
\end{align*} \quad (7-15)
\]

We will now see if this sinusoidal function is a solution to the wave equation, by substitution.

\[
\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}
\]

\[
\frac{\partial^2}{\partial x^2} A\sin (kx - \omega t) = \frac{1}{v^2} \frac{\partial^2}{\partial t^2} A\sin (kx - \omega t) \quad (7-16)
\]

\[-k^2 \sin (kx - \omega t) = \frac{1}{v^2} (-\omega^2) A\sin (kx - \omega t) \]

Figure 7-4. A sinusoidal wave, at \( t = 0 \) and at a later time such that \( \omega t = 1 \).
The factor of \( \sin(kx - \omega t) \) cancels out on both sides, and it is clear that the sine wave is a solution to the wave equation only if \( k \) and \( \lambda \) are related in a certain way:

\[
\frac{\omega}{k} = \frac{\lambda}{T} = v
\]  
(7-17)

If we adopt this relation, we can write the sine wave in the following equivalent ways:

\[
\sin(kx - \omega t) = \sin(2\pi \left( \frac{x - \frac{t}{T}}{\lambda} \right)) = \sin k(x - vt)
\]  
(7-18)

**Example:** Consider transverse waves on the lowest string of a guitar (the E string). Let us suppose that the tension in the string is equal to 300 N (about 60 lbs), and that the 65-cm-long string has a mass of 5 g (about the weight of a nickel). The fundamental resonance of this string has a wavelength equal to twice the length of the string. Calculate the mass density of the string and the speed of transverse waves in the string. Also calculate \( f, \omega \) and \( k \) for this resonance.

**Solution:** The mass density is

\[
\mu = \frac{m}{L} = \frac{0.005 \text{ kg}}{0.65 \text{ m}} = 0.00769 \text{ kg/m}
\]  
(7-19)

The wave speed is then

\[
v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{(300 \text{ kg-m/s}^2)}{(0.00769 \text{ kg/m})}} = 197.5 \text{ m/s}
\]  
(7-20)

For a wavelength of \( \lambda = 1.3 \text{ m} \) we get

\[
k = \frac{2\pi}{\lambda} = \frac{2\pi}{(1.3 \text{ m})} = 4.833 \text{ m}^{-1}
\]

\[
\omega = kv = (4.833 \text{ m}^{-1})(197.5 \text{ m/s}) = 954 \text{ rad/sec}
\]  
(7-21)

\[
f = \frac{\omega}{2\pi} = 152 \text{ Hz}
\]

Is this right for an E string? Using a frequency of 440 Hz for A above middle C, an A two octaves down would have a frequency of \( 440 \text{ Hz}/4 = 110 \text{ Hz} \), and the E above that note would be at \( f = (110 \text{ Hz}) \star 1.5 = 165 \text{ Hz} \). [An interval of an octave corresponds to a factor of two in frequency; a fifth, as between A and E, corresponds to a factor of 1.5.]

So, the frequency of the E string is a bit low. What do you do to correct it? Check the equations to see if tightening the string goes in the right direction.
Example. Water waves are a type of transverse wave, propagating along the water surface. A typical speed for waves traveling over the continental shelf (water depth of 40 m) is \( v = 20 \text{ m/s} \). If the wavelength is \( \lambda = 220 \text{ m} \), find the frequency and period with which a buoy will bob up and down as the wave passes. Also calculate \( \omega \) and \( k \).

Solution. With the velocity and wavelength known, the period of the oscillation and the frequency can be calculated:

\[
T = \frac{\lambda}{v} = \frac{(220 \text{ m})}{(20 \text{ m/s})} = 11 \text{ sec} \quad (7-22)
\]

\[
f = \frac{1}{T} = 0.091 \text{ Hz}
\]

If you want to see if this period is reasonable, go to the web site http://facs.scripps.edu/surf/nocal.html and look at "the swells" off the coast right now.

And \( \omega \) and \( k \).

\[
\omega = 2\pi f = 0.571 \text{ rad/sec} \quad (7-23)
\]

\[
k = \frac{2\pi}{\lambda} = \frac{2\pi}{220 \text{ m}} = 0.0286 \text{ m}^{-1}
\]

As a cross check, see if \( \omega/k \) gives back the wave speed; this is an important relation.

\[
\frac{\omega}{k} = \frac{0.571 \text{ rad/sec}}{0.0286 \text{ m}} = 20 \text{ m/s} \quad (7-24)
\]

It works out.

D. General traveling-wave solutions.

It is heartening to have guessed a solution to the wave equation so easily. But how many other solutions are there, and what do they look like? Is a sine wave the only wave that travels along at speed \( v \), without changing its form?

In fact, it is easy to see that a huge variety of waves have this property. [However, later on we will see that the wave keeps its shape intact only if the wave speed does not depend on the wavelength.] Consider any function whose argument is \( (x-\nu t) \),

\[
y(x,t) = f(x-\nu t) \quad (7-30)
\]

We can take two partial derivatives with respect to \( x \) and two partial derivatives with respect to \( t \), using the chain rule, and see what happens.

\[
\frac{\partial}{\partial x} f(x-\nu t) = f'(x-\nu t) \frac{\partial}{\partial x} (x-\nu t) 
\]

\[
= f'(x-\nu t) \quad (7-31)
\]
Here $f'$ is the derivative of the function $f$ with respect to its argument. Taking the second partial derivative with respect to $x$ gives

$$\frac{\partial^2}{\partial x^2} f(x - vt) = \frac{\partial}{\partial x} f'(x - vt)$$

$$= f''(x - vt) \frac{\partial}{\partial x} (x - vt)$$

$$= f''(x - vt)$$  \hspace{1cm} (7-32)

Taking the time derivative is similar, except that in place of $\frac{\partial}{\partial x} (x - vt) = 1$, we have

$$\frac{\partial}{\partial t} (x - vt) = -v$$. So,

$$\frac{\partial}{\partial t} f(x - vt) = f'(x - vt) \frac{\partial}{\partial t} (x - vt)$$

$$= -vf'(x - vt)$$  \hspace{1cm} (7-33)

and

$$\frac{\partial^2}{\partial t^2} f(x - vt) = \frac{\partial}{\partial t} (-vf'(x - vt))$$

$$= -vf''(x - vt) \frac{\partial}{\partial t} (x - vt)$$

$$= v^2 f''(x - vt)$$  \hspace{1cm} (7-34)

Substituting into the wave equation shows it to be satisfied:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

$$f''(x - vt) = \frac{1}{v^2} \left(v^2 f''(x - vt)\right)$$  \hspace{1cm} (7-35)

$$= f''(x - vt)$$

As an illustration, let's consider the Gaussian function $g(u) = e^{-\frac{u^2}{2\sigma^2}}$ shown in figure 7-5. We can use this function to interpret the meaning of the parameter $v$ in the wave equation. The peak of the Gaussian occurs when the variable $u$ is equal to zero. Suppose the argument of the function is $x - vt$ instead of $u$:

$$g(x - vt) = e^{-\frac{(x-vt)^2}{2\sigma^2}}$$  \hspace{1cm} (7-36)

At $t = 0$, the peak of the function occurs when $x = 0$. But at a later time $t = \Delta t$, the peak as a function of position will occur at a value $\Delta x$ such that the argument $\Delta x - v\Delta t$ vanishes; this gives

$$\Delta x - v\Delta t = 0 \implies \frac{\Delta x}{\Delta t} = v = \text{wave velocity}$$  \hspace{1cm} (7-37)
Thus, the velocity with which the peak of the Gaussian appears to move is just the constant \( v \) which occurs in the wave equation. From now on, we will refer to it as the wave velocity.

**E. Energy carried by waves on a string.**

One of the paradoxical aspects of wave motion has to do with the question of what exactly about the wave is moving. The string itself does not move in the direction of wave propagation. It seems to be just the shape, or the profile made by the displaced string, which moves. But it can be seen that there is energy in the moving string, and an argument can be made that energy is transported by the wave, in the direction of the wave propagation.

**Kinetic energy.**

It is rather clear that there is kinetic energy due to the transverse motion associated with the displacement \( y(\Delta x - v\Delta t) \). Referring to figure 7-6a, we can see that the kinetic energy of the string between \( x \) and \( x+dx \) is given by

\[
d(KE) = \frac{1}{2} \, dm \left( \frac{\partial y(x,t)}{\partial t} \right)^2
\]

\[= \frac{1}{2} \mu dx \left( \frac{\partial y(x,t)}{\partial t} \right)^2 \tag{7-38}
\]

**Potential energy.**

The potential energy due to the displacement of the string from the equilibrium position is equal to the work done in stretching an undisplaced string out to the position of the string at a given instant of time. Figure 7-6b shows how this calculation proceeds. We will write the displacement of the string at this instant of time as \( y(x,t_0) \), where the subscript on \( t_0 \) reminds us that this time is held fixed. We imagine exerting at every point on the string just the force necessary to hold it in position. Then gradually the string is stretched from a straight line out to its shape \( y(x,t_0) \) at time \( t_0 \). Let \( \eta(x) = a f(x,t_0) \) be the displacement of the string during this process. As the parameter \( \alpha \) goes from 0 to 1, the string displacement goes from \( \eta(x) = 0 \) to \( \eta(x) = y(x,t_0) \).

In deriving the wave equation we calculated the force on a length \( dx \) of the string, due to the tension \( T \) and the curvature of the string:

\[
F_y = T \left( \frac{\partial y}{\partial x_{x+dx}} \right)_x - \frac{\partial y}{\partial x_{x}}
\]

\[= Tdx \left( \frac{\partial y}{\partial x_{x+dx}} \right)_x \]

\[= Tdx \frac{\partial^2 y}{\partial x^2} \tag{7-39}
\]
When the displacement is equal to $\alpha y(x,t)$, the force will be equal to $\alpha Tdx \frac{\partial^2 y}{\partial x^2}$. We now calculate the work done on length $dx$ of the string as $\alpha$ goes from 0 to 1.

$$W = \int_{\alpha=0}^{1} -F_y(x,\alpha) d\eta$$

$$= -\int_{\alpha=0}^{1} F_y(x,\alpha) y(x,t_0) d\alpha$$

$$= -\int_{\alpha=0}^{1} \left( \alpha Tdx \frac{\partial^2 y}{\partial x^2} \right) y d\alpha$$

$$= -T dx \ y \frac{\partial^2 y}{\partial x^2} \int_{\alpha=0}^{1} \alpha d\alpha$$

$$= -\frac{1}{2} T y \frac{\partial^2 y}{\partial x^2} dx$$

Thus we have, for a general wave motion $y(x,t)$.
\[
\text{KE per unit length} = \frac{1}{2} \mu \left( \frac{\partial y}{\partial t} \right)^2
\]

(7-41)

\[
\text{PE per unit length} = -\frac{1}{2} T y \frac{\partial^2 y}{\partial x^2}
\]

Example. Calculate the kinetic, potential and total energy per unit length for the sinusoidal traveling wave

\[
y(x,t) = A \sin \left( kx - \omega t \right)
\]

(7-42)

Solution. We will use the relations (7-33). The KE per unit length is

\[
\text{KE per unit length} = \frac{1}{2} \mu \left( \frac{\partial}{\partial t} A \sin \left( kx - \omega t \right) \right)^2
\]

(7-43)

and the PE per unit length is

\[
\text{PE per unit length} = -\frac{1}{2} T y \frac{\partial^2 y}{\partial x^2}
\]

\[
= -\frac{1}{2} T \left( A \sin \left( kx - \omega t \right) \right) \left( \frac{\partial^2}{\partial x^2} A \sin \left( kx - \omega t \right) \right)
\]

\[
= -\frac{1}{2} T \left( A \sin \left( kx - \omega t \right) \right) \left( -k^2 A \sin \left( kx - \omega t \right) \right)
\]

\[
= \frac{1}{2} Tk^2 A^2 \sin^2 \left( kx - \omega t \right)
\]

(7-44)

\[
= \frac{1}{2} \mu \omega^2 A^2 \sin^2 \left( kx - \omega t \right)
\]

In the last step we have used the relations \( \omega = kv \) (general for all sinusoidal waves) and \( v = \sqrt{\frac{T}{\mu}} \) (waves on a string). Note the similarity of the expressions for the two types of energy. The only difference is the presence of the factor of \( \cos^2 \left( kx - \omega t \right) \) in one case, and \( \sin^2 \left( kx - \omega t \right) \) in the other. If we use the well-known property that either \( \sin^2 u \) or \( \cos^2 u \) averages to 1/2 when averaged over a large number of cycles, we see that the average energy per unit length is the same in the two modes of energy storage, and we have

\[
\langle \text{KE per unit length} \rangle_{\text{time}} = \langle \text{PE per unit length} \rangle_{\text{time}} = \frac{1}{4} Tk^2 A^2.
\]

(7-45)

Here the brackets \( \langle \rangle_{\text{time}} \) represent an average over time. The equality of the average energy in these two types of energy is an example of a situation often found in physics where different "degrees of freedom" of a system share equally in the energy. Note that the average could have been carried out over position \( x \) along the string rather than the time, and the result would have been the same.

The total energy per length is constant, since \( \sin^2 \left( kx - \omega t \right) + \cos^2 \left( kx - \omega t \right) = 1 \), for any time or position along the string:
Example. Consider the $E$ string on a guitar as discussed in a previous example, with $T = 300 \text{ N}$, $\mu = 0.00769 \text{ kg/m}$, $k = 4.833 \text{ m}^{-1}$, and $\omega = 954 \text{ rad/sec}$. Take the amplitude of the string’s deflection to be $A = 1 \text{ cm}$, and calculate the average energy density for kinetic and potential energy, and the total energy in the string.

Solution. Just plug into the previous relations.

Average KE per unit length = Average PE per unit length

\[
\frac{1}{4} \mu \omega^2 A^2
\]  \hspace{1cm} (7-47)

\[
\frac{1}{4} (0.00769 \text{ kg/m})(954 \text{ rad/sec})^2 (0.01 \text{ m})^2
\]  \hspace{1cm} = 0.175 \text{ J/m}

The total energy density is twice this, giving for the total energy

\[
\text{total energy} = \text{(total energy density)} \times \text{(length of string)}
\]  \hspace{1cm} (7-48)

\[
= (0.350 \text{ J/m}) \times (0.65 \text{ m})
\]  \hspace{1cm} = 0.228 \text{ J}

This isn't much. If the note dies away in one second, the maximum average audio power would be a quarter of a watt. Aren't guitars louder than that? Oops - what's that great big black box on stage beside the guitar?

F. The superposition principle.

There is a very important property of linear differential equations, like the wave equation, called the superposition principle, which can be stated as follows for the wave equation:

The Superposition principle. Suppose that $f_1(x,t)$ and $f_2(x,t)$ are two solutions to the wave equation as given above. Then any linear combination $g(x,t)$ of $f_1$ and $f_2$, of the form

\[
g(x,t) = \alpha f_1(x,t) + \beta f_2(x,t)
\]  \hspace{1cm} (7-49)

where $\alpha$ and $\beta$ are arbitrary constants, is also a solution to the wave equation.

This is pretty easy to prove. First we calculate the partial derivatives:

\[
\frac{\partial^2 g}{\partial x^2} = \frac{\partial^2}{\partial x^2} \left( \alpha f_1(x,t) + \beta f_2(x,t) \right)
\]  \hspace{1cm} (7-50)

\[
\frac{\partial^2 f_1}{\partial x^2} + \beta \frac{\partial^2 f_2}{\partial x^2}
\]

and
\[
\frac{1}{v^2} \frac{\partial^2 g}{\partial t^2} = \frac{1}{v^2} \frac{\partial^2}{\partial t^2} \left( \alpha f_1(x,t) + \beta f_2(x,t) \right) 
\]
\[
= \alpha \frac{1}{v^2} \frac{\partial^2 f_1}{\partial t^2} + \beta \frac{1}{v^2} \frac{\partial^2 f_2}{\partial t^2} 
\]

Now substitute into the wave equation, in the form
\[
\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} = 0 
\]

For \( g(x,t) \) this becomes
\[
\frac{\partial^2 g}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 g}{\partial t^2} = 0 \quad ? 
\]
\[
\alpha \frac{\partial^2 f_1}{\partial x^2} + \beta \frac{\partial^2 f_2}{\partial x^2} - \left( \alpha \frac{1}{v^2} \frac{\partial^2 f_1}{\partial t^2} + \beta \frac{1}{v^2} \frac{\partial^2 f_2}{\partial t^2} \right) = 0 \quad ? 
\]

Regrouping, we have
\[
\alpha \frac{\partial^2 f_1}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 f_1}{\partial t^2} + \beta \left( \frac{\partial^2 f_2}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 f_2}{\partial t^2} \right) = 0 \quad ? 
\]
\[
\alpha \left( \frac{\partial^2 f_1}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 f_1}{\partial t^2} \right) + \beta \left( \frac{\partial^2 f_2}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 f_2}{\partial t^2} \right) = 0 \quad ? \quad \text{YES!!!} 
\]

Since \( f_1 \) and \( f_2 \) each satisfies the wave equation, the two expressions in parentheses vanish separately, and the equality is satisfied.

This principle gives us a great deal of freedom in constructing solutions to the wave equation. A very common approach to wave problems involves representing a complicated wave solution as the superposition of simpler waves, as illustrated in the next section.

**G. Dispersion; Group and phase velocity.**

The wave equation with constant coefficients as given by equation (7-11) has traveling-wave solutions which all travel at the same speed. This is true for sinusoidal solutions (equation (7-12)) or for arbitrarily shaped pulses (equation (7-30)). A wave medium where this is the case is referred to as a "non-dispersive" medium, for the following reason: while sinusoidal waves propagate without changing their shape, wave pulses do not; in general, as a wave pulse propagates, it changes its shape, in most cases getting broader and broader as it travels. This broadening is referred to as dispersion.

In a dispersive medium, there are two important velocities to be considered. The "phase velocity," which we have represented with the symbol \( v \), is the velocity of a sinusoidal wave of a certain frequency or wavelength. The other velocity is the "group velocity," which we will denote by \( u \). The group velocity has the following definition, which will really only be clear after we have discussed Fourier series and transforms. A pure sinusoidal solution to the wave equation represents a disturbance which is completely delocalized, in the sense that it has the same amplitude for all positions on the string and for all times. If one wanted to represent a short light signal from a laser, for instance, as used in communications, it would make sense to use a modified wave which has a
beginning and an end. In order to do this, a linear superposition of sinusoidal waves of different wavelengths can be used. However, this means superposing waves traveling at different velocities! If they are initially lined up so as to add up to give a narrow pulse, or "wave packet," it makes sense that after some time elapses they will no longer be properly aligned. The result of such a process of superposition, however, is that the wave packet moves at a velocity which is entirely different from the velocity of the component sinusoidal waves!

\[ u \text{ (the group velocity)} \neq v \text{ (the phase velocity)} \quad (7-55) \]

This remarkable fact has to be seen to be believed. There is a rather simple example of superposition of waves which illustrates it. Let's superimpose two sinusoidal waves of equal amplitude \( A \), and slightly different wave numbers \( k_1 = k_0 - \Delta k \) and \( k_2 = k_0 + \Delta k \). that is, the two wave numbers are separated by an amount \( 2\Delta k \), and centered about the value \( k_0 \). We will assume that we know the "dispersion relation:"

\[ \omega = \omega(k) \quad \text{the dispersion relation} \quad (7-56) \]

For non-dispersive media where \( v \) is a constant, this is a very simple linear relation, \( \omega = kv \), obtained from equation (7-17). However, if \( v \) is a function of \( k \), the relation becomes more complicated. In any event, this relation allows the angular frequencies \( \omega_1 = \omega_0 - \Delta \omega \) and \( \omega_2 = \omega_0 + \Delta \omega \) to be determined from \( k_1 \) and \( k_2 \). We now carry out the superposition.

\[ y(x,t) = A \sin(k_1x - \omega_1t) + A \sin(k_2x - \omega_2t) \]

\[ = A \sin((k_0x - \omega_0t) - (\Delta kx - \Delta \omega t)) + A \sin((k_0x - \omega_0t) + (\Delta kx - \Delta \omega t)) \quad (7-57) \]

Now we use the trigonometric identities

\[ \sin(a + b) = \sin a \cos b + \cos a \sin b \]

\[ \sin(-b) = -\sin b \]

\[ \cos(-b) = \cos b \quad (7-58) \]

to obtain the result

\[ y(x,t) = A \sin((k_0x - \omega_0t) - (\Delta kx - \Delta \omega t)) + A \sin((k_0x - \omega_0t) + (\Delta kx - \Delta \omega t)) \]

\[ = A \sin(k_0x - \omega_0t) \cos(\Delta kx - \Delta \omega t) - A \cos(k_0x - \omega_0t) \sin(\Delta kx - \Delta \omega t) \]

\[ + A \sin(k_0x - \omega_0t) \cos(\Delta kx - \Delta \omega t) + A \cos(k_0x - \omega_0t) \sin(\Delta kx - \Delta \omega t) \quad (7-59) \]

\[ = 2A \sin(k_0x - \omega_0t) \cos(\Delta kx - \Delta \omega t) \]

Here is the interpretation of this result. The first sinusoidal factor, \( \sin(k_0x - \omega_0t) \), represents a traveling sinusoidal wave, with velocity

\[ v = \frac{\omega_0}{k_0} \quad (7-60) \]

The second factor is a slowly varying function of \( x \) and \( t \) which modulates the first sine wave, turning it into a sort of wave packet, or rather a series of wave packets. This is illustrated graphically in figure 7-7.
Here we have plotted the solution given in equation \((7-59)\), using the dispersion relation for deep-water gravity waves,
\[
\omega = \sqrt{gk},
\]
where \(g\) is the acceleration of gravity, for two different instants of time. Note that the displacement of the envelope is different from the displacement of the carrier at the central frequency. In this picture, the lobes produced by the envelope are thought of as the wave packets, and the rate of displacement of the envelope is interpreted as the group velocity. From the form of the envelope function,
\[
y_{envelope} = \cos(\Delta k x - \Delta \omega t),
\]
we can see how fast it propagates. For a sinusoid with argument \(kx - \omega t\), the propagation speed is equal to \(\omega/k\). So, for the envelope, we get
\[
v_{envelope} = \frac{\Delta \omega}{\Delta k} = u.
\]

We will interpret this ratio as the derivative of \(\omega\) with respect to \(k\). So, our final result for phase and group velocities is
\[ \omega = \omega(k) \quad \text{dispersion relation} \]
\[ v = \frac{\omega}{k} \quad \text{phase velocity} \quad (7-64) \]
\[ u = \frac{d\omega}{dk} \quad \text{group velocity} \]

**Example.** For deep-water gravity waves such as storm waves propagating across the Pacific Ocean, the velocity of sinusoidal waves is given by
\[ v = \frac{g}{\sqrt{k}}. \quad (7-65) \]
Find the dispersion relation \( \omega(k) \), and calculate the group velocity \( u \). For waves of wavelength 200 m, calculate the phase and group velocities.

**Solution.** From the definitions of \( \omega \) and \( k \) we know that
\[ v = \frac{\lambda}{T} = \frac{\omega}{k}. \]
Solving for omega and using equation 7-65 gives
\[ \omega = kv \]
\[ = k \frac{g}{k} \]
\[ = \frac{g}{k} \quad \text{the dispersion relation} \]
The group velocity is obtained by taking a derivative:
\[ u = \frac{d\omega}{dk} = \frac{d}{dk} \sqrt{kg} = \frac{1}{2} \frac{g}{\sqrt{k}}. \]
Note that the group velocity is exactly equal to half of the phase velocity.

For a wavelength of \( \lambda=200 \) m, the wave number is
\[ k = \frac{2\pi}{\lambda} = \frac{2\pi}{(200 \text{ m})} = 0.0314 \text{ m}^{-1}. \]
The phase and group velocities are then
\[ v = \frac{g}{\sqrt{k}} = \sqrt{\frac{(9.8 \text{ m/s}^2)}{(0.031415 \text{ m}^{-1})}} = 17.7 \text{ m/s} \quad (\approx 35 \text{ mph}) \]
\[ u = \frac{1}{2} v = 8.83 \text{ m/s} \]
[Just in case you ever need it, the general expression for the velocity of gravity waves in water of depth \( h \) is
\[ v = \sqrt{\frac{g}{k} \tanh(kh)} \quad \text{shallow water} \rightarrow \sqrt{gh} \]
\[ \text{deep water} \rightarrow \sqrt{\frac{g}{k}}, \]
reducing correctly to the limiting forms for deep and shallow water given here and in section C above.]
Problems

Problem 7-1. The derivative as a limit. To illustrate the derivative as the limit of a rate of change, consider the function \( f(t) = \frac{1}{2} at^2 \).

(a) Use the standard rules for taking derivatives (learned in your calculus course) to calculate \( \frac{df}{dt} \).

(b) Now use the definition

\[
\frac{df}{dt} \equiv \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}
\]

to calculate \( \frac{df}{dt} \). You will need to make an approximation which is appropriate when \( \Delta t \) is small.

Problem 7-2. Calculation of partial derivatives. In the functions below, \( x, y, z, \) and \( t \) are to be considered as variables, and other letters represent constants.

(a) \( \frac{\partial}{\partial x} 2axy^2 \)

(b) \( \frac{\partial}{\partial y} 2axy^2 \)

(c) \( \frac{\partial^2}{\partial x^2} 2axy^2 \)

(d) \( \frac{\partial^2}{\partial y^2} 2axy^2 \)

(e) \( \frac{\partial^2}{\partial x \partial y} 2axy^2 \)

(f) \( \frac{\partial}{\partial x} A \sin(kx - \omega t) \)

(g) \( \frac{\partial}{\partial t} A \sin(kx - \omega t) \)

(h) \( \frac{\partial^2}{\partial x^2} A \sin(kx - \omega t) \)

(i) \( \frac{\partial}{\partial z} B \exp \left( -\frac{(z-c)^2}{2d^2} \right) \)

Problem 7-3. Solution to the wave equation. Show by direct substitution that the following functions satisfy the wave equation,

\[
\frac{\partial^2}{\partial x^2} f(x,t) = \frac{1}{v^2} \frac{\partial^2}{\partial t^2} f(x,t).
\]
You should assume that the relation $v = \frac{\omega}{k}$ holds.

(a) $f(x,t) = A\cos(kx - \omega t)$

(b) $f(x,t) = C\sin(kx\cos\omega t)$

**Problem 7-4. Derivation of the wave equation for sound.** Waves traveling along a tube of gas can be pictured as shown in the diagram. The small rectangle shows an element of gas, which is considered to be at position $x$ in the absence of any wave motion. As a wave passes, the element of gas is displaced by an amount $y(x,t)$, which depends on time and on the position $x$. The motion of the element of gas is determined by the pressure on either side, as shown in the lower part of the diagram.

(a) There will be no net force on the element of gas as long as the pressure is the same on both sides. However, if there is a variation of pressure with position, there will be a net force $F$ on the element of gas, of width $\Delta x$, where the force is positive on the left-hand side of the gas element and negative on the right-hand side, and follows the general relation $F = PA$. The force will thus be

$$F = (\text{force on left-hand side}) - (\text{force on right-hand side}).$$

Show that this force is given by

$$F = -A\frac{\partial P}{\partial x}\Delta x.$$
(b) We will assume that the pressure in the gas varies by only a small amount from \( p_0 \), the ambient background pressure ("atmospheric pressure"):

\[
P = p_0 + p(x, t).
\]

The variation of \( p(x, t) \) with position is in turn related to the displacement of the gas, through the change in the volume \( V \) of the element of gas. A uniform displacement of the gas such that every part of the gas moves by the same amount does not change the volume of a gas element. However, if \( y \) is a function of \( x \), there is a change in volume. We can use the adiabatic gas law, \( PV^{\gamma} = \text{constant} \), to relate the small pressure variation \( p \) to the corresponding change in volume of the element of gas:

\[
dP = -\gamma \frac{p_0}{V_0} \Delta V = p(x, t).
\]

Here \( p_0 \) and \( V_0 \) are the pressure and volume of the element of gas in the absence of a wave, and \( \gamma \) is a constant related to the number of degrees of freedom of the gas molecules (\( \gamma_{\text{air}} = \frac{5}{3} \)). In this case, the change in the volume of the element of gas is \( \Delta V = A(y(x + \Delta x) - y(x)) \). Show that this relation and the relation from the adiabatic gas law lead to the partial-derivative expression

\[
p = -\gamma p_0 \frac{\partial y}{\partial x}.
\]

(c) Now combine the results for parts (a) and (b) and use Newton’s 2nd law to find the wave equation for the displacement \( y(x,t) \). Show that the wave velocity \( v \) can be given in terms of the ambient pressure \( p_0 \) and the gas density \( \rho = m/V_0 \) by

\[
v = \sqrt{\frac{\gamma p_0}{\rho}}.
\]

**Problem 7-5. Wave speed in telephone wires.** Suppose someone wants to send telegraph signals across the country using the transmission of transverse wave pulses along the wires, rather than electrical signals. Make a reasonable estimate of the maximum tension in the wires and of their mass density, and calculate the wave velocity. Assume that the waves run along the wires between telephone poles, and that they pass right by each telephone pole without reflection. How long would it take for such a signal to propagate from New York to San Francisco (about 5000 km)?

**Problem 7-6. Energy density in a string.** The expressions derived in the text for kinetic and potential energy density are

\[
\text{KE per unit length} = \frac{1}{2} \mu \left( \frac{\partial y}{\partial t} \right)^2 \quad \text{(7-41)}
\]

\[
\text{PE per unit length} = -\frac{1}{2} T y \frac{\partial^2 y}{\partial x^2}
\]

It is a bit surprising that derivatives with respect to time and position enter in such a different way, considering how symmetrical their role is in the wave equation. You can fix this up.

The total potential energy over the interval \( A < x < B \) is obtained by integrating the energy density given above:

\[
\text{PE between A and B} = \int_{x=A}^{x=B} \left( -\frac{1}{2} T y \frac{\partial^2 y}{\partial x^2} \right) dx
\]
Do an integration by parts with respect to $x$, and show that an equally good expression for the potential energy density is

$$\text{PE per unit length} = \frac{1}{2} T \left( \frac{\partial \psi}{\partial x} \right)^2$$

You must explain why this is reasonable. Be careful about the evaluation at the endpoints that is involved in integrating by parts.

**Problem 7-7. Phase and group velocity for the deBroglie wave.** In 1923 Louis deBroglie advanced the idea that a free particle of mass $m$ could be described by a wave function, of the form

$$\Psi_{deB} (x,t) = Ae^{\frac{i(px-Et)}{\hbar}}$$

$$= Ae^{i(kx-\omega t)}$$

where $\hbar$ is Planck's constant, and we will take the non-relativistic forms for the momentum and energy of the particle,

$$p = mv_{part},$$

$$E = \frac{1}{2}mv_{part}^2,$$

where $v_{part}$ is the particle's velocity.

(a) Find expressions for $k$ and $\omega$ in terms of the particle velocity $v_{part}$.

(b) Show that the dispersion relation for this wave is

$$\omega = \omega (k) = \frac{\hbar}{2m} k^2.$$

(c) Calculate the phase velocity and the group velocity for these waves and show how each one relates to the particle velocity $v_{part}$.

(d) It is generally argued that the group velocity is the velocity with which the waves carry energy. Do your answers to (c) support this argument?

**Problem 7-8. Phase and group velocity for the deBroglie wave of a relativistic particle.** In 1923 Louis deBroglie advanced the idea that a free particle of mass $m$ could be described by a wave function, of the form

$$\Psi_{deB} (x,t) = Ae^{\frac{i(p^\mu x_\mu)}{\hbar}}$$

$$= Ae^{\frac{Et-px}{\hbar}}$$

$$= Ae^{i(\omega t-kx)}$$

where $\hbar$ is Planck's constant. Note that the phase of the particle, $p^\mu x_\mu = Et - px$ is a relativistic invariant. We will take the relativistic forms for the momentum and energy of the particle,
\[ E = \gamma mc^2, \]
\[ p = \eta mc^2, \]
where \( \gamma = \frac{1}{\sqrt{1 - \beta^2}} \), \( \eta = \gamma \beta \), and \( \beta = \frac{v}{c} \), with \( v \) the particle velocity.

(a) Find expressions for \( k \) and \( \omega \) in terms of \( \gamma \), \( \eta \) and \( \beta \).

(b) Show that the dispersion relation for this wave is
\[ \omega^2 = \omega^2(k) = k^2 + \left( \frac{mc^2}{\hbar} \right)^2. \]

(c) Calculate the phase velocity \( \frac{\omega}{k} \) and the group velocity \( \frac{d\omega}{dk} \) for these waves. (It will be easiest to work with the results of part (a).) The result, in terms of \( \beta \), is very simple and surprising.

(d) Particles are not supposed to go faster than the speed of light. What do your results of part (c) say about this?