Chapter 5. The Inverse; Numerical Methods

In the Chapter 3 we discussed the solution of systems of simultaneous linear algebraic equations which could be written in the form

$$\overline{A}\vec{x} = \vec{C} \qquad (5\text{-}1)$$

using Cramer's rule. There is another, more elegant way of solving this equation, using the *inverse matrix*. In this chapter we will define the inverse matrix and give an expression related to Cramer's rule for calculating the elements of the inverse matrix. We will then discuss another approach, that of Gauss-Jordan elimination, for solving simultaneous linear equations and for calculating the inverse matrix. We will discuss the relative efficiencies of the two algorithms for numerical inversion of large matrices.

A. The inverse of a square matrix.

Definition of the inverse. The inverse of a scalar number *c* is another scalar, say *d*, such that the product of the two is equal to 1: c∗d=1. For instance, the inverse of the number 5 is the number 0.2 . We have defined multiplication of one matrix by another in a way very analogous to multiplication of one scalar by another. We will therefore make the following definition.

> *Definition:* For a given square matrix $\overline{A}$, the matrix $\overline{B}$ is said to be the inverse of $\overline{A}$ if
>
> $$\overline{B}\overline{A} = \overline{A}\overline{B} = \overline{I} \qquad (5\text{-}2)$$
>
> We then write $\overline{B} = \overline{A}^{-1}$.

Notice that we have not guaranteed that the inverse of a given matrix exists. In fact, many matrices do not have an inverse. We shall see below that the condition for a square matrix $\overline{A}$ to have an inverse is that its determinant not be equal to zero.

Use of the inverse to solve matrix equations. Now consider the matrix equation just given,

$$\overline{A}\vec{x} = \vec{C} \qquad (5\text{-}1)$$

We can solve this equation by multiplying on both sides of the equation by $\overline{A}^{-1}$:

$$\begin{aligned} \overline{A}^{-1}\overline{A}\vec{x} &= \overline{A}^{-1}\vec{C}; \\ \overline{I}\vec{x} &= \overline{A}^{-1}\vec{C}; \\ \vec{x} &= \overline{A}^{-1}\vec{C}. \end{aligned} \qquad (5\text{-}3)$$

Thus, knowing the inverse of the matrix $\overline{A}$ lets us immediately write down the solution $\vec{x}$ to equation (5-1).

As an example, let us consider a specific example, where $\overline{A}$ is a 2x2 matrix.

$$\overline{A}\vec{x} = \vec{C};$$
$$\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \qquad (5\text{-}4)$$

If we knew the inverse of $\overline{A}$, we could immediately calculate $\vec{C} = \overline{A}\vec{x}$. In this simple case, we can guess the inverse matrix. We write out the condition for the inverse,

$$\overline{A}\overline{A}^{-1} = \bar{I};$$
$$\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} * & * \\ * & * \end{pmatrix} = \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \left(I_{ij}\right) \qquad (5\text{-}5)$$

As a first guess we try to make I_{12} come out to zero; one possibility is

$$\overline{A}\overline{A}^{-1} = \bar{I};$$
$$\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} * & 2 \\ * & -1 \end{pmatrix} = \begin{pmatrix} ? & 0 \\ ? & ? \end{pmatrix} = \left(I_{ij}\right) \qquad (5\text{-}6)$$

Now we arrange for I_{21} to be zero:

$$\overline{A}\overline{A}^{-1} = \bar{I};$$
$$\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} ? & 0 \\ 0 & ? \end{pmatrix} = \left(I_{ij}\right) \qquad (5\text{-}7)$$

If we now look at the diagonal elements of $\bar{I}$, they come out to be $I_{11} = 3$ and $I_{22} = -3$. We can fix this up by changing the sign of the (1,2) and (2,2) elements of the inverse, and by multiplying it by 1/3. So we have

$$\overline{A}\overline{A}^{-1} = \bar{I};$$
$$\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}\frac{1}{3}\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \left(I_{ij}\right); \qquad (5\text{-}7)$$
$$\overline{A}^{-1} = \frac{1}{3}\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$$

Now that we have the inverse matrix, we can calculate the values x_1 and x_2:

$$\begin{aligned} \vec{x} &= \overline{A}^{-1}\vec{C} \\ &= \frac{1}{3}\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} -6 \\ 9 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 3 \end{pmatrix} \end{aligned} \qquad (5\text{-}8)$$

So, the solution to the two simultaneous linear equations is supposed to be $x_1 = -2$, $x_2 = 3$. We will write out the two equations in long form and substitute in.

$$\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix};$$

$$\begin{Bmatrix} x_1 + 2x_2 = 4 \\ -x_1 + x_2 = 5 \end{Bmatrix};$$

$$\begin{Bmatrix} -2 + 2*3 = 4 \\ -(-2) + 3 = 5 \end{Bmatrix};$$

$$\begin{Bmatrix} 4 = 4 \\ 5 = 5 \end{Bmatrix}.$$

It checks out!

The inverse matrix by the method of cofactors. Guessing the inverse has worked for a 2x2 matrix - but it gets harder for larger matrices. There is a way to calculate the inverse using cofactors, which we state here without proof:

$$\begin{aligned}\left(\overline{A}^{-1}\right)_{ij} &= \frac{\mathrm{cof}_{ji}(\overline{A})}{\left|\overline{A}\right|} \\ &= \frac{(-1)^{j+i} M_{ji}(\overline{A})}{\left|\overline{A}\right|}\end{aligned} \qquad (5\text{-}9)$$

(Here the *minor* $M_{pq}(\mathbf{A})$ is the determinant of the matrix obtained by removing the p-th row and q-th column from the matrix **A**.)

Note that you cannot calculate the inverse of a matrix using equation (5-9) if the matrix is singular (that is, if its determinant is zero). This is a general rule for square matrices:

$$\left|\overline{A}\right| = 0 \iff \text{inverse does not exist}$$

Example: Find the inverse of the matrix

$$\overline{A} = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} \qquad (5\text{-}10)$$

Here are the calculations of the four elements of $\overline{A}^{-1}$. First calculate the determinant:

$$\left|\overline{A}\right| = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 1 - (-2) = 3 \qquad (5\text{-}11)$$

Then the matrix elements:

$$\left(\bar{A}^{-1}\right)_{11} = \frac{cof_{11}(\bar{A})}{|\bar{A}|} = \frac{(-1)^{1+1}A_{11}}{|\bar{A}|} = \frac{1}{3};$$
$$\left(\bar{A}^{-1}\right)_{12} = \frac{cof_{21}(\bar{A})}{|\bar{A}|} = \frac{(-1)^{2+1}A_{12}}{|\bar{A}|} = -\frac{2}{3};$$
$$\left(\bar{A}^{-1}\right)_{21} = \frac{cof_{12}(\bar{A})}{|\bar{A}|} = \frac{(-1)^{1+2}A_{21}}{|\bar{A}|} = \frac{1}{3};$$
$$\left(\bar{A}^{-1}\right)_{22} = \frac{cof_{22}(\bar{A})}{|\bar{A}|} = \frac{(-1)^{2+2}A_{11}}{|\bar{A}|} = \frac{1}{3};$$
(5-12)

So,

$$\bar{A}^{-1} = \frac{1}{3}\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \qquad (5\text{-}13)$$

Check that this inverse works:

$$\bar{A}\bar{A}^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \bar{I};$$
$$\bar{A}^{-1}\bar{A}\,\frac{1}{3}\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \bar{I}$$
(5-14)

Example: Calculate the inverse of the following 3x3 matrix using the method of cofactors:

$$\bar{A} = \begin{pmatrix} 2 & 4 & 3 \\ 1 & 3 & 2 \\ 1 & 3 & 1 \end{pmatrix} \qquad (5\text{-}15)$$

Solution: This is getting too long-winded. We will just do two representative elements of $\bar{A}^{-1}$.

$$|\bar{A}| = \begin{vmatrix} 2 & 4 & 3 \\ 1 & 3 & 2 \\ 1 & 3 & 1 \end{vmatrix} = 2(3-6) - 4(1-2) + 3(3-3) = -2;$$

$$\left(\bar{A}^{-1}\right)_{11} = \frac{cof_{11}(\bar{A})}{|\bar{A}|} = \frac{(-1)^{1+1}\begin{vmatrix} 3 & 2 \\ 3 & 1 \end{vmatrix}}{|\bar{A}|} = \frac{-3}{-2} = \frac{3}{2};$$
$$\left(\bar{A}^{-1}\right)_{23} = \frac{cof_{32}(\bar{A})}{|\bar{A}|} = \frac{(-1)^{3+2}\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}}{|\bar{A}|} = \frac{-1}{-2} = \frac{1}{2};$$
(5-16)

B. Time required for numerical calculations.

Let's estimate the computer time required to invert a matrix by the method of cofactors. The quantity of interest is the number of floating-point operations required to carry out the inverse. The inverse of a *nxn* matrix involves calculating n^2 cofactors, each of them requiring the calculation of the determinant of an (n-1)x(n-1) matrix. So we need to know the number of operations involved in calculating a determinant. Let's start with a 2x2 determinant. There are two multiplications, and an addition to add the two terms. n=2 gives 3 FLOPs. (FLOP = Floating-Point Operation.) To do a 3x3 determinant, the three elements in the top row are each multiplied by a 2x2 determinant and added together: 3x(3 FLOPs) + 2 FLOPs for addition; n=3 requires 3x3 + 2 FLOPs. Now we can proceed more or less by induction. It is pretty clear that the determinant of a 4x4 matrix requires 4 calculations of a 3x3 determinant: --> 4x3x3 FLOPs. And for a 5x5 determinant, 5x4x3x3 operations. It is a pretty good approximation to say the following:

$$\text{No. of operations for nxn determinant} = n! \qquad (5\text{-}17)$$

This means that calculating the inverse by the cofactor method (n^2 cofactors) requires $n^2 n!$ FLOPs.

A fast PC can today do about 10 GigaFLOPs/sec. This leads to the table given below showing the execution time to invert matrices of increasing dimension.

dim. n	for determinant (n!)	for inverse (method of cofactors) (n^2*n!)	time (sec) (PC)	for inverse (Gauss-Jordan) (4n^3)	time (sec) (PC)
2	2	8	8E-10	32	3.2E-09
3	6	54	5.4E-09	108	1.08E-08
4	24	384	3.84E-08	256	2.56E-08
5	120	3000	0.0000003	500	0.00000005
6	720	25920	0.000002592	864	8.64E-08
7	5040	246960	0.000024696	1372	1.372E-07
8	40320	2580480	0.000258048	2048	2.048E-07
9	362880	29393280	0.002939328	2916	2.916E-07
10	3628800	362880000	0.036288	4000	0.0000004
11	39916800	4829932800	0.48299328	5324	5.324E-07
12	479001600	68976230400	6.89762304	6912	6.912E-07
13	6227020800	1.05237E+12	105.2366515	8788	8.788E-07
14	8.7178E+10	1.70869E+13	1708.694508	10976	1.0976E-06
15	1.3077E+12	2.94227E+14	29422.67328	13500	0.00000135
16	2.0923E+13	5.35623E+15	535623.4211	16384	1.6384E-06
17	3.5569E+14	1.02794E+17	10279366.67	19652	1.9652E-06
18	6.4024E+15	2.07437E+18	207436908.1	23328	2.3328E-06
19	1.2165E+17	4.39139E+19	4391388125	27436	2.7436E-06
20	2.4329E+18	9.73161E+20	97316080327	32000	0.0000032
21	5.1091E+19	2.25311E+22	2.25311E+12	37044	3.7044E-06
22	1.124E+21	5.44016E+23	5.44016E+13	42592	4.2592E-06
23	2.5852E+22	1.36757E+25	1.36757E+15	48668	4.8668E-06
24	6.2045E+23	3.57378E+26	3.57378E+16	55296	5.5296E-06

Table 5-1. Floating-point operations required for calculation of $n\,x\,n$ determinants and inverses of $n\,x\,n$ matrices, and computer time required for the matrix inversion. Results

are given for two different numerical methods. (As a useful conversion number, the number of seconds in a year is about 3.14 x 10^7.)

It can be seen from the table that the inversion of a 24x24 matrix could take a time on a fast computer about equal to the age of the Universe. This suggests that a more economical algorithm is desirable for inverting large matrices!

Teasing Mathematica: Try this calculation of a determinant.

```
n=500
m=Table[Random[],{n},{n}];
Det[m]
```

Does this suggest that the algorithm used for Table 5-1 is not the fastest known?

C. The Gauss-Jordan method for solving simultaneous linear equations.

There is a method for solving simultaneous linear equations that avoids the determinants required in Cramer's method, and which takes many fewer operations for large matrices. We will illustrate this method for two simultaneous linear equations, and then for three. Consider the 2x2 matrix equation solved above,

$$\overline{A}\vec{x} = \vec{C};$$
$$\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \qquad (5\text{-}4)$$

This corresponds to the two linear equations

$$\begin{aligned} x_1 + 2x_2 &= 4 \\ -x_1 + x_2 &= 5 \end{aligned} \qquad (5\text{-}18)$$

A standard approach to such equations would be to add or subtract a multiple of one equation from another to eliminate one variable from one of the equations. If we add the first equation to the second, we get

$$\begin{pmatrix} x_1 + 2x_2 = 4 \\ -x_1 + x_2 = 5 \end{pmatrix} \xrightarrow{\text{addt eq. (1) to eq. (2)}} \begin{pmatrix} x_1 + 2x_2 = 4 \\ 0 + 3x_2 = 9 \end{pmatrix} \qquad (5\text{-}19)$$

Now we eliminate x_2 from the top equation, by subtracting 2/3 x the bottom equation:

$$\begin{pmatrix} x_1 + 2x_2 = 4 \\ -x_1 + x_2 = 5 \end{pmatrix} \xrightarrow{\text{add eq. (1) to eq. (2)}} \begin{pmatrix} x_1 + 2x_2 = 4 \\ 0 + 3x_2 = 9 \end{pmatrix}$$
$$\xrightarrow{\text{subtract (2/3)x eq. (2) from eq. (1)}} \begin{pmatrix} x_1 + 0 = -2 \\ 0 + 3x_2 = 9 \end{pmatrix} \qquad (5\text{-}20)$$

And finally, multiply the second equation by 1/3:

$$\begin{pmatrix} x_1 + 2x_2 = 4 \\ -x_1 + x_2 = 5 \end{pmatrix} \xrightarrow{\text{add eq. (1) to eq. (2)}} \begin{pmatrix} x_1 + 2x_2 = 4 \\ 0 + 3x_2 = 9 \end{pmatrix}$$
$$\xrightarrow{\text{subtract (2/3)x eq. (2) from eq. (1)}} \begin{pmatrix} x_1 + 0 = -2 \\ 0 + 3x_2 = 9 \end{pmatrix} \xrightarrow{\text{multiply eq. (2) by 1/3}} \begin{pmatrix} x_1 + 0 = -2 \\ 0 + x_2 = 3 \end{pmatrix} \qquad (5\text{-}21)$$

So we have found that $x_1 = -2$ and $x_2 = 3$, as determined earlier in the chapter using the inverse.

Note that the same operations could have been carried out using just the coefficients of the equations, and omitting x_1 and x_2, as follows. The assembly of the coefficients of x_1 and x_2 and the constants on the right of the equation is refered to as the *augmented matrix.*

$$\left(\begin{array}{cc|c} 1 & 2 & 4 \\ -1 & 1 & 5 \end{array}\right) \xrightarrow{\text{add eq. (1) to eq. (2)}} \left(\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 3 & 9 \end{array}\right)$$
$$\xrightarrow{\text{subtract (2.3)x eq. (2) from eq. (1)}} \left(\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 3 & 9 \end{array}\right) \xrightarrow{\text{multiply eq. (2) by 1/3}} \left(\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 3 \end{array}\right) \tag{5-22}$$

The results for x_1 and x_2 appear in the column to the right.

Example: Use the Gauss-Jordan method to solve the system of linear equations represented by

$$\overline{A}\vec{x} = \vec{C};$$
$$\begin{pmatrix} 2 & 4 & 3 \\ 1 & 3 & 2 \\ 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} \tag{5-23}$$

Solution: We set up the augmented matrix, and then set about making the matrix part of it diagonal, with ones on the diagonal. This is done in the following systematic fashion. First use the first equation to eliminate A_{21} and A_{22}. Next use the second equation to eliminate A_{32}.

$$\left(\begin{array}{ccc|c} 2 & 4 & 3 & 1 \\ 1 & 3 & 2 & 1 \\ 1 & 3 & 1 & 4 \end{array}\right) \xrightarrow{\text{subtract 1/2 x (1) from (2) and from (3)}} \left(\begin{array}{ccc|c} 2 & 4 & 3 & 1 \\ 0 & 1 & 1/2 & 1/2 \\ 0 & 1 & -1/2 & 7/2 \end{array}\right)$$
$$\xrightarrow{\text{subtract (2) from (3)}} \left(\begin{array}{ccc|c} 2 & 4 & 3 & 1 \\ 0 & 1 & 1/2 & 1/2 \\ 0 & 0 & -1 & 3 \end{array}\right) \tag{5-24}$$

Next we work upwards, using equation (3) to eliminate A_{23} and A_{13}. After that, equation (2) is used to eliminate A_{12}. At this point the matrix is diagonal. the final step is to multiply equations (1) and (3) by a constant which makes the diagonal elements of **A** become unity.

$$\left(\begin{array}{ccc|c} 2 & 4 & 3 & 1 \\ 0 & 1 & 1/2 & 1/2 \\ 0 & 0 & -1 & 3 \end{array}\right) \xrightarrow[\text{add } 3*(3) \text{ to}(1)]{\text{add } 1./2*(3) \text{ to}(2),} \left(\begin{array}{ccc|c} 2 & 4 & 0 & 10 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 3 \end{array}\right)$$

$$\xrightarrow[\text{from}(1)]{\text{subtract } 4 \text{ x } (2)} \left(\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 3 \end{array}\right) \xrightarrow[\text{and } (3) \text{ by } -1]{\text{multiply}(1) \text{ by } 1.2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right) \quad (5\text{-}25)$$

The solution for the unknown x's is thus $x_1 = 1$, $x_2=2$, $x_3 = -3$.

SUMMARY: Work your way through the matrix, zeroing the off-diagonal elements, IN THE ORDER SHOWN BELOW, zeroing ONE, then TWO, then THREE, etc. If you try to invent your own scheme of adding and subtracting rows, it may or may not work.

$$. \begin{pmatrix} \bullet & SIX & FIVE \\ ONE & \bullet & FOUR \\ TWO & THREE & \bullet \end{pmatrix}$$

D. The Gauss-Jordan method for inverting a matrix.

There is a very similar procedure which leads directly to calculating the inverse of a square matrix. Suppose that $\overline{B}$ is the inverse of $\overline{A}$. Then

$$\overline{A}\overline{B} = \bar{I};$$

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (5\text{-}26)$$

This can be thought of us three sets of three simultaneous linear equations:

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} B_{11} \\ B_{21} \\ B_{31} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} B_{12} \\ B_{22} \\ B_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad (5\text{-}27)$$

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} B_{13} \\ B_{23} \\ B_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

These three sets of equations can be solved simultaneously, using a larger augmented equation, as follows:

$$\left(\begin{array}{ccc|ccc} 2 & 4 & 3 & 1 & 0 & 0 \\ 1 & 3 & 2 & 0 & 1 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{subtract 1/2 x (1) from (2) and from (3)}} \left(\begin{array}{ccc|ccc} 2 & 4 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1/2 & -1/2 & 1 & 0 \\ 0 & 1 & -1/2 & -1/2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{\text{subtract (2) from (3)}} \left(\begin{array}{ccc|ccc} 2 & 4 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1/2 & -1/2 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{array}\right) \xrightarrow[\text{add 3*(3) to(1)}]{\text{add 1./2*(3) to(2),}} \left(\begin{array}{ccc|ccc} 2 & 4 & 0 & 1 & -3 & 3 \\ 0 & 1 & 0 & -1/2 & 1/2 & 1/2 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{array}\right)$$

$$\xrightarrow[\text{from (1)}]{\text{subtract 4 x (2)}} \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 3 & -5 & 1 \\ 0 & 1 & 0 & -1/2 & 1/2 & 1/2 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{array}\right) \xrightarrow[\text{and (3) by -1}]{\text{multiply(1) by 1/2}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/2 & -5/2 & 1/2 \\ 0 & 1 & 0 & -1/2 & 1/2 & 1/2 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right)$$

(5-28)

So, the result is

$$\overline{B} = \overline{A}^{-1} = \frac{1}{2}\begin{pmatrix} 3 & -5 & 1 \\ -1 & 1 & 1 \\ 0 & 2 & -2 \end{pmatrix} \qquad (5\text{-}29)$$

The check is to multiply $\overline{A}$ by its inverse:

$$\overline{B}\,\overline{A} = \frac{1}{2}\begin{pmatrix} 3 & -5 & 1 \\ -1 & 1 & 1 \\ 0 & 2 & -2 \end{pmatrix}\begin{pmatrix} 2 & 4 & 3 \\ 1 & 3 & 2 \\ 1 & 3 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (5\text{-}29)$$

So the inverse just calculated is correct.

Time for numerical inverse. Let us estimate the time to invert a matrix by this numerical method. The process of zeroing out one element of the left-hand matrix requires multiplying the line to be subtracted by a constant ($2n$ FLOPs), and subtracting it ($2n$ FLOPs). This must be done for (approximately) n^2 matrix elements. So the number of floating-point operations is about equal to $4n^3$ for matrix inversion by the Gauss-Jordan method. Consulting Table 5-1 shows that, for a 24x24 matrix, the time required is less than a milli-second, comparing favorably with 10^{11} years for the method of cofactors.

Number of operations to calculate the inverse of a nxn matrix.	
method	number of FLOPs
cofactor	$n^2*n!$
Gauss-Jordan	$4n^3$

PROBLEMS

Problem 5-1. (a) Use the method of cofactors to find the inverse of the matrix

$$\bar{C} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

(b) Check your result by verifying that $\bar{C}\bar{C}^{-1} = \bar{I}$.

Problem 5-2. Use the Mathematica function *Inverse* to find the inverse of the matrix

$$\bar{C} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

(See Appendix C for the necessary Mathematica operations.) Check your result.

Problem 5-3. ***Prove*** that if an operator $\bar{A}$ has both a left inverse (call it $\bar{B}$) and a right inverse (call it $\bar{C}$), then they are the same; that is, if

$$\bar{B}\bar{A} = \bar{I}$$

and

$$\bar{A}\bar{C} = \bar{I}$$

then

$$\bar{B} = \bar{C} .$$

[Be careful to assume only the properties of $\bar{B}$ and $\bar{C}$ that are given above. It is *not* to be assumed that $\bar{A}$, $\bar{B}$ and $\bar{C}$ are matrices.]

Problem 5-5. Suppose that $\bar{B}$ and $\bar{C}$ are members of a group with distributive multiplication defined, each having an inverse (both left-inverse and right-inverse). Let $\bar{A}$ be equal to the product of $\bar{B}$ and $\bar{C}$, that is,

$$\bar{A} = \bar{B}\bar{C} .$$

Now consider the group member $\bar{D}$, given by

$$\bar{D} = \bar{C}^{-1}\bar{B}^{-1} .$$

Show by direct multiplication that $\bar{D}$ is both a left inverse and a right inverse of $\bar{A}$.
[Be careful to assume only the properties of $\bar{B}$ and $\bar{C}$ that are given above. It is *not* to be assumed that $\bar{A}$, $\bar{B}$ and $\bar{C}$ are matrices.]

Problem 5-6. (a) Use the method of Gauss-Jordan elimination to find the inverse of the matrix

$$\bar{C} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

(b) Check your result by verifying that $\bar{C}\bar{C}^{-1} = \bar{I}$.