Chapter 4. Practical Examples.

In this chapter we will discuss solutions to two physics problems where we make use of techniques discussed in this book. In both cases there are multiple masses, coupled to each other so that their motions are not independent. This leads to coupled linear equations, which are naturally treated using matrices.

A. Simple harmonic motion - a review.

We are going to discuss masses coupled by springs and a compound pendulum. Let us start by reviewing the mathematical description of the oscillations of a single mass on a spring or a simple pendulum.

Figure 4-1 shows the two simple systems which form the basis for the more complex systems to be studied.

In each case there is a restoring force proportional to the displacement:

\[ F \propto \text{displacement} \]  

(4-1)

If we combine this with Newton's law of motion,

\[ \sum \text{applied forces} = ma \]  

(4-2)

we obtain

\[ \text{acceleration} = \frac{d^2x}{dt^2} = -(\text{some constant})/m \]  

(4-3)

or

\[ \frac{d^2x}{dt^2} = -\omega_0^2 x \]  

\text{SHM}  

(4-4)

You can easily show that for the mass on a spring, \( \omega_0 = \sqrt{\frac{k}{m}} \), and for the pendulum,

\[ \omega_0 = \sqrt{\frac{g}{L}} \], two famous relations.
So, how do we find a function \( x(t) \) satisfying equation (4-4)? Its graphical interpretation is the following: the second derivative of a function gives the curvature, with a positive second derivative making the function curve up, negative, down. So, equation (4-4) says that the function always curves back towards the \( x = 0 \) axis, as shown in figure 4-2. Look like a sine wave?

The equation (4-4) cannot be simply integrated to give \( x(t) \). Too bad. Second best is to do what physicists usually do - try to guess the solution. What familiar functions do we know which come back to the same form after two derivatives?

\[
\begin{align*}
\sin t & \quad \text{and} \quad \text{real constants} \\
\cos t & \\
e^{it} & \\
e^{-it} & 
\end{align*}
\]

The first set of functions are the ones to use here, though they are closely related to the second set. The general solution to equation (4-4) can be written as

\[
x(t) = C \cos (\omega_0 t - \phi) \tag{4-5}
\]

where \( C \) and \( \phi \) are arbitrary constants. (Second-order differential equations in time always leave two constants to be determined from initial conditions.) It is fairly easy to show (given as a homework problem) that the following forms are equivalent to that given in equation (4-5).

\[
\begin{align*}
x(t) &= A \sin \omega_0 t + B \cos \omega_0 t, \quad A \text{ and } B \text{ real constants} \\
x(t) &= D e^{i\omega_0 t} + E e^{-i\omega_0 t}, \quad E^* = D \text{ complex constants} \\
x(t) &= \Re (Fe^{i\omega_0 t}), \quad F \text{ a complex constant}
\end{align*}
\]
It turns out that the exponential forms are the easiest to work with in many calculations, and the very easiest thing is to set

\[ x(t) = a e^{i \omega t}. \]  

(4-7)

This looks strange, since observables in physics have to be real. But what we do is to use this form to solve any (linear) differential equation, and take the real part afterwards. It works. We will use this form for the general solution in the examples to follow.

**B. Coupled oscillations - masses and springs.**

Many complex physical systems display the phenomenon of resonance, where all parts of the system move together in periodic motion, with a frequency which depends on inertial and elastic properties of the system. The simplest example is a single point mass connected to a single ideal spring, as shown in figure 4-1a. The mass has a sinusoidal displacement with time which can be described by the function given in equation (4-7), with \( \omega_0 = \sqrt{\frac{k}{m}} \) as the resonant frequency of the system, and \( a \) a complex amplitude. It is understood that the position of the mass is actually given by the real part of the expression (4-7); thus the magnitude of \( a \) gives the maximum displacement of the mass from its equilibrium position, and the phase of \( a \) determines the phase of the sinusoidal oscillation.

**A system of two masses.** A somewhat more complicated system is shown in figure 4-3. Here two identical masses are connected to each other and to rigid walls by three identical springs. The motions of masses 1 and 2 are described by their respective displacements \( x_1(t) \) and \( x_2(t) \) from their equilibrium positions. The magnitude of the force exerted by each spring is equal to \( k \) times the change in length of the spring from the equilibrium position of the system, where it is assumed that the springs are unstretched. For instance, the force exerted by the spring in the middle is equal to \( k(x_2 - x_1) \). Taking the positive direction to be to the right, its force on \( m_1 \) would be equal to \(+ k(x_2 - x_1)\), and its force on \( m_2 \) would be equal to \(-k(x_2 - x_1)\). Newton's second law for the two masses \( m_1 \) and \( m_2 \) then leads to the two equations

![Figure 4-3. System of two coupled oscillators.](image-url)
\[ m\ddot{x}_i = -kx_i + k(x_2 - x_i) \]  
\[ m\ddot{x}_2 = -k(x_2 - x_i) - kx_2, \]  
(4-8)

or, in full matrix notation,
\[ \ddot{\vec{x}} = m\dddot{x} = -k\vec{K}\vec{x} \]  
(4-8a) (generalized Hooke's law).

where
\[ \vec{K} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \]  
(4-8b)

In the absence of external forces the masses will vibrate back and forth in some complicated way. A mode of vibration where both masses move at the same frequency, in some fixed phase relation, is called a normal mode, and the associated frequencies are referred to as the resonant frequencies of the system. Such a motion is described by
\[ x_1(t) = a_1 e^{i\omega t} \]  
\[ x_2(t) = a_2 e^{i\omega t}, \]  
(4-9)

or
\[ \vec{x}(t) = \vec{a} e^{i\omega t}, \]  
(4-9a)

Note that the frequency \( \omega \) is the same for both masses, but the amplitude and phase, determined by \( a_1 \) or \( a_2 \), is in general different for each mass.

Substituting (4-9a) into (4-8a), and using the fact that \( \frac{d^2}{dt^2} e^{i\omega t} = -\omega^2 e^{i\omega t} \), we obtain two coupled linear equations for the two undetermined constants of the motion \( a_1 \) or \( a_2 \):
\[ -2a_1 + a_2 = -\lambda a_1 \]  
\[ a_1 - 2a_2 = -\lambda a_2, \]  
(4-10)

or
\[ \vec{K}\vec{a} = \lambda \vec{a}, \]  
(4-10a)

Here we have introduced a dimensionless constant
\[ \lambda = \frac{\omega^2}{\omega_0^2}, \]  
(4-11)

where \( \omega \) is the angular frequency of this mode of oscillation, and
\[ \omega_0 = \sqrt{\frac{k}{m}} \]  
(4-12)

is a constant characteristic of the system, with the dimensions of an angular frequency. Note that \( \omega_0 \) is not necessarily the actual frequency of any of the normal modes of the system; the frequency of a given normal mode will be given by \( \omega = \omega_0 \lambda^{1/2} \).

Equation (4-10a) is the eigenvalue equation for the matrix \( \vec{K} \), and the eigenvalues are determined by re-writing (4-13) as
\[ (\vec{K} - \lambda \vec{1})\vec{a} = \vec{0}. \]  
(4-16)
This system of linear equations will have solutions when the determinant of the matrix
\[ K - \lambda I \] is equal to zero. This leads to the characteristic equation:
\[
\begin{vmatrix}
2 - \lambda & -1 \\
-1 & 2 - \lambda
\end{vmatrix}
= (\lambda - 2)^2 - 1.
\]
\[ = \lambda^2 - 4\lambda + 3 \]
\[ = (\lambda - 1)(\lambda - 3) = 0 \]

There are thus two values of \( \lambda \) for which equation (4-16) has a solution: \( \lambda^{(1)} = 1 \) and \( \lambda^{(2)} = 3 \), corresponding to frequencies of oscillation \( \omega^{(1)} = \omega_0 \) and \( \omega^{(2)} = \sqrt{3}\omega_0 \). We will investigate the nature of the oscillation for each of these resonant frequencies.

**Case 1.** \( \omega^{(1)} = \omega_0 \). This is the same frequency as for the single mass-on-a-spring of figure 4-1a. How can the interconnected masses resonate at this same frequency? A good guess is that they will move with \( a_1 = a_2 \), so that the distance between \( m_1 \) and \( m_2 \) is always equal to the equilibrium distance, and the spring connecting \( m_1 \) and \( m_2 \) exerts no force on either mass. To verify this, we substitute \( \lambda = 1 \) into equation (4-16) and solve for \( a_1 \) and \( a_2 \):
\[
(\overline{K} - \lambda I)\ddot{a} = \begin{pmatrix} 2 - \lambda & -1 \\ -1 & 2 - \lambda \end{pmatrix} \ddot{a} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \ddot{a} = 0.
\]
giving two equations for two unknowns:
\[
a_1 - a_2 = 0
\]
\[-a_1 + a_2 = 0.
\]
Both equations tell us the same thing:
\[ a_1 = a_2 .
\]
Both masses have the same displacement at any given time, so the spring joining them never influences their motion, and their resonant frequency is the same as if the central spring was not there.

**Case 2.** \( \omega^{(2)} = \sqrt{3}\omega_0 \). This frequency is higher than for the single mass-on-a-spring of figure 4-1a, so the middle spring must be stretched in such a way as to reinforce the effect of the outer springs. We might guess that the two masses are moving in opposite directions. Then as they separated, the middle spring would pull them both back towards the center, while the outside springs pushed them back towards the center. The acceleration would be greater and the vibration faster. We can see if this is right by substituting \( \lambda = 3 \) into equation (4-16) and solve for \( a_1 \) and \( a_2 \):
\[
(\bar{K} - \lambda^{(2)} \bar{I}) \bar{a} = \begin{pmatrix} 2 - \lambda^{(2)} & -1 \\ -1 & 2 - \lambda^{(2)} \end{pmatrix} \bar{a} \]
\[
= \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \bar{a} = 0 
\]
giving the equations
\[
-a_1 - a_2 = 0 \\
-a_1 - a_2 = 0 ,
\]
confirming that
\[
a_1 = -a_2 .
\]
Thus we have the following eigenvalues and eigenvectors for the matrix \( \bar{K} \):
\[
\bar{a}^{(1)} = \frac{1}{\sqrt{2}} (1 \ 1) \quad \leftrightarrow \quad \lambda^{(1)} = 1 \\
\bar{a}^{(2)} = \frac{1}{\sqrt{2}} (1 \ -1) \quad \leftrightarrow \quad \lambda^{(2)} = 3
\]
The equations above only determined the ratios of components of \( \bar{a} \); I have added the factor of \( 1/\sqrt{2} \) to normalize the vectors to a magnitude of 1.

Three interconnected masses. With three masses instead of two, at positions \( x_1, x_2 \) and \( x_3 \), the three coupled equations still have the form of equation (4-13), with
\[
\bar{K} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}
\]
and characteristic equation
\[
|\bar{K} - \lambda \bar{I}| = \begin{vmatrix} 2 - \lambda & -1 & 0 \\ -1 & 2 - \lambda & -1 \\ 0 & -1 & 2 - \lambda \end{vmatrix} = 0 .
\]
It will be left to the problems to find the three normal-mode frequencies and to determine the way the masses move in each case.

Systems of many coupled masses. A long chain of masses coupled with springs is a commonly used model of vibrations in solids and in long molecules. It would not be too
hard to write down the matrix $K$ corresponding to such a long chain. However, analyzing the solutions requires more advanced methods which we have not yet developed.

C. The triple pendulum

There is an interesting problem which illustrates the power (and weaknesses) of the trained physicist. Consider three balls, suspended from a fixed point, as shown in figure 4-5a.

![Figure 4-5](image)

**Figure 4-5.** Three balls, forming a compound pendulum. (a) Hanging from the ceiling, at rest. (b) Oscillating in the first normal mode. (c) Free-body diagram for ball 2.

4-5a. If the balls are displaced from equilibrium and released, they can move in rather complicated ways. A further amusing problem is to imagine making the point of support move back and forth, or in a circle. We may not get quite this far, for lack of time.

To make a tractable problem, take the usual scandalous physics approach of simplifying the problem, as follows:

1. Consider only motion in a plane, consisting of the vertical direction and a transverse direction.
2. Consider only small displacements. The idea is to be able to make the small-angle approximation to trigonometric functions.
3. Take all three masses to be equal, given by $m$, and take the three string lengths to be equal, given by $L$.

Now the problem looks like figure 4-5b. The three variables of the problem are the transverse positions of the three balls. The forces on the three balls are not too hard to
calculate. For instance, the free-body diagram for ball 2 is shown in Figure 4-5c. In the small-angle approximation,
\[
\begin{align*}
\theta_2 \approx \sin \theta_2 \approx \left( x_2 - x_i \right) / L \\
\theta_3 \approx \sin \theta_3 \approx \left( x_3 - x_2 \right) / L.
\end{align*}
\] (4-27)

Also, reasoning that the string tensions mainly just hold the balls up, they are given by
\[
\begin{align*}
T_1 & \approx 3mg \\
T_2 & \approx 2mg. \\
T_3 & \approx mg
\end{align*}
\] (4-28)

The vertical forces automatically cancel. For forces in the horizontal direction, Newton's second law for this ball then gives
\[
"ma = F"
\]
\[
\begin{align*}
mx_2 &= -T_2 \sin \theta_2 + T_3 \sin \theta_3 \\
&= -\frac{2mg}{L} \left( x_2 - x_i \right) + \frac{mg}{L} \left( x_3 - x_2 \right) \\
&= -2ma_0^2 \left( x_2 - x_i \right) + ma_0^2 \left( x_3 - x_2 \right)
\end{align*}
\] (4-29)

Here we have used the fact that a simple pendulum consisting of a mass \( m \) on a string of length \( L \) oscillates with an angular frequency of
\[
\omega_0 = \sqrt{\frac{g}{L}}.
\] (4-30)

Similar reasoning for the other two masses leads to the three coupled equations in three unknowns,
\[
\begin{align*}
mx_1 &= -3ma_0^2 x_1 + 2ma_0^2 \left( x_2 - x_1 \right) \\
mx_2 &= -2ma_0^2 \left( x_2 - x_1 \right) + ma_0^2 \left( x_3 - x_2 \right) \\
mx_3 &= -ma_0^2 \left( x_3 - x_2 \right)
\end{align*}
\] (4-31)

We now look for normal modes, where
\[
\ddot{x} = \ddot{a} e^{i\omega t}.
\] (4-32)

Substituting into equation (4-31) gives a factor of \(-\omega^2\) on the left-hand side, suggesting that we define a dimensionless variable as before,
\[
\lambda \equiv \frac{\omega^2}{\omega_0^2}.
\] (4-33)

giving
\[
\lambda \ddot{\bar{a}} = \bar{K} \ddot{a}.
\] (4-34)

with
\[
\bar{K} = \begin{pmatrix}
5 & -2 & 0 \\
-2 & 3 & -1 \\
0 & -1 & 1
\end{pmatrix}.
\] (4-35)
This is the classic eigenvector-eigenvalue equation,

\[ \mathbf{K} \mathbf{a} = \lambda \mathbf{a} \quad \text{(4-36)} \]

(You might want to fill in the steps yourself leading from equation (4-31) to this point.)

In this way, the physical concept of a search for stationary patterns of relative displacements of the masses translates into the mathematical idea of finding the eigenvectors of the matrix \( \mathbf{K} \).

As with the coupled masses, we write this equation in the form

\[
\left( \mathbf{K} - \mathbf{\lambda I} \right) \mathbf{a} = \begin{pmatrix} 5 - \mathbf{\lambda} & -2 & 0 \\ -2 & 3 - \mathbf{\lambda} & -1 \\ 0 & -1 & 1 - \mathbf{\lambda} \end{pmatrix} \mathbf{a} = \mathbf{0} \\
\]

(4-37)

Solutions will exist if and only if the determinant of the matrix \( \mathbf{K} - \mathbf{\lambda I} \) vanishes, leading to the "characteristic equation" for the eigenvalues,

\[
|\mathbf{\lambda I} - \mathbf{K}| = \begin{vmatrix} \mathbf{\lambda} - 5 & 2 & 0 \\ 2 & \mathbf{\lambda} - 3 & 1 \\ 0 & 1 & \mathbf{\lambda} - 1 \end{vmatrix} = 0
\]

\[
(\mathbf{\lambda} - 5)((\mathbf{\lambda} - 3)(\mathbf{\lambda} - 1) - 4(\mathbf{\lambda} - 1)) = 0.
\]

\[
\mathbf{\lambda}^3 - 9\mathbf{\lambda}^2 + 18\mathbf{\lambda} - 6 = 0 \quad \text{(4-38)}
\]

This is a cubic, with three roots, and is hard to solve analytically. There is in principle a closed-form solution, but it is pretty hairy. Here is how Mathematica does it:

\[
\text{NSolve}[x^3 - 9x^2 + 18x - 6 == 0, x]
\]

({{x -> 0.415775}, {x -> 2.29428}, {x -> 6.28995}})

Another pretty good way, however, is just to calculate values using Excel until you get close. In the spreadsheet to the right, you can see that the cubic goes through zero somewhere near \( \lambda = 0.4 \), and again near \( \lambda = 2.2 \). You can easily make the step smaller and pin down the values, as well as finding the third root. The values are given in Table I.

<table>
<thead>
<tr>
<th>lambda</th>
<th>dlambda</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.1</td>
</tr>
<tr>
<td>0.1</td>
<td>-4.289</td>
</tr>
<tr>
<td>0.2</td>
<td>-2.752</td>
</tr>
<tr>
<td>0.3</td>
<td>-1.383</td>
</tr>
<tr>
<td>0.4</td>
<td>-0.176</td>
</tr>
<tr>
<td>0.5</td>
<td>0.875</td>
</tr>
<tr>
<td>0.6</td>
<td>1.776</td>
</tr>
<tr>
<td>0.7</td>
<td>2.533</td>
</tr>
<tr>
<td>0.8</td>
<td>3.152</td>
</tr>
<tr>
<td>0.9</td>
<td>3.639</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>1.1</td>
<td>4.241</td>
</tr>
<tr>
<td>1.2</td>
<td>4.368</td>
</tr>
<tr>
<td>1.3</td>
<td>4.387</td>
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<td>4.304</td>
</tr>
<tr>
<td>1.5</td>
<td>4.125</td>
</tr>
<tr>
<td>1.6</td>
<td>3.856</td>
</tr>
<tr>
<td>1.7</td>
<td>3.503</td>
</tr>
<tr>
<td>1.8</td>
<td>3.072</td>
</tr>
<tr>
<td>1.9</td>
<td>2.569</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
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<tr>
<td>2.1</td>
<td>1.371</td>
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<td>2.2</td>
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<td>2.3</td>
<td>-0.043</td>
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<tr>
<td>2.4</td>
<td>-0.816</td>
</tr>
<tr>
<td>2.5</td>
<td>-1.625</td>
</tr>
<tr>
<td>2.6</td>
<td>-2.464</td>
</tr>
</tbody>
</table>
Next, for each of the three eigenvalues, we must determine the corresponding eigenvector. This amounts to solving the system of three homogeneous linear equations,

\[
\begin{pmatrix}
5 - \lambda^{(i)} & -2 & 0 \\
-2 & 3 - \lambda^{(i)} & -1 \\
0 & -1 & 1 - \lambda^{(i)}
\end{pmatrix} \mathbf{a}^{(i)} = \mathbf{0}.
\]  

(4-39)

Here $\lambda^{(i)}$ and $\mathbf{a}^{(i)}$ are the $i$-th eigenvalue and eigenvector, respectively. For instance, for the first eigenvalue given above, this gives

\[
\begin{pmatrix}
4.5842 & -2 & 0 \\
-2 & 2.5842 & -1 \\
0 & -1 & 0.5842
\end{pmatrix} \mathbf{a}^{(1)} = \mathbf{0}.
\]  

(4-40)

The magnitude of the eigenvector is not determined, since any multiple of the eigenvector would still be an eigenvector, with the same eigenvalue. So, let’s take the first component of $\mathbf{a}$ to be equal to 1. The we can find the ratios $a_2/a_1$ and $a_3/a_1$ from

\[
\begin{pmatrix}
4.5842 & -2 & 0 \\
-2 & 2.5842 & -1 \\
0 & -1 & 0.5842
\end{pmatrix} \begin{pmatrix} 1 \\ a_2 \\ a_3 \end{pmatrix} = \mathbf{0}.
\]  

(4-41)

For instance, the equation from the first line of the matrix is

\[4.5842*1 - 2*a_2 = 0.\]  

(4-42)

giving

\[a_2 = 2.2921.\]  

(4-43)

Next, multiply the third line in the matrix by 2.5842 and add it to the second line, to give

<table>
<thead>
<tr>
<th>motion</th>
<th>eigenvalue $\lambda$</th>
<th>normalized frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>(single ball)</td>
<td></td>
<td>1.0000</td>
</tr>
<tr>
<td>mode 1</td>
<td>0.4158</td>
<td>0.64487</td>
</tr>
<tr>
<td>mode 2</td>
<td>2.2943</td>
<td>1.5147</td>
</tr>
<tr>
<td>mode 3</td>
<td>6.2899</td>
<td>2.5080</td>
</tr>
</tbody>
</table>

Table I. Eigenvalues for the three normal modes of the three-ball system, and the corresponding frequency, given in terms of the frequency for a single ball on a string of length $L$. 

For instance, the equation from the first line of the matrix is

\[4.5842*1 - 2*a_2 = 0.\]  

(4-42)

giving

\[a_2 = 2.2921.\]  

(4-43)
\[
\begin{pmatrix}
4.5842 & -2 & 0 \\
-2 & 0 & 0.5097 \\
0 & -1 & 0.5842
\end{pmatrix}
\begin{pmatrix}
1 \\
a_2 \\
a_3
\end{pmatrix}
= \begin{pmatrix}
0 \\
\tilde{a}_1 \\
\tilde{a}_2
\end{pmatrix}.
\]  
(4-44)

The equation from the second line is
\[-2 + 0.5097 a_3.
\]  
(4-45)

giving
\[a_3 = 3.9240.
\]  
(4-46)

Or,
\[
\tilde{a}^{(1)} = \begin{pmatrix} 1 \\ 2.2921 \\ 3.9240 \end{pmatrix}.
\]  
(4-47)

for the first eigenvector! In this mode, the coordinates of three balls are given by
\[
\tilde{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \tilde{a}^{(1)} e^{i\omega t} = \begin{pmatrix} \cos \omega t \\ 2.2921 \cos \omega t \\ 3.9240 \cos \omega t \end{pmatrix}.
\]  
(4-48)

Note that the balls all move in the same direction, in this mode.

The other eigenvectors can be found in a similar way. The exact values are left to the problems. But figure 4-6 shows the displacements of the balls in the three modes. The higher the mode (and the higher the frequency), the more the balls move in opposite directions.
PROBLEMS

Problem 4-1. (a) Using identities from Appendix A, show that

$$\cos(\omega_0 t - \phi) = A \cos \omega_0 t + B \cos \omega_0 t$$

and find $A$ and $B$ in terms of $C$ and $\phi$.

(b) Using identities from Appendix A, show that

$$\cos(\omega_0 t - \phi) = D e^{i \omega_0 t} + E e^{-i \omega_0 t}$$

and find $D$ and $E$ in terms of $C$ and $\phi$. (Here $C$ is taken to be real.)

Problem 4-2. Find the normal-mode frequencies $\{\omega_i, i = 1,3\}$ for the problem described in the text (see fig. 4-4) of three identical masses connected by identical springs. Express the frequencies in terms of $\lambda \equiv \frac{\omega^2}{\omega_0^2}$, where $\omega_0 = \sqrt{\frac{k}{m}}$.

Problem 4-3. Find the normal modes for the problem described in the text (see figure 4-4) of three masses connected by springs.

Problem 4-4. Consider a system of two masses and three springs, connected as shown in figure 4-3, but with the middle spring of spring constant equal to $2k$. 

Figure 4-6. The three normal modes for the triple pendulum. The balls are shown at maximum displacement, when they are all (momentarily) at rest.
(a) Try and guess what the normal modes will be - directions of motion of the masses and frequencies.
(b) Write the equations of motion, find the characteristic equation, and solve it, and so determine the frequencies of the two normal modes. Compare with your guesses in part (a).

**Problem 4-5.** Find the eigenvectors $\vec{a}_2$ and $\vec{a}_3$ for the triple pendulum corresponding to the second and third eigenvalues, $\lambda_2$ and $\lambda_2$. Give a qualitative interpretation, in terms of the co- or counter-motion of the balls, with respect to the first one.

**Problem 4-6.** Repeat the analysis of the multiple pendulum in the text, but for two balls, rather than three. You should determine the two normal-mode frequencies $\omega_i$ and the normal-mode eigenvectors $\vec{a}_i$. In this case it should be possible to find the eigenvalues exactly, without having to resort to numerical methods. Discuss the solution.