1. Introduction

It is generally believed that non-magnetized accretion disks with Keplerian velocities are stable to all physically relevant perturbations. A disk’s stability has been attributed (Balbus & Hawley 1998) to Rayleigh’s centrifugal stability theorem because the disk’s angular momentum per unit
mass increase with increasing radius (Rayleigh 1917). However, Rayleigh’s theorem applies only to
a limited class of flows, so Keplerian disks can have purely hydrodynamic (i.e., without magnetic
fields) instabilities, even if their relevance to astrophysical disks is controversial. Some disk insta-
bilities are extensions of the Goldreich-Schubert-Fricke instability (c.f., Nelson et al. (2013); Klahr
& Hubbard (2014)); some are due to radial entropy gradients (c.f., Klahr & Bodenheimer (2003);
Lesur & Papaloizou (2010)); some are due to mode resonances in stratified flows (c.f., Yavneh et al.
(2001)), and some require local extrema in the radial distribution of vortensity or potential vorticity
(c.f., Lovelace et al. (1999). The arguments that these and other purely hydrodynamic instabilities
may not be relevant to accretion disks include: unphysical boundary conditions, large ad hoc initial
perturbations, and special conditions on cooling times. Initial perturbations in Keplerian disks can
be nonlinearly amplified by the Keplerian shear and create large transient growths (Tevzadze et al.
2003, 2008; Salhi & Cambon 2010; Volponi 2010; Salhi et al. 2013; Afshordi et al. 2005), but no
mechanism has been found by which the initial perturbations can regenerate themselves and create
sustained turbulence. There have been numerous conjectures, based on analogies to other flows
with strong shear, such as circular Couette flow and Pouseille flow, that Keplerian disks might be
finite-amplitude unstable to noise [we need some references here]. However, sustained turbulence
has not been observed in numerical simulations of purely hydrodynamic Keplerian disks (Balbus
et al. 1996, hereafter BHS96; Shen et al. 2006, hereafter SSG06; Rincon et al. 2007) or laboratory
experiments (Ji et al. 2006; though see Balbus 2011; Paoletti & Lathrop 2011; Schartman et al.
2012 for some recent controversy).

In high-energy accretion disks and in some parts of disks around protostars i.e., protoplanetary
disks (PPDs), turbulence can be generated and sustained by the magneto-rotational instability
(MRI; Balbus & Hawley 1991), first proposed by Velikhov (1959) & Chandrasekhar (1960). How-
ever, some regions of PPDs are too cold to sufficiently ionize and couple efficiently to a magnetic
field (Gammie 1996). These regions are stable to the MRI, and are thus referred to as “dead zones”.
In support of the idea that large regions near the mid-planes of PPDs remain “dead” even when
surrounded by MRI-active regions, several groups (Fleming & Stone 2003; Fromang & Papaloizou
2006; Oishi & Mac Low 2009) simulated the nonlinear evolution of vertically stratified Keplerian
disks composed of magnetized plasma whose magnetic resistivity becomes large near the mid-plane.
Although the magneto-rotational instability (MRI) was present far from the mid-plane, the large
magnetic resistivity prevented the plasma from efficiently coupling to the magnetic field near the
mid-plane. Thus, “dead zones” appear to stubbornly remain in large regions of PPDs near their
mid-planes.

Because angular momentum transport plays such an important role in the evolution of a PPD
and because the transport by collisional viscosity in a PPD is many orders of magnitude too small
to be consistent with the inferred rate of transport needed for star formation (Shakura & Sunyaev
1973), we reconsider here in an astrophysical context a new purely hydrodynamic instability that
we discovered in Boussinesq flows, which we called the zombie instability (Marcus et al. 2013,
hereafter MPHJ13). Here we examine how this instability is triggered from weak noise and creates
self-sustaining turbulence in a PPD. We postpone to future papers an analyses of how the turbulence created from the zombie instability transports angular momentum and of how the turbulent zombie vortices can serve as locations where dust accumulates and agglomerates into planetesimals.

In this paper, we study the destabilizing effects of the vertical stratification that occurs naturally in a Keplerian PPD with a non-zero vertical component of gravity from its central protostar. Vertical gravity and stratification were ignored in many previous studies of disk stability. We build on our previous study of disk stability, which spontaneously produced vortices, but in which we did not recognize that they were created by the zombie instability) (Barranco & Marcus 2005, hereafter BM05) and Barranco & Marcus (2006, hereafter BM06). Here, we examine a Keplerian disk whose initial unperturbed equilibrium is isothermal and in vertical hydrostatic equilibrium with a stable density stratification with a buoyancy or (Brunt-Väisälä) frequency $N$. Unlike BM05 and BM06, to better understand the physics, we approximate the disk’s vertical gravity as constant. We shall show that perturbations consisting of random noise with no initial coherent structures and with initial rms Mach numbers as small as $10^{-6}$ create three-dimensional, space-filling, zombie turbulence with sustained rms Mach numbers of 0.2 - 0.3 and with large turbulent vortical structures dominating the flow. The turbulence is self-sustaining because it draws its energy from the Keplerian shear. We shall show that this purely hydrodynamic instability is not a numerical artifact (it occurs in spectral simulations of Boussinesq and anelastic flows (BM06), as well as in simulations of a fully compressible ideal gas using the Athena code (Gardiner & Stone 2008; Stone et al. 2008)); nor is the instability a result of non-physical ad hoc assumptions: no unusual boundary conditions are required (the instability and subsequent turbulence was computed using both a shearing box boundary condition and with periodic radial boundaries); nor does the instability require a fast or specially tuned cooling time (for the instabilities presented here, the flows are dissipationless, but we have computed the zombie instability in laboratory Couette flows with small viscosities and in disks with finite radiative cooling times).

The paper is organized as follows. In section 2 we review the fully compressible, anelastic and Boussinesq equations, list their equilibrium solutions, define their control parameters and discuss the conditions under which the equations are valid. In section 3 we summarize our numerical results that show the disks are unstable by showing how the non-Keplerian component of the energy grows in time. In section 4 we show that the zombie instability is triggered by a threshold vorticity rather than a threshold velocity, and we show how the properties of the zombie instability, of disks, and of turbulence work together so that the Mach number of the energy of the initial turbulent noise that is required to trigger the instability can be as small as $10^{-6}$. In section 5 we analyze the fundamental fluid dynamics of the instability, and our Discussion is in section 6.

2. Equations for the fluid motion in a disk in the local plane-parallel approximation

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The stability of flows in protoplanetary disks have been examined using a variety of approximations, but here we limit ourselves to those in which the curvature of the disk is ignored and in which the unperturbed azimuthal flow is expanded locally around a fiducial cylindrical radius $R_0$. Hill (1878) was the first to carry out this type of expansion.

### 2.1. Ideal, fully compressible flow

Consider a disk in which the unperturbed steady flow is only in the azimuthal direction with an angular velocity $\Omega(R)$ such that

$$\Omega(R) \propto R^{-q}. \quad (2-1)$$

In the local Cartesian approximation around $R_0$, Euler’s equation for an observer in a frame rotating with angular velocity $\Omega_0 \equiv \Omega(R_0)$ around the $z$-axis is:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla P - 2\Omega_0 \hat{z} \times \mathbf{v} + 2q\Omega_0^2 x \hat{x} - g(z) \hat{z}, \quad (2-2)$$

where $\mathbf{v}(x, y, z)$ is the gas velocity written in the Cartesian approximation, where $P$ and $\rho$ are the gas pressure and density, $-2\Omega_0 \hat{z} \times \mathbf{v}$ is the Coriolis term, $2q\Omega_0^2 x \hat{x}$ is the tidal acceleration that arises from the difference between the centrifugal acceleration $-R\Omega^2(R)$ and the quantity $R_0\Omega^2(R_0)$, $-g(z)$ is the acceleration of gravity in the $z$ direction, and where a “hat” above a coordinate means the unit vector in that coordinate’s direction. The Cartesian approximation in eq. (2-2) uses $x \ll R_0$, where the Cartesian $x$ coordinate corresponds to the cylindrical radial direction with $x \equiv R - R_0$, where the Cartesian $y$ coordinate corresponds to the azimuthal $\phi$ coordinate with $y \equiv -R_0\phi$, and where $z$ is identical in the Cartesian and cylindrical coordinate systems. Using the same local Cartesian approximation, the continuity equation becomes

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (2-3)$$

and the energy equation becomes

$$\frac{\partial \rho \epsilon}{\partial t} = -\nabla \cdot (\rho \epsilon \mathbf{v}) - P (\nabla \cdot \mathbf{v}), \quad (2-4)$$

where $\epsilon$ is the internal energy of the gas. Following BHS96, we write $\epsilon \equiv (3/2)RT$, where $R$ is the gas constant, $T$ is the gas temperature, and the ideal gas equation of state is

$$P = (\gamma - 1)\rho \epsilon = RT, \quad (2-5)$$

where $\gamma$ is the ratio of the specific heats at constant pressure and constant volume and set to 5/3. The steady, unperturbed equilibrium velocity (written with an overbar) that satisfies eqs. (2-2) – (2-5) is

$$\bar{v}_x = \bar{v}_z = 0 \quad (2-6)$$
$$\bar{v}_y = -q\Omega_0 x. \quad (2-7)$$
The steady equilibrium pressure and density, that satisfy eqs. (2-2) – (2-5) are functions of $z$ only and obey the hydrostatic equation:

$$d\bar{P}(z)/dz = -\bar{\rho}(z)g(z).$$

The steady equilibrium internal energy and temperature are also functions of $z$ only and satisfy the equation of state: $\bar{\epsilon}(z) = (3/2)\bar{P}(z)/\bar{\rho}(z)$ and $\bar{T}(z) = \bar{P}(z)/[R\bar{\rho}(z)]$.

Because there is no thermal radiation, diffusion, or dissipation in equations (2-2) – (2-5), there is a degeneracy of the allowable steady equilibrium thermodynamic solution. In general, one thermodynamic quantity, $\bar{P}(z)$, $\bar{T}(z)$, $\bar{\rho}(z)$, or $\bar{\epsilon}(z)$ can be arbitrarily specified (but see the one exception to this degeneracy explained in the next paragraph). Once that quantity is specified the others follow uniquely (up to a constant of integration).

In the Cartesian approximation of a Keplerian disk in which the self-gravity of the gas is ignored, but the vertical $z$ component of the gravity from the central object is included, $g(z) = -\Omega_0^2 z$, where $z = 0$ is the mid-plane of the disk. This vertical gravity was used in BM05 and BM06 along with the choice that $\bar{T}(z)$ is constant (motivated by models such as Chiang & Goldreich (1997)). This calculation produced zombie vortices. In contrast, the disk stability studies in BHS96, and in SGS06 and in (MORE REFERENCES NEEDED FOR AUTHORS WHO DID THIS) use equations eqs. (2-2) – (2-5) with $g(z) \equiv 0$, and, as we show in § 6, the latter approximation prohibits the zombie instability. When $g = 0$, the steady state equilibrium pressure corresponding to the steady velocity in eqs. (2-6) – (2-7) must be constant, and therefore cannot be an arbitrary function of $z$. However, $\bar{T}(z)$ or $\bar{\rho}(z)$ or $\bar{\epsilon}(z)$ can still be arbitrarily specified. In the stability calculations in BHS96, and in SGS06 and in (MORE REFERENCES NEEDED FOR AUTHORS WHO DID THIS) with $g = 0$, the steady equilibrium internal energy, temperature and density were all chosen to be constants (see § 3), so that the unperturbed disk flow is barotropic.

### 2.2. Anelastic flows

The anelastic approximation to eqs. (2-2) – (2-5) is commonly used in atmospheric flows (Ogura & Phillips 1962; Gough 1969; Bannon 1996) where there is a reference density or steady equilibrium density $\bar{\rho}(z)$ that varies with $z$. In the anelastic approximation $\bar{\rho}(z)$ can be an arbitrary function of $z$, and the changes in $\bar{\rho}(z)$ with respect to $z$ can be arbitrarily large. However, the anelastic approximation has two requirements for all locations and for all time: (1) $|\rho(x, y, z, t) - \bar{\rho}(z)| \ll \bar{\rho}(z)$, and (2) $|v|$ must be much less than the isothermal speed of sound $C_s \equiv \sqrt{RT}$ (or that the Mach number $Ma$ must be small). The latter requirement is not satisfied in many astrophysical flows, so the anelastic equations are not commonly used in astrophysics. However, it should be noted that both of the requirements above are fulfilled in the computations of disk stability in BHS96 and BM05.

One computational nicety of using the anelastic approximation in an initial-value code is
that at every time step conditions (1) and (2) can be examined, and it can be determined if the approximations required by the anelastic equations are still satisfied. In our anelastic disk study in BM05, we carried out these checks frequently, and in the anelastic calculations presented here, we frequently did consistency checks to verify that the flow had not evolved past its anelastic restrictions.

The anelastic equations are usually written in terms of the steady equilibrium density $\bar{\rho}(z)$ and pressure $\bar{P}(z)$, where $\bar{P}(z)$ and $\bar{\rho}(z)$ satisfy the hydrostatic equation (2-8). The anelastic equations for disk flow were derived in BM06 from eqs. (2-2) – (2-5) by expanding in powers of $|\rho(x, y, z, t) - \bar{\rho}(z)|/\bar{\rho}(z)$. For a disk in which the angular velocity of the unperturbed steady equilibrium flow is proportional to $R^{-q}$, the anelastic version of the local Cartesian Euler equation (2-2) in the rotating frame becomes: DO NOT USE THE NEXT EQUATION BECAUSE IT DOES NOT CONSERVE ENERGY

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\bar{\rho}(z)} \nabla[P(x, y, z, t) - \bar{P}(z)]$$

$$-2\Omega_0 \hat{z} \times \mathbf{v} + 2q\Omega_0^2 x \hat{x} - \frac{\rho(x, y, z, t) - \bar{\rho}(z)}{\bar{\rho}(z)} g(z) \hat{z}. \quad (2-9)$$

Using the anelastic approximation, the continuity equation (2-3)) becomes

$$\nabla \cdot [\bar{\rho}(z)\mathbf{v}] = 0, \quad (2-10)$$

The equation of state and energy equation are most conveniently written in terms of the potential temperature (see BM05 and BM06). However, for the study presented here, we need not complicate the equations with potential temperature. The dissipationless anelastic equations, like the dissipationless fully compressible equations, have a degeneracy in the steady equilibrium thermodynamic solution so that the equilibrium temperature $\bar{T}(z)$ is arbitrary. Here, as in BM05, BHS96, and SGS06, we choose a constant $\bar{T}(z) = T_0$. For $\bar{T}(z) = T_0$, eq. (2-9) reduces to (see BM06):

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla \frac{P(x, y, z, t) - \bar{P}(z)}{\bar{\rho}(z)} - 2\Omega_0 \hat{z} \times \mathbf{v} + 2q\Omega_0^2 x \hat{x} + \frac{T(x, y, z, t) - T_0}{T_0} g(z) \hat{z}. \quad (2-11)$$

For $\bar{T}(z) = T_0$, we still use the continuity equation (2-10). Energy equation (2-4) and the equation of state (2-5) become

$$\frac{\partial T(x, y, z, t)}{\partial t} = -(\mathbf{v} \cdot \nabla) T - T v_z N^2(z)/g(z), \quad (2-12)$$

and

$$\frac{P(x, y, z, t) - \bar{P}(z)}{\bar{P}(z)} = \frac{\rho(x, y, z, t) - \bar{\rho}(z)}{\bar{\rho}(z)} + \frac{T(x, y, z, t) - T_0}{T_0}, \quad (2-13)$$
where \( N(z) \equiv \sqrt{g(z)/(1/\gamma)}(dlnP/dz) - dln\bar{\rho}/dz \) is the Brunt-Väisälä frequency of the unperturbed steady equilibrium flow (Schwartzchild 1958; Kundu 1990). For \( \bar{T}(z) = T_0 \),

\[
N(z) = \sqrt{(1/\gamma - 1)}g(z)(dln\bar{\rho}/dz)
\]  

(2-14)

Equations (2-10) – (2-14) are the governing anelastic equations for constant \( \bar{T} \). Note that the steady equilibrium solution to eqs. (2-10) – (2-14) is the same as the steady equilibrium solution to the fully compressible equations where \( \overline{\mathbf{v}} \) is given by eqs. (2-6) – (2-7), and the relation between \( \bar{P}(z) \) and \( \bar{\rho}(z) \) is given by the hydrostatic equation (2-8), and \( \bar{T} = T_0 \).

For \( \bar{T} = T_0 \) and the Keplerian vertical gravity \( g(z) = \Omega_0^2 z \) used in BM05 and BM06, \( \bar{P}(z) \) and \( \bar{\rho}(z) \) are Gaussian functions of \( z \) with

\[
\bar{P}(z) = P_0 \exp \left[ -z^2/(2H^2) \right],
\]

(2-15)

\[
\bar{\rho}(z) = \rho_0 \exp \left[ -z^2/(2H^2) \right],
\]

(2-16)

where \( \rho_0 \equiv \bar{\rho}(z = 0) \) and where \( P_0 \equiv \bar{P}(z = 0) = R \rho_0 T_0 \), and \( H \equiv \sqrt{KT_0}/\Omega_0 \), where \( H \) is the defined as the fiducial vertical pressure scale height (equal to the actual vertical scale height only at \( z = H \)). Note that

\[
C_s = H \Omega_0 = (g/\Omega_0)(H/z),
\]

(2-17)

where \( C_s \equiv \sqrt{R/T_0} \) is the isothermal sound speed. With Keplerian vertical gravity, the Brunt-Väisälä frequency is linear in \( z \) with \( N(z) = \Omega_0^2 z/\sqrt{KT_0}/(\gamma - 1) = (\Omega_0^2 z/C_s)\sqrt{1 - 1/\gamma} \).

For simplicity, the study presented here uses \( \bar{T} = T_0 \) with a constant vertical gravity \( g(z) = g_0 \), so \( \bar{P}(z) \) and \( \bar{\rho}(z) \) are exponential functions of \( z \) with

\[
\bar{P}(z) = P_0 \exp (-z/H),
\]

(2-18)

\[
\bar{\rho}(z) = \rho_0 \exp (-z/H),
\]

(2-19)

where \( \rho_0 \equiv \bar{\rho}(z = 0) \) and where \( P_0 \equiv \bar{P}(z = 0) = R \rho_0 T_0 \), and \( H \equiv R T_0/g_0 = C_s^2/g_0 \) is the vertical pressure scale height. With constant vertical gravity, the Brunt-Väisälä frequency is constant

\[
N = N_0 = g_0/\sqrt{KT_0}/(\gamma - 1) = (g_0/C_s)\sqrt{1 - 1/\gamma}.
\]

For the anelastic equations with constant vertical gravity, it is useful to define the dimensionless constant

\[
\beta \equiv g_0/(H \Omega_0^2),
\]

(2-20)

so that

\[
C_s = \beta^{1/2}H \Omega_0 = \beta^{-1/2}(g_0/\Omega_0).
\]

(2-21)

When \( \beta \) is unity, \( C_s = H \Omega_0 \), as in eq. (2-17), the case for Keplerian gravity.

In this paper, all of the equations are solved in a Cartesian domain of size \( L_x \times L_y \times L_z \). The dimensionless equations for anelastic flow (and for fully compressible flow) with constant equilibrium \( \bar{T} \) and constant vertical gravity contain three dimensionless numbers: \( \gamma \) (which is always 5/3 in this
paper), $q$ (which is the negative of the shear of the steady equilibrium flow in eqs. (2-6) – (2-7) in units of $\Omega_0$), and $\beta$ (which is equal to $\gamma N_0^2/\left(\gamma - 1\right)$ in units of $\Omega_0^2$ and is therefore a dimensionless measure of the vertical stratification), that is

$$N_0/\Omega_0 = \sqrt{\beta/\left(\gamma - 1\right)/\gamma}. \tag{2-22}$$

In addition, three dimensionless number describe the size of the computational domain: $H/L_x$, $L_y/L_x$ and $L_z/L_x$. Unless otherwise specified in this paper $H/L_x = L_y/L_x = L_z/L_x = 1$.

The anelastic approximation removes acoustic and sound waves from their solutions. For numerically computing weather in the Earth’s atmosphere (which has low Mach numbers and which was the motivation for the development of the anelastic equations), the filtering has been shown, in general, to have no deleterious effects on the computation of atmospheric instabilities, eddies, thermal convection, and other large scale flows, nor does the anelastic approximation have an adverse effect on computing Rossby, inertial, internal-gravity, or Poincaré waves (WE NEED SOME GOOD REFERENCES HERE). The anelastic approximation has also been used successfully in computing thermal convection and other low-Mach number flows in stars (WE NEED SOME GOOD REFERENCES HERE). The anelastic approximation prevents turbulence created by the zombie instability from launching acoustic waves. Because acoustic waves are efficient in transporting momentum and have been considered by many authors as the main mechanism for transporting angular momentum outward in protoplanetary disks (Johnson & Gammie 2005; Shen et al. 2006; Lesur & Papaloizou 2010; Lyra & Klahr 2011; Raettig et al. 2013), the anelastic equations are not suitable for computing angular momentum transport rates and $\alpha$ in disks, even if they are suitable for computing zombie instabilities.

### 2.3. Boussinesq flows

The Boussinesq equations are another commonly used approximation for vertically stratified flows and were used in MPJH13 and will be used in §5 to determine the fundamental physics of the zombie instability. Boussinesq equations are frequently used as the governing equations for the oceans and for laboratory flows in which there are vertical density stratifications either due to the fluid temperature or salt (Kundu 1990). The Boussinesq equations are commonly used in the study of laboratory convection (Chandrasekhar 1981) and often for models of convection in stars (Spiegel 1971). The Boussinesq equations are valid when there is an average density $\rho_0$ such that for all space and time $|\rho(x, y, z, t) - \rho_0| \ll \rho_0$. Here, we consider the case where the density variations in the fluid are due only to compositional changes (say, for example the density of salt dissolved in water) and not due to temperature. The Boussinesq equations for disk flow can be obtained from eqs. (2-2) – (2-5) by expanding in powers of $|\rho(x, y, z, t) - \rho_0|/\rho_0$. For a disk in which the angular velocity of the unperturbed steady equilibrium flow is proportional to $R^{-q}$, the Boussinesq version
of the local Cartesian Euler equation (2-2) in the rotating frame becomes:

\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{\nabla P(x, y, z, t) - \bar{P}(z)}{\rho_0} - 2\Omega_0 \mathbf{\hat{z}} \times \mathbf{v} + 2q\Omega_0^2 x \mathbf{\hat{x}} - \frac{\rho(x, y, z, t) - \bar{\rho}(z)}{\rho_0} g(z) \mathbf{\hat{z}},
\]

(2-23)

where the steady equilibrium pressure \(\bar{P}(z)\) and steady equilibrium density \(\bar{\rho}(z)\) satisfy the hydrostatic equation (2-8). The Boussinesq version of the continuity equation (2-3) is

\[
\nabla \cdot \mathbf{v} = 0,
\]

(2-24)

and the Boussinesq version of the fully compressible energy equation (2-4) or anelastic energy equation (2-12) is

\[
\frac{\partial \rho(x, y, z, t)}{\partial t} = - (\mathbf{v} \cdot \nabla)\rho = - (\mathbf{v} \cdot \nabla)[\rho(x, y, z, t) - \bar{\rho}(z)] + \rho_0 v_z N^2(z)/g(z),
\]

(2-26)

where the Boussinesq Brunt-Väisälä frequency of the steady equilibrium flow is defined as

\[
N(z) = \sqrt{-g(z)(d\bar{\rho}/dz)/\rho_0}
\]

(2-27)

Equation (2-25) is the diffusionless advection equation for the total density \(\rho\). Due to the fact that this equation has no diffusion, there is a degeneracy in the steady equilibrium thermodynamic solution, and in this Boussinesq case this means that \(\bar{\rho}(z)\) is an arbitrary function of \(z\). Notice that the Boussinesqs equations (2-23) – (2-24) and (2-26) – (2-27) do not include \(T\) and that there is no equation of state. The steady velocity equilibrium solution to these Boussinesq equations is the same as the steady equilibrium solution to the fully compressible equations and to the anelastic equations given by eqs. (2-6) – (2-7) and hydrostatic equilibrium (2-8).

For constant \(N(z) = N_0\) and for constant gravity \(g(z) = g_0\), \(\bar{\rho}(z)\) is linear in \(z\), and eq. (2-27) is often used to parameterize the steady equilibrium density:

\[
\bar{\rho}(z) = \rho_0(1 - N_0^2 z/g_0).
\]

(2-28)

In the case of constant \(N\), the dimensionless Boussinesq equations of motion depend on only two dimensionless numbers: \(q\) (which, as in the anelastic equations, is the negative of the shear of the steady equilibrium flow in units of \(\Omega_0\)), and \(N_0/\Omega_0\). As we did with the anelastic equations, we set the two dimensionless numbers that describe the computational domain, \(L_y/L_x\) and \(L_z/L_x\) equal to unity.

For the case in which \(g = 0\), the Brunt-Väisälä frequency \(N\) is also equal to zero, \(\bar{P}(z) = P_0\) is constant, but \(\bar{\rho}(z)\) is still an arbitrary (and dynamically unimportant) function of \(z\). For the case \(g = 0\) and \(\bar{\rho}(z) = \rho_0\) is constant, the Boussinesq and anelastic equations become identical.
In most laboratory experiments, the diffusion time of salt is very long compared to any other physically relevant time, so the equations of motion effectively have no diffusion. The degeneracy in $\bar{\rho}(z)$ is frequently exploited in laboratory experiments, and experiments are often initialized with a steady equilibrium in which $\bar{\rho}(z)$ is chosen arbitrarily to suit the experimenter’s needs (c.f., Aubert et al. (2012)). Note that the inclusion of the tidal acceleration term $2q\Omega^2_0 x \hat{x}$ in eq.(2-23) is somewhat misleading because the equation can be re-written without it by defining the pressure head $\Pi \equiv [P(x,y,z,t) - \bar{P}(z)]/\rho_0 - q\Omega^2_0 x^2$ as

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = - \nabla \Pi - 2\Omega_0 \hat{z} \times \mathbf{v} - \frac{\rho(x,y,z,t) - \bar{\rho}(z)}{\rho_0} g(z) \hat{z}, \quad (2-29)$$

indicating that dropping the tidal acceleration does not affect the velocity $\mathbf{v}$ (but it does change the value of the pressure $P$). Thus, the Boussinesq equations (2-24) and (2-26) – (2-29) govern the velocity and density of rotating, salt-stratified laboratory flows in which the rotation axis, gravity and the equilibrium density gradients are aligned in the $z$ direction and in flows in which viscosity and salt diffusion can be ignored. Those flows include channel flows or unbounded flows in which the unperturbed equilibrium is $\bar{v}_x = \bar{v}_z = 0$ and $\bar{v}_y(x)$ is an arbitrary function of $x$. An example of the latter is plane Couette flow, for which $\bar{v}_y(x) = -q\Omega_0 x$ is a linear function of $x$, and thus the Boussinesq equations that govern plane Couette flow are the same as those that govern Boussinesq disk flow. Unbounded, rotating plane Couette flows with constant Brunt-Väisälä frequency $N_0$ make up a two-parameter family of flows, with dimensionless parameters $q$ and $N_0/\Omega_0$. These plane Couette flows are all thought to be linearly neutrally stable meaning that if the equations are linearized about the steady equilibrium Couette flow in hydrostatic equilibrium the eigenmodes neither grow nor decay exponentially in time.

3. Evidence of Instability in Protoplanetary Disks with Vertical Gravity

3.1. Temporal growth and decay of an initial energy fluctuation

One of the most cited pieces of evidence that protoplanetary disks are stable to purely hydrodynamic instabilities is given by Figure 1 in BHS96, which we reproduce below. This figure shows the growth or decay of the kinetic energy fluctuations of the flow as a function of time and $q$, where $q$ is defined in eq. (2-1) and where the kinetic energy fluctuations is defined as $[\mathbf{v}(x,y,z,t) - \bar{\mathbf{v}}_y \hat{y}]^2/2$, where $\bar{\mathbf{v}}_y \hat{y}$ is the steady equilibrium flow in eq. (2-7). For $q = 3/2$, the energy fluctuation is defined to be the kinetic energy of the non-Keplerian component of the flow. The initial-value calculation that produced Figure 1 used the fully compressible equations (2-2) – (2-5) starting with the steady equilibrium flow in eqs. (2-6) – (2-8) perturbed with small-amplitude noise. The initial value of the noise’s rms velocity was $\sim 0.1 L_x \Omega_0$. The vertical gravity $g$ was set equal to zero in the calculations that produced Figure 1, and the values of steady equilibrium temperature, density and
pressure were all chosen to be constants; $\bar{T}(z) = T_0$; $\bar{\rho}(z) = \rho_0$; $\bar{P}(z) = P_0$. The equations were solved with periodic boundary conditions in $y$ and $z$ and shearing box boundaries in $x$ (Goldreich & Lynden-Bell 1965; Marcus & Press 1977). Because the energy fluctuations all increase in time for $q \geq 2$ and decrease in time for $q < 2$, Figure 1 is used to support the hypothesis that the flow is stable (unstable) to all perturbations when the angular momentum per unit mass of the flow, $R^2 \Omega(R) \propto R^{(2-q)}$, increases (decreases) in the radially outward direction. This hypothesis is consistent with the Rayleigh’s centrifugal stability theorem; however, it must be noted that Rayleigh’s theorem was proved only for the case of constant density fluids (Rayleigh 1917) and therefore may not be applicable to astrophysical flows in disks. The curve labeled with “shr” in Figure 1 corresponds to the case with $q = 3/2$ and with the Coriolis and tidal acceleration terms dropped from eq. (2-2), which would be appropriate for fully compressible flow in a channel with cross-stream shear, but no rotation.

Figure 2 shows a nearly identical set of curves as those in Figure 1, but in Figure 2, the flows are computed with a spectral numerical code (BM06) using the anelastic equations (2-10) – (2-14). The boundary and initial conditions were approximately the same for the flows in Figures 1 and 2 (but see BM06 for a discussion of computational details and see § 4 for details of the initial noise.) However, the initial amplitude of the noise’s rms velocity in Figure 2 is $\sim 0.04 \ L_x \Omega_0$, which is smaller than that in Figure 1. In Figure 1 the fluctuating kinetic energies of all of the curves have an initial precipitous decay from their initial values. We chose the amplitude of the initial noise in Figure 2 to match the value of the local minimum amplitude of the noise of the growing modes in Figure 1 at the approximate time of $t = 0.1$. As in Figure 1, the flows in Figure 2 were computed with $g = 0$, $\bar{T}(z) = T_0$, $\bar{\rho}(z) = \rho_0$, and $\bar{P}(z) = P_0$. We remind the reader that the Boussinesq and anelastic equations become identical when $g = 0$ and $\bar{T}(z) = T_0$; $\bar{\rho}(z) = \rho_0$; $\bar{P}(z) = P_0$, so Figure 2 would be the same if we had used the Boussinesq rather than the anelastic equations. Figure 2 illustrates our point that the stability of unstratified disks as a function of $q$ is independent of whether the flow is fully compressible, anelastic or Boussinesq.

Figure 3 is similar to Figures 1 and 2, but its three curves are all for the Keplerian case with $q = 3/2$. The dashed broken black curve in Figure 3 is the same as the curve in Figure 2 for $q = 3/2$, but plotted for a much longer time. The blue unbroken curve is computed with the anelastic equations (2-10) – (2-14). The black unbroken curve is computed with the Boussinesq equations (2-24) – (2-27). The blue and black unbroken curves are both computed using the same equation, boundary , and initial conditions as in Figure 2 but with one important difference. In Figure 3, the blue and black unbroken curves were computed with $g = g_0 \neq 0$ and $N = N_0 = 2\Omega_0$ (or, equivalently, $\beta = 10$ for the anelastic flow) with $\bar{P}$ and $\bar{\rho}$ given by eqs. (2-18) and (2-19), and $\bar{T} = T_0$ for the anelastic flow. For the Boussinesq flow, $\bar{\rho}$ is given by eq. (2-28). To emphasize the weak rms velocity needed to trigger instability in vertically stratified flow, the initial noise in Figure 3 has an rms velocity that is $1/8^{th}$ the value in used Figure 1. (See the next section for how small we can make the initial kinetic fluctuation and still make the flow go unstable.)
Fig. 1.— This is Figure 1 from BHS96, which shows the temporal evolution of the fluctuation kinetic energy per unit mass, defined in § 3.1, where time is in units of “years” \((2\pi/\Omega_0)\) and the kinetic energy per unit mass is in units of \((L_x\Omega_0)^2\). The time evolutions are for different values of \(q\) as defined by eq. (2-1). These are fully-compressible simulations with \(g = 0, N = 0, \gamma = 5/3\). The size of the computational domain is \(L_x = L_y = L_z\). The numerical code was Zeus with a spatial grid of \(64^3\) points. The initial rms velocity amplitude was \(\sim 0.1(\Omega_0 L_x)\). The initial unperturbed equilibrium flow had uniform pressure, density, and temperature. The curve labeled with “shr” in Figure 1 corresponds to the case with \(q = 3/2\) and with the Coriolis and tidal acceleration terms dropped from eq. (2-2). The growth and decay of the fluctuation kinetic energy as a function of \(q\) supports Rayleigh’s theorem that the flow is stable for \(q < 2\) for flows with constant density.
Fig. 2.— Time evolution of the fluctuation kinetic energy per unit mass with different $q$ as in Fig. 1 and with the same parameter values and units as in Fig. 1, but using the anelastic equations, which are identical to the Boussinesq equations when $g \equiv 0$, as is the case here. Unlike the flows in Fig. 1, the flows here were initialized with a smaller amplitude noise (see text for details). The initial 3D spectrum of the energy fluctuations used in this figure was homogeneous and isotropic, but unlike the initialization in Figure 1, the energy spectrum was Kolmogorov, rather than Gaussian (see § 4 for details). The Boussinesq/anelastic simulations used $g = 0, N = 0$. The spatial resolution of the spectral calculations used $128^3$ Fourier modes. The stability of the anelastic and computed flows as a function of $q$ are the same as shown in Fig. 1.
Fig. 3.— figure caption 3 on next page
Figure 3 caption Time evolution of the fluctuation kinetic energy per unit mass (which in this case is the non-Keplerian kinetic energy) for anelastic and Boussinesq flows for $q \equiv 3/2$. Blue unbroken line - anelastic calculation with vertical density stratification. Black unbroken line – Boussinesq with vertical density stratification. Black broken dashed line - Boussinesq/anelastic flow with $g = 0$ and $N = 0$, which is the same calculation as shown in Fig 2 labeled with “1.5”, but integrated for a much longer time. The figure shows that with vertical density stratification, flows with $q = 3/2$ are unstable. In the two simulations with stratified density, we set $N_0/\Omega_0 = 2$ or $\beta = 10$. The spatial resolution is $256^3$ Fourier modes. To guide the eye, and to remove fast oscillations in the energy that are due to the shearing box boundary conditions, the energies in this figure and in Figure 2 are moving-averages-in-time, with a window size of 10 yrs. The anelastic simulation has an initial rms Mach number of $Ma = 4.3 \times 10^{-3}$ based on the isothermal sound speed. All three flows shown here were initially perturbed with Kolmogorov noise with an rms velocity that is $1/8^{th}$ of the value of the initial noise in Figure 1. The time evolution of kinetic energy can be divided into 3 parts: From $t = 0$ to 50 yrs the flow adjusts from the initial condition with most of the initial vorticity destroyed by hyperviscosity. The hyperviscous dissipation causes the initial fast decrease in the non-Keplerian kinetic energy. From $t = 50$ to 250 yrs the non-Keplerian kinetic energy increases approximately exponentially. During this time, the critical layers are strongly excited (see § 5), turn into vortex layers, and roll-up into zombie vortices. From $t = 250$ yrs onward, the growth of the non-Keplerian energy slows until it reaches a statistically steady state.
The two curves in Figure 4 show the non-Keplerian energies per unit mass as functions of time using the same parameters values and initial conditions as the anelastic flow in Figure 3 (with the exception that the rms velocity of the initial noise is approximately 1.5 times larger). The dashed broken [solid unbroken] curve was computed with the anelastic [fully compressible] equations using the Athena code. Figure 4 also differs from Figures 2 and 3 in that the flows are integrated forward in time for only 200 years rather than 1200 years because the Athena code is much more computer resource intensive than the spectral codes. None the less, Figure 4 shows that the fully compressible flow with \( q = \frac{3}{2} \) is also unstable. In fact for the first 200 years, the fully compressible flow grows faster than the anelastic flow. This is due to the vertical boundary conditions used in the anelastic spectral code (see BM06), which are highly dissipative. At late times, i.e., at 1000 yrs the non-Keplerian energy of the flow anelastic flow computed in Figure 4 (but not shown at late times) is approximately the same as the late-time anelastic flow in Figure 3. The fact that the late-time anelastic flows in Figures 3 and 4 are similar despite the fact that they were initialized with different magnitude perturbation is one of many numerical indications that after a flow goes unstable to the zombie instability, it evolves to an attracting turbulent solution whose gross properties are independent of the details of the initial conditions.

Fig. 4.— As in Figure 3 with \( \beta = 10 \), but with both of the plotted flows having an rms velocity of the initial noise approximately 1.5 that of the blue curve in Figure 3. The dashed broken curve is computed with the anelastic equations, and the unbroken curve computed with the fully compressible equations using the Athena code. The reason why the anelastic kinetic energy is relatively small is due to the anelastic code’s vertical boundary damping (see BM06).
3.2. The vertical vorticity of the zombie instability

The growth of the non-Keplerian kinetic energies in Figures 3 and 4 is evidence of instability, but spatial plots of the relative vorticity of the flow, defined as $\omega \equiv \nabla \times (\mathbf{v} - \bar{\mathbf{v}}) = \nabla \times \mathbf{v} + q \Omega_0 \hat{z}$ are more useful in illustrating the zombie instability’s strength and ubiquity throughout the computational domain. In particular, the point-wise Rossby number, defined in terms of the relative vertical vorticity as $\text{Ro}(x, y, z, t) \equiv [\hat{z} \cdot \omega(x, y, z, t)]/(2\Omega_0)$ will be shown in § 4 and § 5 to be much more indicative of the zombie instability than, say, the Mach number, because the threshold for instability depends on Ro, and because the late-time zombie turbulence has a characteristic $|\text{Ro}|$ of 0.25, regardless of the values of the parameters of the flow. The onset of the zombie instability is best shown in plots of $\text{Ro}(x, y, z, t)$ as functions of time and space when the initial perturbation of the steady equilibrium flow $\bar{\mathbf{v}}$ is a single, isolated vortex. However, that type of perturbation is not relevant to protoplanetary disks, so we defer a study of a single-vortex initial condition to § 5 where we examine the onset of the instability. The focus of this section is to show how $\text{Ro}(x, y, z, t)$ develops in a Keplerian flow when the initial perturbation is Kolmogorov noise.

Figure 5 shows $\text{Ro}(x, y, z, t)$ for an anelastic flow in the $x-z$ plane at four different times and at an arbitrary stream-wise location in $y$. (Because the equations, boundary conditions and initial conditions are invariant under translation in $y$, the flow in all $x-z$ planes is statistically the same for all time.) Figure 6 shows $\text{Ro}(x, y, z, t)$ in the $x-y$ plane for the same flow at $z = 0$, which is midway between the upper and lower boundaries. The parameter values, initial conditions and boundary conditions of the flow in Figures 5 and 6 are identical to the stratified anelastic flow shown in Figure 3 with the exception that $N_0/\Omega_0 = 1$, rather than 2, or, equivalently, with $\beta = 2.5$, rather than 10. Note that the initial perturbing velocity has a Kolmogorov energy spectrum in which the velocity phases are random, so there are no coherent features of any kind in the initial flow. As discussed in § 4.1, the relative vorticity field $\omega(x, y, z, t)$ and $\text{Ro}(x, y, z, t)$ of Kolmogorov noise is dominated by the Fourier modes with the smallest lengthscales, so Figures 5a and 6b are dominated by the smallest scales, and in fact, the sizes of the patches in the panels are equal to the spatial resolution of the calculation, which is $L_x/256$ in each direction. Figures 5a and 6a look the same because the initial perturbation of noise is homogeneous and isotropic. Much of the initial vorticity in Figures 5a and 6a is quickly destroyed by the numerical code’s hyperviscosity, which is largest at the smallest lengthscales in the numerical calculations. By time $t = 2.5 \text{ yrs}$ (Figures 5b and 6b) most of the surviving initial vorticity has inverse-cascaded to larger lengthscales and the initial $\bar{\mathbf{v}}$ has smeared out the vorticity in the stream-wise $y$ direction to form elongated vortical structures. At time $t = 2.5 \text{ yrs}$, the asymmetry between cyclonic and anticyclonic vorticity $\omega_z$ – one of the signatures of the zombie instability – is apparent. Relative vorticity is defined as cyclonic [anticyclonic] when $\omega_z$ and $\Omega_0$ have the same [opposite] signs, or equivalently, when $\text{Ro}(x, y, z, t) > 0$ [$\text{Ro}(x, y, z, t) < 0$]. In all of the figures in this paper, cyclonic [anticyclonic] vorticity is shown as red [blue] in color-plots and as white [black] in grey-scale plots. As we shall elaborate in § 5, “stripes” or layers of cyclonic vorticity aligned in the stream-wise direction are linearly stable. In contrast, stripes or layers of anticyclonic vorticity aligned in the stream-wise direction are linearly
unstable; the anticyclonic vortex layers roll-up into stable anticyclonic vortices. The instability of the anticyclonic vortex layers is primarily cause of the cyclone/anticyclone asymmetry in Figures 5 and 6.

By 50.9 yrs, the zombie instability is well underway. As shown in § 5, one of the first signatures of the instability is the excitation of critical layers (defined and reviewed in § 5) and their accompanying dipolar vortex layers, which are easily identified because they occur as a pair of “stripes” in the x-y plane with a layer of cyclonic relative vorticity immediately adjacent to a layer of anticyclonic relative vorticity. A dipolar vortex layer aligned in the stream-wise direction can be seen in Figure 6c at $x = 0.44/\Delta$. [PHIL: YOU MUST DEFINE \Delta BEFORE THIS.] It is unusual, especially with initial perturbation consisting of noise, to find dipolar vortex layers at late times due to the fast instability of the anticyclonic component of the dipolar layer. At the time of Figure 6c, the critical layer at $x = 0.44/L_x$ has only just recently been excited and formed a dipolar vortex layer. Shortly after the time shown in Figure 6c, the anticyclonic part of the dipolar vortex layer became unstable and rolled up into anticyclonic vortices. (See § 5.) At late times, $t = 1370$ yrs in Figures 5d and 6d, the flow has reached a statistically steady state of zombie turbulence. Here the flow has formed a pattern that appears to have cross-stream or x wavenumber of 7. This pattern is especially clear in Figure 6d. The relative vorticity, although very turbulent, has developed some spatial coherence. The cyclonic vorticity has formed approximately 2-dimensional layers that are approximately aligned in the y-z planes. Between these planes are approximately ellipsoidally-shaped turbulent anticyclones. The aspect ratio $\chi$ (defined as the stream-wise diameter of an anticyclone in the y direction divided by its cross-stream diameter in x) is approximately the same as the laminar vortices studied by Moore & Saffman (1971), where $\chi$ is given by

$$- \frac{\omega_z}{q \Omega_0} = -\frac{2Ro}{q} = \frac{\chi + 1}{\chi - 1} \quad (3-1)$$

The Moore-Saffman relation was derived for a steady two-dimensional vortex with uniform relative vorticity embedded in flow with uniform shear. However, the turbulent zombie vortices have aspect ratios similar to that in eq. (3-1) because the relation is the quantification of the fact that a large relative vorticity tends to make a vortex “round” and a large background shear tends to elongate a vortex in its stream-wise direction. At late times the characteristic magnitude of Ro of the anticyclones in zombie turbulence is always $\sim -0.25$, so regardless of the parameters of the flow, eq. (3-1) shows that in a Keplerian disk, the aspect ratios $\chi$ of zombie vortices is between 4 and 5. The general vortex patterns in Figures 5d and 6d with a near periodicity in the x direction is a signature of zombie turbulence and the periodicity’s wavenumber is a predictable property of the flow (see § 5). Figures 5d and 5d are highly turbulent, but the magnitudes of the Rossby numbers of the anticyclonic vortices and cyclonic layers persist indefinitely. We have carried out several sets of initial-value calculations in which zombie turbulence is created in fully compressible flows (Figure 7), anelastic, and Boussinesq for a wide variety of parameters and the Ro($x, y, z, t$) always look like Figures 5d and 6d.
Fig. 5.— figure 5 caption see next page
caption for figure 5 Time evolution of point-wise Rossby number \( Ro(x, y, z, t) \) in the \( x-z \) plane. The figure has been cropped in the \( z \) direction so it does not show the damping regions at the vertical boundaries. The unperturbed anelastic flow has \( q = 3/2 \) and \( N_0/\Omega_0 = 1 \), or \( \beta = 2.5 \). The initial noise has a Kolmogorov \( (k^{-5/3}) \) spectrum, and has the same rms velocity as the flows in Figure 3. The color-map ranges from \(-0.25\) to \(0.25\), with blue [red] for anticyclones [cyclones] with \( Ro < 0 [Ro > 0] \). Green corresponds to \( Ro = 0 \). (In grey-scale plots, the blackest pixels have \( Ro = -0.25 \) and whitest have \( Ro = 0.25 \).) a) \( t = 0 \) yr. Relative vorticity dominated by the smallest lengthscale, so the image is pixelated at the resolution length. The color and grey scales are over-saturated in this panel with the minimum \( Ro \) of this initial condition being 2.3 and maximum being 2.4. b) \( t = 2.5 \) yrs. Decay of much of the initial relative vorticity and stretching in the stream-wise direction by the Keplerian shear. c) \( t = 50.9 \) yrs. Inverse cascading to large scales and asymmetry between cyclonic and anticyclonic relative vorticity. d) \( t = 1370.0 \) yrs. Zombie turbulence with zombie vortices filling up the whole domain with \( Ro \simeq -0.3 \). The near spatial periodicity, here with wavenumber between 6 and 7, of the turbulent flow in the \( x \) direction is one of the signatures that makes zombie turbulence easy to identify.
Fig. 6.— Same as Figure 5 but in the $x$-$y$ plane at $z = 0$. Panel a looks like Figure 5a because the initial noise is isotropic and homogeneous.
Fig. 7.— you must define the x and y and z units for this and all subsequent figures in term of \( \Delta \). Point-wise Rossby number \( Ro(x, y, z, t) \) at \( t = 190 \) yrs using the same colormap range as in Figures 5 and 6. a) As in Figure 5d; b) as in Figure 6d. The flow here is the same fully compressible flow computed with the Athena code shown in Figure 4 with \( \beta = 10 \) or \( N/\Omega_0 = 2 \). No damping at the vertical boundaries is used in this simulation so the flow is uncropped and shows the full computational domain. Although the zombie instability is well underway, the turbulence is not fully developed. The pattern is still evolving and the non-Keplerian kinetic energy is still growing. See the Appendix for a precise recipe to produce zombie vortices and zombie turbulence using the Athena code.

3.3. Space-filling properties of the zombie turbulence

Figures 8–10 show how the rms Mach numbers (based on the isothermal sound speed) \( Ma_{rms}(t) \) and rms Rossby numbers \( Ro_{rms}(t) \) evolve in time for three anelastic flows. For all three flows, the values of \( Ma_{rms}(t) \) and rms Rossby numbers \( Ro_{rms}(t) \) initially plummet due to the small-scale hyperviscosity, but then grow after the zombie instability sets in. All of our calculations with zombie turbulence have late-time values of \( Ro_{rms}(t) \) between 0.2 and 0.3. Figures 9 and 10 have the same flow parameter values of \( \beta \) (or \( N_0/\Omega_0 \)), \( \gamma \) and \( q \), but the initial Kolmogorov noise has different amplitudes. The fact that the statistical properties of the late-time flows in Figures 9 and 10 shows that the flows evolve to a common attracting solution that is independent of the initial conditions.

To understand how the Mach and Rossby numbers evolve at late times, it is first necessary to
show that their values are not independent. The rms Rossby number is approximately

\[ Ro_{\text{rms}} \simeq \frac{V_{\text{eddy}}(L_\omega)}{(L_\omega \Omega_0)}, \]

where \( L_\omega \) is the characteristic lengthscale of the flow where the vorticity has its maximum value and \( V_{\text{eddy}}(L) \) is the characteristic velocity of a turbulent eddy of diameter \( L \) (rigorously defined in § 4.1). The rms Mach number is approximately

\[ Ma_{\text{rms}} \simeq \frac{V_{\text{eddy}}(L_v)}{C_s} = \frac{V_{\text{eddy}}(L_v)}{(\beta^{1/2} H \Omega_0) L_x} = \frac{R o_{\text{rms}} \beta^{-1/2}}{V_{\text{eddy}}(L_\omega)}, \]

where \( L_v \) is the characteristic lengthscale of the flow where the velocity has its maximum value and where we used eq. (2-21). For the flow in Figure 8, \((L_x/L_\omega) \simeq 7\) (i.e., there is a near periodicity in the \( x \) direction with wavenumber 7), and \( L_v \simeq L_\omega \), so \( V_{\text{eddy}}(L_v) \simeq V_{\text{eddy}}(L_\omega) \). Using \( \beta = 2.5 \) and \( L_x/L_\omega = 7 \), eq. (3-3) gives \( Ma_{\text{rms}} \simeq 0.09 R o_{\text{rms}} \), which is qualitatively consistent with Figures 8a and 8b at late times. For the flows in Figures 9 and 10, with \( \beta = 10 \) and \((L_x/L_\omega) \simeq 4\), eq. (3-3) gives \( Ma_{\text{rms}} \simeq 0.08 R o_{\text{rms}} \), which is qualitatively consistent with the figures at late times. However, Figures 9 and 10 show that the values of \( R o_{\text{rms}}(t) \) have plateaued at late times, but that the values of \( Ma_{\text{rms}}(t) \) have not. This suggests that the flow has not yet reached a statistically steady state. However, eq. (3-3) implies that the only way in which \( Ma_{\text{rms}}(t) \) can grow while keeping \( R o_{\text{rms}}(t) \) fixed is if \( V_{\text{eddy}}(L_v)/V_{\text{eddy}}(L_\omega) \) is still growing, and the latter is indicative that the inverse cascade of energy is still continuing at late times in the flows in Figures 9 and 10.

In protoplanetary disks with Keplerian vertical gravity, eqs. (2-17) and (2-21) show that \( \beta \equiv 1 \). With \( \beta = 1 \), eq.(3-3) shows that at late times \( Ma_{\text{rms}} = R o_{\text{rms}} \frac{V_{\text{eddy}}(L_v)}{V_{\text{eddy}}(L_\omega)} \frac{L_\omega}{H} \). If, as argued by others [WE NEED REFERENCES HERE], the turbulent flow in a disk inverse cascades until it reaches lengthscale \( H \), and if therefore at late times, the lengthscale \( L_\omega \) of the dominant vortices and eddy velocities are approximately equal to \( H \), then

\[ Ma_{\text{rms}} \simeq R o_{\text{rms}}. \]

We shall show in the next section and Discussion that eq. (3-4) will have profound effects on the late-time turbulence and the rate of angular momentum transport in a disk.

Figures 5d, 6d, and 7 show that at late times the zombie turbulence fills the computational domain. Figure 11 quantifies this property by defining a spatial filling factor for the turbulent vorticity: \( f_{Ro}(\delta, t) \) is the volume fraction of the computational domain that has \( |Ro(x,y,z,t)| \geq \delta \). We further quantify the space-filling properties of the turbulence by defining a spatial filling factor for the turbulent velocity: \( f_{Ma}(\delta, t) \) is the volume fraction of the computational domain that has \( Ma(x,y,z,t) \geq \delta \). Figure 11 shows that for the anelastic flow in Figures 3 and 9 that approximately 10% of the flow is filled with vortices with Rossby numbers with magnitudes greater than 0.3; 30% with magnitudes greater than 0.2; and almost 60% with magnitudes greater than 0.1. Figure 12
demonstrates the space-filling property of the turbulent velocity in terms of $f_{Ma}(\delta, t)$. The filling factors in Figures 11 and 12 are representative of all of our anelastic calculations.

![Graphs](image)

Fig. 8.— Time evolution of the rms Mach number (based on the isothermal sound speed) $Ma_{rms}(t)$ (panel a) and $Ro_{rms}(t)$ (panel b) for the anelastic flow in Figures 5 and 6. The values of rms Mach and Rossby numbers both rapidly plummet due to hyperviscosity but grow after the zombie instability sets in and eventually plateau. All of our calculations with zombie turbulence have late-time values of $Ro_{rms}(t)$ between 0.2 and 0.3. At late times, the value of $Ma_{rms}(t)$ is slaved to the value of $Ro_{rms}(t)$ – see § 3.3 for details.
Fig. 9.— Time evolution of $Ma_{rms}$ and $Ro_{rms}$ as plotted in Figure 8, but for the anelastic flow in Figure 3, so this flow has the same initial Kolmogorov noise as the flow in Figure 8, but has $\beta = 10$ rather than 2.5 (or $N_0/\Omega_0 = 2$, rather than unity). The late-time $Ro_{rms}$ is slightly smaller than that in Figure 8. The flow at $t = 1000$ yrs is not yet in equilibrium as indicated by the fact that $Ma_{rms}(t)$ is still increasing at that time. However, the fact that $Ro_{rms}(t)$ has reached a plateau at that time shows that the inverse cascade of energy is still active - see § 3.3 for details.
Fig. 10.— Mach and Rossby numbers as function of time as in Figure 9. As in Figure 9, $\beta = 10$ (and $N_0/\Omega_0 = 2$), but here the rms Mach number of the initial Kolmogorov noise is two-thirds the value in Figure 9. After $t \simeq 500$ yrs, many of the statistical properties of the flows in Figures 9 and 10 are nearly the same, which shows that the flows are being drawn to an attractor that is independent of the details of the initial conditions.

Fig. 11.— Time evolution of the spatial filling factor $f_{Ro}\left(\delta, t\right)$ for the flow in Figures 3 and 9. Dotted line for $\delta = 0.1$; dashed line for $\delta = 0.2$; unbroken line for $\delta = 0.3$. These filling factors are typical of all of our anelastic calculations.
4. Threshold for finite amplitude instability

4.1. Initial Perturbations

Our first study of the zombie instability (MPJH13) focussed on initial perturbations of steady equilibrium flows perturbed by a single vortex, and we determined that the instability was not linear, but required a finite-amplitude trigger. We found that the amplitude of the initial perturbing vortex was set by its vorticity or Rossby number, rather than its velocity. For Boussinesq flows with \( q = 3/2 \), the necessary initial \( |Ro| \) was between 0.2 and 0.3. Here, to be relevant to protoplanetary disks, we examine the amplitude of the three-dimensional noise that is needed to trigger the zombie instability in flows with \( q = 3/2 \). We designed three sets of numerical experiments to determine whether the noise’s initial Rossby number, Mach number, energy, or some other property determines the trigger for instability and to quantify the trigger’s amplitude.

4.2. Review of Turbulent spectra, eddy velocities, eddy vorticities, and Fourier modes

To better understand how an instability is triggered from initial noise, we briefly review the nomenclature used in describing homogeneous, isotropic turbulence (which is how define initial “noise” in this paper). We consider the differential kinetic energy spectrum per unit mass \( E(k) \)
as a function of spatial wave number $k \equiv |k| = \sqrt{k_x^2 + k_y^2 + k_z^2}$. Often, the spectrum has a power-law dependence on $k$, so $E(k) = E_0 k^{-\alpha}$ with $E_0$ constant, and we define $\alpha$ as the spectral index. Kolmogorov turbulence has $\alpha = 5/3$. To simplify our analysis, we consider fluids with constant (unity) density (which is reasonable approximation for lengthscales in a disk that are less than the disk’s vertical pressure scale height). The turbulence has kinetic energy per unit mass $E \equiv \int_0^\infty E(k) \, dk$, an rms velocity equal to $\sqrt{2E} \equiv C_s \, Ma_{rms}$, and $Ro_{rms} \equiv [\int_0^\infty E(k) k^2 \, dk]^{1/2}$.

It is useful to think of the turbulent velocity as a sequence of eddies with each eddy half the size of the preceding eddy in the sequence (Tennekes & Lumley 1972). An eddy with wavenumber $k$ and lengthscale $l \equiv 2\pi/k$ has kinetic energy $\int_k^{2k} E(k') \, dk'$ and and an rms eddy velocity of $V_{eddy}(l) = [2 \int_k^{2k} E(k') \, dk']^{1/2}$. For an $E(k)$ with spectral index $\alpha$:

$$V_{eddy}(l) = V_{eddy}(L) \left( \frac{l}{L} \right)^{(\alpha-1)/2}, \quad (4-1)$$

where $L$ is the largest lengthscale of the flow.

We define the Rossby number $\tilde{Ro}(k)$ and the Mach number $\tilde{Ma}(k)$ of the eddy with wavenumber $k = 2\pi/l$ as:

$$2\Omega_0 l \tilde{Ro}(k) \equiv C_s \tilde{Ma}(k) \equiv V_{eddy}(l), \quad (4-2)$$

so

$$\tilde{Ma}(k) \propto k^{(1-\alpha)/2}$$

$$\tilde{Ro}(k) \propto k^{(3-\alpha)/2}. \quad (4-3)$$

For the Kolmogorov spectrum with $\alpha = 5/3$,

$$V_{eddy}(l) = V_{eddy}(L) \left( \frac{l}{L} \right)^{1/3} \quad (4-4)$$

$$\tilde{Ma}(k) \propto k^{-1/3} \quad (4-5)$$

$$\tilde{Ro}(k) \propto k^{2/3} \quad (4-6)$$

For turbulence with spectral index $1 < \alpha < 3$, equations (4-3) show that an eddy’s kinetic energy and velocity decrease with decreasing lengthscale $l$, while its vorticity increases. Equivalently, with increasing $k$, $\tilde{Ro}(k)$ increases and $\tilde{Ma}(k)$ decreases. The implication of this is that most of the energy is at the large lengthscales, while most of the enstrophy is at the smallest. The largest lengthscale eddies contribute the most to the rms Mach number, while the smallest eddies contribute most to the rms Rossby number. For a turbulent spectrum with a large inertial range (i.e., where the ratio of its largest wavenumber to its smallest is large), the ratio of the rms velocity of the largest lengthscale eddy to the rms velocity of the total flow is

$$[1 - (1/2)(\alpha-1)]^{1/2}, \quad (4-7)$$
so for Kolmogorov turbulence, the rms Mach number of the largest eddy is 61% that of the rms Mach number of the total flow. If the smallest lengthscale of the turbulence is determined by viscous dissipation, then that length is \( l_\nu \equiv \nu / V_{\text{eddy}}(l_\nu) \), where \( \nu \) is the kinematic viscosity, and eq. (4-1) shows that
\[
l_\nu / L = Re^{-2/(1+a)},
\]
where \( Re \equiv [L V_{\text{eddy}}(L)]/\nu \) is the Reynolds number of the flow. For Kolmogorov turbulence, \( l_\nu \) is called the Kolmogorov length and is equal to \( L Re^{-3/4} \).

An eddy is not equivalent to a Fourier mode \( \tilde{v}_k \) of the velocity, but rather it is the sum or integral of a band of Fourier modes with different wavenumbers centered around wavenumber \( |k| \). Unfortunately, there have been some confusion in the literature that incorrectly states that \( |\tilde{v}_k| \) scales with \( k \) the same way as \( V_{\text{eddy}}(2\pi/k) \) scales. If a turbulent velocity with spectral index \( a \) is written as a discrete sum of Fourier modes,
\[
v(x) = \sum_{k_x} \sum_{k_y} \sum_{k_z} \tilde{v}_k e^{i k \cdot x},
\]
then
\[
|\tilde{v}_k| \propto k^{-(a+2)/2}.
\]
This scaling is due to the fact that there are \( 7 (4\pi/3)|k|^3 \) Fourier modes in an eddy made up of the Fourier modes between \( k \) and \( 2k \), so that the energy of the eddy with wavenumber \( k \) is \( V_{\text{eddy}}(2\pi/k)^2/2 = |\tilde{v}_k|^2 7 (2\pi/3)k^3 \). The velocity of the turbulent initial noise used in the calculations in this paper were created using eqs. (4-9) and (4-10) where the \( \tilde{v}_k \) have random phases.

Figures 13 and 14 illustrate how the spectral index \( a \) affects the spatial pattern and lengthscale of the vertical velocity and vertical vorticity of the initial noise in our calculations. Figure 13 is computed with Kolmogorov noise but shows the general behavior of turbulence with \( 1 < a < 3 \): the velocity is gathered into large scale features, while the vorticity is dominated by the smallest scales (as also shown in Figures 5a and 6a). In Figure 14, \( a = 5 \), so in accord with eqs. (4-3), \( \tilde{Ro}(k) \) and \( \tilde{Ma}(k) \) both decrease with increasing \( k \), and the velocity and the vorticity are both gathered into large scale features.
4.3. Threshold for the zombie instability

We have found that initial perturbations of noise can trigger the zombie instability in flows with a wide range of flow parameter values. To determine the necessary threshold amplitude of the initial noise to trigger instability in an anelastic flow with $q = 3/2$, with uniform gravity $g_0$, and $\beta = 10$ or $N_0/\Omega_0 = 2$, we carried out three sets of numerical experiments in which we varied the properties of the initial noise but not the equilibrium flow parameters. The thick dashed line in Figure 15a shows the initial spectrum $\tilde{Ro}(k)$ of the Kolmogorov noise that we used in a numerical experiment that did not produce zombie turbulence, but in which the initial non-Keplerian kinetic energy decayed. Because the initial noise has a Kolmogorov energy spectrum, $\tilde{Ro}(k)$ increases with wavenumber as $k^{2/3}$ (i.e., with a slope in this log-log plot of 2/3); while $\tilde{Ma}(k)$ (not plotted) decreases as $k^{-1/3}$. The vertical dotted line in Figure 15a shows the resolution wavenumber $k_{\text{res}}$ of the calculation, i.e., wavenumbers with $k > k_{\text{res}}$ are excluded from the numerical calculation. In
Figure 15a $k_{res} = 128(2\pi/L_x)$. To find a $\tilde{Ro}(k)$ of initial Kolmogorov noise that did not stabilize the flow, we began with the Kolmogorov noise that destabilized the anelastic flow shown in Figure 3, and then carried out a sequence of runs in which we steadily decreased the amplitude of the noise, while keeping all of the flow parameters and computational parameters fixed, until we found a sufficiently low amplitude that the flow was not destabilized. Then, we carried out a binary-chop search on the amplitude of the initial noise to find the minimum threshold amplitude of the initial noise that triggered the zombie instability. Figure 15a illustrates the search. The flow with the thick unbroken line in Figure 15a is the $\tilde{Ro}(k)$ of an initial Kolmogorov noise that destabilized the flow. Therefore, the $\tilde{Ro}(k)$ of the noise with the minimum value for instability is bracketed between the thick dashed broken and unbroken lines in Figure 15a. By carrying out the other two sets of numerical experiments (discussed below), we concluded that the criterion for instability is set by the maximum value of $\tilde{Ro}(k)$ of the initial noise, and for $\beta = 10$ and $q = 3/2$, we determined that the minimum value $\tilde{Ro}^*$ of the initial $\tilde{Ro}(k)$ needed to destabilize the flow is $\sim 0.19$. The horizontal broken lines in all three panels of Figure 15 show $\tilde{Ro}^*$. Because $\tilde{Ro}(k)$ increases with $k$ in all of the curves shown in Figure 15, the maximum value of each $\tilde{Ro}(k)$ is $\tilde{Ro}(k_{res})$. Therefore, $\tilde{Ro}^*$ is bracketed by the two values of $\tilde{Ro}(k_{res})$ shown in Figure 5a.

To convince ourselves that the maximum value of the initial $\tilde{Ro}(k)$ is what determines whether the noise destabilizes the flow and that $\tilde{Ro}^* \simeq 0.19$, we carried out a second series of numerical experiments for an anelastic flow with $\beta = 10$, $q = 3/2$. In these experiments, the kinetic energy of the initial noise was held fixed along with the values of the parameters of the equilibrium flow and the computational parameters. We chose the value of the kinetic energy of the initial noise to be the same as in the dashed curve in Figure 15a, which failed to destabilize the flow. In this second set of experiments, the only quantity that was varied was the spectral index $a$ of the initial noise. In Figure 15b, the thick dashed line is identical to the thick dashed line in Figure 15a, which has a Kolmogorov spectral index of $a = 5/3$. The unbroken line is $\tilde{Ro}(k)$ for the initial noise with $a = 1$ and has a slope of $(3 - a)/2 = 1 > 2/3$. This initial noise destabilized the flow. By carrying out a binary-chop on the minimum value $a^*$ of needed to destabilize the flow, we determined the threshold value of the spectral index in this set of experiments that was needed to destabilize the flow. However, the value of $a^*$ is not important; its value is an artifact of the numerical resolution and initial energy of the noise – see below). What is important is that when $a$ is equal to $a^*$, we find that $\tilde{Ro}(k_{res})$ is equal to $\tilde{Ro}^*$. In Figure 15b, $\tilde{Ro}^*$ is bracketed by the values of $\tilde{Ro}(k_{res})$ of the noise that destabilizes the flow (unbroken line) and the noise that fails to destabilize it (dashed line). These experiments support (but do not prove) our hypothesis that it is the maximum value of the initial $\tilde{Ro}(k)$ that determines whether the noise triggers the zombie instability and that $\tilde{Ro}^* \simeq 0.19$ for flows with $\beta = 10$, $q = 3/2$. However, this set of experiments does prove that neither the amplitude of the energy nor the rms Mach number of the initial noise determines whether the noise triggers the instability because the values of the initial noise’s kinetic energy and rms Mach number are the same in all of the runs in this second set of experiments.

In the third set of numerical experiments, the flows all have $\beta = 10$, $q = 3/2$, and the initial
noise in all of the runs is Kolmogorov with identical energy spectra, \( E(k) = E_0 k^{-5/3} \), where the value of \( E_0 \), as well as \( a \), are the same in all of the experiments. Because \( E_0 \) is held fixed, the largest lengthscale eddies in the initial noise have the same Mach numbers and same Rossby numbers in all of our experiments. Equivalently, for \( k \to 0 \), the \( \tilde{Ro}(k) \) and \( \tilde{Ma}(k) \) of the initial noise is the same in all experiments. In this third set of experiments, we vary the values of \( k_{res} \). In Figure 15c the initial Kolmogorov noise represented by the thick dashed line is the same as represented by the dashed lines in Figures 15a and 15b, has \( k_{res} = 256 (2\pi/L_x) \), and failed to destabilize the flow. The initial Kolmogorov noise corresponding to the unbroken line in Figure 15c has \( k_{res} = 256 (2\pi/L_x) \), and it destabilizes the flow. By carrying out a binary chop search on \( k_{res} \), we found that the minimum value \( k^*_{res} \) of the \( k_{res} \) that destabilizes the flow in this set of experiments. However, the value of \( k^*_{res} \) is unimportant; its value is an artifact of the spectral index and energy of the initial noise. What is important is that when \( k_{res} = k^*_{res} \), we find that \( \tilde{Ro}(k_{res}) = \tilde{Ro}^* \). In Figure 15c, \( \tilde{Ro}^* \) is bracketed by the values of \( \tilde{Ro}(k_{res}) \) for the noise that destabilizes the flow (unbroken line) and for the noise that fails to destabilize it (dashed line). These experiments add further support (but still do not rigorously prove) our hypothesis that it is the maximum value of \( \tilde{Ro}(k) \) that determines whether the noise triggers the \textit{zombie} instability and that \( \tilde{Ro}^* \simeq 0.19 \) for flows with \( \beta - 10, q = 3/2 \). Howevrer, the experiments prove that neither the Mach number nor the Rossby number of largest eddies in the initial noise determines whether the noise triggers the instability because the values of the Mach and Rossby numbers of the initial noise’s largest lengthscale eddies are the same in all of the runs. Although the values of the kinetic energy of the initial noise are not held fixed in this series of numerical experiments, for all practical purposes they are nearly the same: the difference of the runs. Although the values of the kinetic energy of the initial noise are not held fixed in this series of numerical experiments, for all practical purposes they are nearly the same: the difference in the initial energies of experiments with resolutions of \( k_{res} \) and \( k'_{res} \) is \( \int_{k_{res}}^{k'_{res}} E_0 k^{-5/3} dk \), which is negligible compared to the total energy of the initial noise, \( \int_{2\pi/L}^{k_{res}} E_0 k^{-5/3} dk \).

This third set of experiments has important implications for astrophysical flows. Generally, linear instabilities are viewed as more “reliable” in destabilizing a flow than finite-amplitude instabilities because the threshold for the latter might be too large. For the \textit{zombie} instability this is not a problem. Three properties of protoplanetary disks and turbulence conspire to make the energies and Mach numbers of the needed noise to trigger the \textit{zombie} instability in a protoplanetary disk extraordinarily small: (1) the trigger for \textit{zombie} instability depends on the maximum value of the Rossby number of the eddies in the noise; (2) in turbulence with a spectral index with \( 1 < a < 3 \), the Mach numbers and kinetic energies of eddies decrease with increasing wavenumber \( k \), while their Rossby numbers increase with \( k \); and (3) the viscosity of the fluid is relatively small. If the requirement to trigger the \textit{zombie} instability is that for some \( k, \tilde{Ro}(k) \geq \tilde{Ro}^* \), and if \( a < 3 \), then this requirement becomes

\[
\tilde{Ro}(k_{\succ}) > \tilde{Ro}^*, \tag{4-11}
\]

where \( k_{\succ} \) is the largest wavenumber in the flow (which in a numerical calculation would be \( k_{res} \), and in a viscously damped flow would be \( 2\pi/l_0 \)). Using eq. (4-3), eq. (4-11) becomes

\[
\left[ \frac{L k_{\succ}}{2\pi} \right]^{3-a/2} \tilde{Ro} \left( \frac{L}{2\pi} \right) > \tilde{Ro}^*, \tag{4-12}
\]
and using eq. (4-2), eq. (4-12) becomes

\[
\left[ \frac{2\Omega_0 L}{C_s} \right] \left[ \frac{L k_>}{2\pi} \right]^{\frac{3-a}{2}} \tilde{Ma} \left( \frac{L}{2\pi} \right) > \tilde{Ro}^*.
\] (4-13)

Using eq. (4-7), eq. (4-13) becomes

\[
[1 - (1/2)(a-1)]^{1/2} \left[ \frac{2\Omega_0 L}{C_s} \right] \left[ \frac{L k_>}{2\pi} \right]^{\frac{3-a}{2}} Ma_{rms} > \tilde{Ro}^*.
\] (4-14)

For Kolmogorov turbulence and with \( k_> = 2\pi/l_\nu \), eq. (4-14) becomes

\[
Ma_{rms} > 0.8 \beta^{1/2} \left[ \frac{H}{L} \right] \tilde{Ro}^* Re^{-1/2},
\] (4-15)

where we used eq. (2-21) to eliminate \( C_s \). Many authors have argued (c.f., BM05) that due to the shear in a protoplanetary disk, it is difficult for coherent objects to have \( L \) much greater than \( H \) and that vortices will merge and inverse cascade their energies until \( L \approx H \). Therefore, in a protoplanetary disk with \( L \approx H \), eq. (4-15) shows that the condition for Kolmogorov noise to have a sufficiently large amplitude to trigger the zombie instability is

\[
Ma_{rms} > Re^{-1/2},
\] (4-16)

where we used the fact that in a fully compressible flow, \( \beta = 1 \). Typical Reynolds numbers \( Re \) of the noise in protoplanetary disks are \( \sim 10^{12} \), so disks are unstable to the zombie instability if \( Ma_{rms} > 10^{-6} \). Thus, although the zombie instability is formally a nonlinear instability and requires a finite-sized perturbation to trigger it, the requirement on the amplitude of the noise is so small that for practical purposes it can be considered to act like a linear instability. Another way of showing how small the amplitude is that is needed to trigger the instability is to re-write eq. (4-16) as

\[
Ma_{rms} > \frac{\nu}{L C_s} \approx \frac{\Lambda}{L},
\] (4-17)

where we have used the definitions of the Mach and Reynolds numbers, and the fact that for most ideal gases \( \nu \approx \Lambda C_s \), where \( \Lambda \) is the mean free path of the gas.

Note that if the energy spectrum \( E(k) \) of the initial turbulence is so steep that \( \tilde{Ro}(k) \) decreases, rather than increases, with \( k \), then the noise must have a much larger Mach number, of order \( \tilde{Ro}^* \), to destabilize the flow. Thus if the spectral index \( a \) of the initial turbulent noise were greater than 3, then the noise must have a significant Mach number to destabilize the flow. In Figure 1, the initial noise had an energy spectrum in which the Fourier modes \( \tilde{v}_k \) scaled with \( k \) as \( |\tilde{v}_k|^2 \propto k^2 e^{-k^2} \), which is so steep that \( \tilde{Ro}(k) \) decreases with \( k \). Thus, even if vertical gravity and density stratification had been included in the calculations in Figure 1, the initial Mach number would have had to been 6 orders of magnitude large than \( 10^{-6} \) to trigger the zombie instability.
Fig. 15.— caption for figure 15 on next page
Caption for Figure 15: Plot of $\tilde{\text{Ro}}(k)$ for three sets of experiments. All of the unperturbed steady equilibrium flows are anelastic with $\beta = 10$, (i.e., $N_0/\Omega_0 = 2$). In all three panels, the thin broken horizontal dashed line is $\tilde{\text{Ro}}^* = 0.1913$, which corresponds to the lowest Rossby number of the initial noise that destabilizes the flow. The thin vertical dotted line is $k = 128 \left(2\pi/L_x\right)$, which is the spatial resolution of the calculations in panels (a) and (b). For all the calculations shown here, the largest value of $\tilde{\text{Ro}}(k)$ occurs when $k = k_{res}$, the largest wavenumber of the computed flow. In all three panels, the thick dashed broken straight line with slope $2/3$ is the $\tilde{\text{Ro}}(k)$ of a calculation that has initial Kolmogorov turbulence with an amplitude that is a little too weak to destabilize the flow, and has $\tilde{\text{Ro}}(k_{res}) < \tilde{\text{Ro}}^*$. a) The thick unbroken straight line is the $\tilde{\text{Ro}}(k)$ of initial Kolmogorov turbulence with an amplitude that is large enough to destabilize the flow. That line has $\tilde{\text{Ro}}(k_{res}) > \tilde{\text{Ro}}^*$. b) The thick unbroken straight line is the $\tilde{\text{Ro}}(k)$ of initial turbulence with a spectral index $a = 1$. This noise destabilizes the flow. The kinetic energies of the initial noise represented by the thick unbroken and broken lines are equal, but the thick unbroken curve has $\tilde{\text{Ro}}(k_{res}) > \tilde{\text{Ro}}^*$. c) The thick unbroken straight line is the $\tilde{\text{Ro}}(k)$ of initial Kolmogorov turbulence that destabilizes the flow. This initial turbulence has an energy spectrum $E(k) = E_0 k^{-5/3}$ that is identical to the spectrum of the thick broken line (which is plotted with slight vertical displacement so that it is not covered by the thick unbroken line). However, the flow with the initial noise represented by the thick unbroken line was computed with a $k_{res}$ that was greater than the resolution of the calculation represented by the thick dashed broken curve. The thick unbroken line has $\tilde{\text{Ro}}(k_{res}) > \tilde{\text{Ro}}^*$. 
5. Review of the Physics of the Zombie Instability

The zombie instability is due to the excitation of a neutrally-stable eigenmode followed by the linear instability of a vortex layer. It is most easily analyzed using the Boussinesq equations (2-24) – (2-29) with a spatially uniform gravity $g_0$ and Brunt-Väisälä frequency $N_0$ and with an unperturbed, steady, equilibrium density that is linear in $z$ as in eq. (2-28). The analysis is the same in anelastic and fully compressible flows and with flows in which $g$ and $N$ are not uniform, but these more complex cases require a WKB expansion (Mathews & Walker 1970).

5.1. Neutrally stable linear eigenmodes and critical layers

When the Boussinesq eqs. (2-24), (2-26), and (2-29) are linearized about a steady equilibrium velocity $\bar{v}(x)$, pressure $\bar{P}$, and density $\bar{\rho}(z)$ in eq. (2-28), with $g = g_0$ and $N = N_0$, the eigenmodes are proportional to $e^{i(k_y y + k_z z - st)}$. The eigen-equation for the eigenmodes of this linearization is a generalization of Rayleigh’s equation (Drazin & Reid 1981) and is a 2nd-order o.d.e in which the coefficient of the highest-derivative term is

$$[\bar{v}_y(x) - s/k_y] \{(\bar{v}_y(x) - s/k_y)^2 - (N_0/k_y)^2\}.$$

(5-1)

It should be noted that any velocity field $\bar{v}_y(x)$ for any function of $x$ with $\bar{v}_z = \bar{v}_x = 0$ is a steady equilibrium solution of the Boussinesq equations (2-24), (2-26), and (2-29) with $\bar{\rho}(z)$ given by eq. (2-28). The coefficient given by expression (5-1) is valid not only for $\bar{v}_y = -q \Omega_0 x$, but also for arbitrary $\bar{v}_y(x)$. When the initial steady equilibrium density $\bar{\rho}(z)$ is stably-stratified or constant and when $v = (3/2) \Omega_0 x \dot{y}$ (i.e., Keplerian flow), the flow is neutrally linearly stable (i.e., $s$ is real, and eigenmodes neither grow nor decay). Eigenmodes of an o.d.e. are singular at locations $x^*$ where the coefficient of the highest-derivative term of the eigen-equation is zero (Mathews & Walker 1970). At $x = x^*$ the eigenmodes have critical layers (Drazin & Reid 1981). Although these eigenmodes are singular, they are not just mathematical curiosities and are relevant to flows in protoplanetary disks and in the laboratory: in fluids with viscosity $\nu$, the “infinities” in the eigenmodes are replaced by terms proportional to $\nu^{-1/2}$. For neutrally stable fluids with uniform density $\rho_0$, eq. (5-1) shows that locations $x^*$ of the critical layers obey $\bar{v}_y(x^*) = s/k_y$. We refer to these as barotropic critical layers. Laboratory experiments and numerical computations show that neutrally stable eigenmodes with barotropic critical layers are difficult to excite, and, with the exception of contrived conditions in boundary layers, never form vortices. However, eq. (5-1) shows that there is another class of neutrally stable eigenmodes with critical layers that have $\bar{v}_y(x^*) - s/k_y \pm N_0/k_y = 0$, and we call them baroclinic critical layers. Boulanger et al. (2007) created weak baroclinic critical layers in a non-rotating, stratified flow in the laboratory. From MPJH13, we now know that those laboratory experiments could not produce strong critical layers and create zombie vortices due to the lack of rotation in the experiments. Rotation’s important role in creating vertical vorticity is seen by taking the curl of eq. (2-29):

$$\partial \omega_z / \partial t = -(v \cdot \nabla)\omega_z + (\mathbf{\omega} \cdot \nabla)v_z + (2 - q) \Omega_0 (\partial v_z / \partial z),$$

(5-2)
where for now and the remainder of this section we restrict ourselves to linearly neutrally stable flows $\bar{v}_y(x) = -q \Omega_0$. Vortex layers form at baroclinic critical layers because $v_z$, the $z$-component of the velocity eigenmode, and $(\partial v_z/\partial z)$ are large (in fact, singular) there. Equation (5-2) shows that the generalized Coriolis term $(2 - q) \Omega_0 (\partial v_z/\partial z)$ creates $\omega_z$. In contrast, barotropic critical layers do not create vortex layers because the $y$, rather than the $z$, component of their velocity eigenmodes are singular, and the barotropic eigenmodes' $(\partial v_z/\partial z)$ are finite and too weak to create vorticity via the Coriolis term.

Within a baroclinic critical layer, the singular $\partial v_z/\partial z$ is nearly anti-symmetric about $x = x^*$; on one side of the layer $\partial v_z/\partial z \to \infty$, and on the other $\partial v_z/\partial z \to -\infty$; thus, the last term in Eq. (5-2) creates a large-amplitude vortex layer centered at $x^*$ that is made of dipolar segments with one side cyclonic $(\omega_z/\Omega_0 > 0)$ and the other anticyclonic $(\omega_z/\Omega_0 < 0)$ (c.f., Fig. 16(a)). Vortex layers that are embedded in a background shearing flow, like those in a protoplanetary disk are, in general, linearly stable [unstable] when the relative vertical vorticity of the layer $\omega_z$ has the opposite [same] sign as the vertical vorticity of the background shearing flow. For a vortex layer embedded in a Keplerian disk, this means that a vortex layer with cyclonic $\omega_z$ is stable, while the anticyclonic layer is unstable. The linear instability of vortex layers is a generalization of the study of Kelvin-Helmholtz instability and is amenable to the same type of analyses (Drazin & Reid 1981). We examined the instabilities of embedded vortex layers that were aligned in the stream-wise direction of background shearing flows when we carried out studies to determine the conditions under which the Jovian zonal (east-west) flows become linearly unstable and create Great-Red-Spot-like vortices (Marcus 1993). We found that when a vortex layer goes unstable, its edges become wavy, the waves amplify, the layer breaks up into a stream-wise series of vortices, and eventually the vortices merge together into one, Moore-Saffman-like vortex (Marcus 1988, 1990). An analysis of vortex layer stability and subsequent roll-up that was nearly identical to our study was carried out later by Lovelace et al. (1999) in the context of the Rossby wave instability in accretion disks.

5.2. Vortex spacing

Figures 16 and 17 show how a linearly neutrally stable baroclinic critical layer forms a vortex layer and how the layer produces zombie vortices. However, one of the most important pieces of physics shown in these two figures is how the near-periodic behavior develops in the cross-stream $x$ direction. In expression (5-1) $k_y \equiv 2\pi m/L_y$ is the wavenumber in the stream-wise direction, where $m$ is an integer and $L_y$ is the domain size in $y$ (which in a protoplanetary disk would be its circumference). Baroclinic critical layers have $k_y \neq 0$, and expression (5-1) shows that the locations of the critical layers are

$$x^* = -[(s \pm 1)/m] \Delta,$$

where

$$\Delta \equiv (L_y N_0)/(2\pi |q|\Omega_0).$$
Eq. (5-3) should not be misunderstood. It does not mean that $x^*$ is the radius in a protoplanetary disk where critical layers form, rather, $x^*$ is the distance in $x$, or radial distance in a PPD, between a perturbation and the location of the critical layer that it excites. The governing equations (2-24), (2-26), and (2-29) and their shearing box boundary conditions are invariant under translations in $y$ and $z$, and also under translation in $x$ by $\delta$ when accompanied by a stream-wise boost in velocity of $(-q\Omega_0)\delta$. The latter symmetry is shift-and-boost symmetry (c.f., Goldreich & Lynden-Bell (1965); Marcus & Press (1977)) and is the symmetry that is exploited that allows the use of shearing box boundary conditions. Due to the shift-and-boost symmetry, the origin of the $x$-axis is not unique, so in eq. (5-3) $x^*$ must be the relative distance between two features, in this case, a perturbation and the critical layer it excites. To demonstrate that this is the correct interpretation of $x^*$, we simulated flows in which the flow was perturbed by either a compact wave generator or by a single compact vortex. Figure 16 shows $\omega_z(x,y,z,t)$ in the $x$-$y$ plane for $z \neq 0$ at four times where the initial perturbing anticyclone is at the origin (so that the initial perturbing vortex lies in a plane distinct from the one shown in the figure and therefore does not appear in Figure 16). The perturbing vortex is nearly in a steady equilibrium with background flow $\bar{v}$, so it primarily excites critical layers with frequencies $s = 0$. (This is confirmed by taking a time series of the velocity at locations inside the critical layers and determining their temporal Fourier components with an all poles method.) The critical layers in Figure 16a are at $|x^*|/\Delta = 1/|m|$ for $|m| = 1, 2$ and 3, and with no critical layers at $|x|/\Delta > 1$, in accord with the fact that the perturbation is at $x = 0$ and with eq. (5-3). Each critical layer has produced a dipolar vortex layer aligned in the stream-wise direction, and the vorticity in the dipolar layer at $x/\Delta = 1/|m|$ appears as $|m|$ segments (i.e., dominated by $k_y = 2\pi|m|/L_y$) of dipolar stripes aligned in the stream-wise $y$ direction. Figure 16b shows cyclonic vortex layers that are wavy but that are more-or-less continuous and aligned in the stream-wise direction, indicating that they are stable; whereas the anticyclonic layers are clearly unstable, have roll-up into discrete anticyclones, and are starting to merge into one large anticyclone at each critical layer. In particular, the anticyclonic vorticity at $x/\Delta = 1/3$ has rolled up and merged into a single anticyclone (near $y/\Delta = 1.5$). The anticyclonic vorticity at $x/\Delta = 1/2$ has rolled up into an anticyclone near $y/\Delta = -0.5$. In contrast, the cyclonic $\omega_z$ near $x/\Delta = 1/2$ has formed a continuous, meandering filament. At later times (Figure 16c) the anticyclones near $x/\Delta = 1/3$ (and near $y/\Delta = 2$) and near $x/\Delta = 1/2$ (and near $y/\Delta = -1$) have become larger. The $x$-$y$ plane in Fig. 16 is at a $z$ where the $|m| = 2$ anticyclones are strongest, so the $|m| = 3$ and 1 anticyclones appear artificially weak. Figures 16c and 16d show anticyclonic vortices at $|x|/\Delta > 1$, which according to eq.(5-3) cannot be created by a perturbations at $x = 0$. The layers at $|x|/\Delta > 1$ are due to the self-replication of $1^{st}$-generation vortices at $|x|/\Delta \leq 1$. A vortex at any location will excite critical layers in a manner exactly like the original perturbing vortex due to the shift-and-boost symmetry (and will have $s = 0$ when viewed in the frame moving with the perturbing vortex). Figure 16c shows $2^{nd}$-generation critical layers at $1 < |x|/\Delta \leq 2$ that were excited by the $1^{st}$-generation vortices at $|x|/\Delta \leq 1$. Figure 16d shows $3^{rd}$-generation critical layers at $2 < |x|/\Delta \leq 3$ and $4^{th}$-generation critical layers forming at $|x|/\Delta > 3$. At late times, the vortices have cross-stream diameters of order $\Delta$ (but see below with respect to how the spacing
decreases at very late times due to the inverse cascade of energy and vortex mergers). Within each zombie vortex the density mixes so that it is in accord with its near hydrostatic and geocyclostrophic equilibrium (c.f., Hassandzadeh et al. (2012)). Our calculations show that there is strong horizontal, but very weak vertical, mixing of density outside the vortices, so the background vertical density stratification and \( N \) remain within 1% of their initial unperturbed values. The lack of vertical mixing, despite strong horizontal mixing, was seen in our earlier simulations (BM06) and also in our laboratory experiments (Aubert et al. 2012) of vortices in rotating, stratified flows.

Figure 16 shows that each generation of vortices excites new critical layers in an adjacent unperturbed region, which spawn the next generation of vortices. The spawning of new generations of new critical layers from old critical layers, the self-replication of the vortices, the eventual takeover of the entire domain by the vortices, and the fact that the vortices are created in a “dead” zone, were the traits that led us in MPJH13 to naming them zombie vortices.

Figure 17 shows the same flow as in Fig. 16 but viewed in the \( x-z \) plane at \( y = 0 \). At late times the domain fills with turbulent anticyclones. Because the initial flow is homogeneous with uniform shear and \( N \), the vortices form a regular lattice despite the flow’s turbulence. As time progresses in Fig. 17, the vortex population spreads out from the perturbing vortex at the origin. At early times (Figure 17a) the flow has 1\(^{st}\)-generation critical layers, with \(|m| = 1, 2, \text{ and } 3\) being most apparent. In this first generation, and all subsequent generations, a vortex perturbs the flow and creates four new prominent vortices at its \(|m| = 1\) critical layers at locations in \( x \) that are \( \pm l_x \) distant from itself and at locations in \( z \) that are \( \pm l_z \) distant from itself. (\( l_x \) is physically set by, and equal to, the distance in \( x \) from a perturbing vortex to the anticyclonic piece of the vortex layer formed by its \(|m| = 1\) critical layer; this distance is slightly greater than \( \Delta \).) The 2\(^{nd}\)-generation \( m = 1 \) critical layers created by the 1\(^{st}\)-generation vortices with \(|m| = 1, 2, \text{ and } 3\) are faintly visible in Fig. 17(b) and much more so in Fig. 17(c). At later times (Fig. 17(d)), the \(|m| = 1\) vortices descended from the 1\(^{st}\)-generation \(|m| = 1\) vortices dominate and form a lattice of zombie vortices located at \([x = 2n l_x, z = 2j l_z]\) and at \([x = (2n + 1)l_x, z = (2j + 1)l_z]\), for all integers \( n \) and \( j \).

The creation of a lattice of zombie vortices in an artifact of having one localized initial perturbation, and lattices do not form from initial noise. With noise, perturbations at random locations excite critical layers, and the vortices spawned from the different critical layers interact with each other, merge and inverse cascade their energies to larger length scales. None the less, the spacing \( \Delta \) in \( x \) between a perturbation and the fundamental critical layer it excites with \( m = 1 \) is evident in the early evolution of zombie turbulence created from initial noise. This spacing \( \Delta \) decreases at late times due to vortex mergers and the inverse cascade of energy, but is the main cause for the near periodicity in \( x \) of the turbulence in Figures 5, 6 and 7.
Fig. 16.— Boussinesq flow with constant gravity and Brunt-Väisälä frequency with \( N_0/\Omega_0 = 2 \). Relative vertical vorticity \( \omega_z \) of the anticyclonic (blue) vortices and cyclonic (red) vortex layers in the \( x-y \) plane. The initial perturbing vortex at the origin is above the plane shown here (\( z/\Delta = -0.4 \)). The first generation zombie vortices form at \( |x|/\Delta \leq 1 \), and sweep outward in \( x \). The Rossby number \( Ro \) of these vortices is \( \sim -0.2 \). (The color is reddest at \( \omega_z/\Omega_0 = 0.4 \), bluest at \( \omega_z/\Omega_0 = -0.4 \), and green at \( \omega_z = 0 \)). \( \Omega_0/N_0 = 0.5 \) and \( q = 3/2 \). The \( x-y \) domain is \( |x|/\Delta \leq 4.7124; |y|/\Delta \leq 2.3562 \), and is larger than shown. a) \( t = 64/N_0 \). b) \( t = 256/N_0 \). c) \( t = 576/N_0 \). d) \( t = 2240/N_0 \).
Fig. 17.— Zombie vortices sweep outward from the perturbing vortex at the origin in the $x$–$z$ plane (at $y = 0$). Anticyclonic $\omega_z$ is black (darkest is $\omega_z/\Omega_0 = -0.4$) and cyclonic is white (lightest is $\omega_z/\Omega_0 = 0.4$). This is the same flow as in Figure 16. The domain has $|z|/\Delta \leq 4.7124$ and is larger than shown. a) $t = 128/N_0$. Critical layers and young zombie vortices with $s = 0$ and $|m| = 1, 2,$ and $3$ are visible. Diagonal lines are internal inertia-gravity waves with shear, not critical layers. b) $t = 480/N_0$. 1$^{st}$-generation vortices near $|x|/\Delta = 1$ and $1/2$ have rolled-up from critical layers with $s = 0$ and $|m| = 1$ and $2$, respectively. c) $t = 1632/N_0$. 2$^{nd}$-generation vortices have spawned from the 1$^{st}$ generation vortices. d) $t = 3072/N_0$. 1$^{st}$, 2$^{nd}$ and 3$^{rd}$ generation vortices.

6. Discussion

We have shown that Keplerian disks are unstable when a vertical component of gravity from the central star or protostar is included in the equations of motion. The zombie instability acts when the vertical density of the disk is stably-stratified (i.e., when there is a well-defined Brunt-Väisälä frequency). This instability is not subtle; its growth is fast, requiring only a few hundred local “years” to produce large-magnitude, space-filling turbulence. Even if the instability is initially spatially confined to a small region, the entire domain fills rapidly with turbulence with rms Rossby numbers between 0.2 and 0.3. When the turbulence created by the zombie instability reaches a statistically-steady state, it is dominated by its large length scales due to vortex merging and the inverse cascade of energy from the initial lengthscale of the instability (i.e., $\Delta$ in equation (5-4)) to larger length scales (as shown in Figures 5 – 7). If the inverse cascade stops at the pressure-scale height $H$, as argued by many authors (c.f. ?), then the rms Mach and Rossby numbers of
the turbulence will be the same (c.f., eq. (3-4)), so the space-filling turbulence will have a Mach number of 0.2 – 0.3. Our calculations show that the turbulence is robust and does not decay in time. In MPJH13, the permanence of the turbulence was shown to be due to the fact that it draws its energy from the energy stored in the Keplerian shear. Disks filled with statistically-steady of zombie turbulence appear to be attracting solutions that are independent of the initial conditions that trigger the turbulence; that is to say, their properties, such as the energy of the non-Keplerian velocity, the energy spectrum as a function of wave number, and the probability distributions of the Mach and Rossby numbers appear to be independent of the initial conditions that trigger the instability. Although the zombie instability is not a linear instability, the finite amplitude of the rms Mach number of the initial Kolmogorov turbulence needed to trigger it is so small, \( \sim 10^{-6} \), that for practical purposes, the zombie instability acts like a linear one.

The lack of subtlety of the zombie instability and its turbulent consequences begs the question of how it was missed in previous studies of Keplerian disks. Part of the answer is that the zombie instability has three necessary ingredients: rotation, horizontal shear, and vertical density stratification. These ingredients are all present in a protoplanetary disk, but many previous studies omitted initial vertical density stratification, and in several cases did not include a vertical component of gravity. Any stability study that does not include all three ingredients – whether a numerical calculation in an ideal gas without vertical gravity, or a laboratory experiment that uses a constant-density fluid – cannot produce the zombie instability. Another reason that the zombie instability was not observed previously (except in BM06 where it was present, but not understood to be a new type of instability) was that many stability studies used noise as the initial perturbation. As shown in § 4, the spectral index of the noise matters. Initial noise with an energy spectrum in which \( E(k) \) falls off faster than \( k^{-3} \), such as the near-Gaussian spectrum used in BHS95, requires a substantial Mach number (\( Ma > 0.2 \)) to trigger instability. In contrast, initial noise with a spectral index between 1 and 3, requires a much smaller initial Mach number to destabilize the flow. Kolmogorov turbulence, with an index of 5/3, requires an initial rms Mach number of order \( Re^{-1/2} \approx 10^{-6} \) to trigger the zombie instability.

Another reason that the zombie instability may have been overlooked in previous numerical studies is that sufficient numerical resolution and/or lack of numerical dissipation is required to resolve its critical layers. Figures 6 and 16 show these are thin. Our simulations show that the zombie instability is present only when a 128 or more grid points or modes per pressure scale height are used in the radial direction. Thus the instability could not be observed in the simulations of vertically stratified disks by Fromang & Papaloizou (2006) who used 30 radial points per pressure scale height or by Fleming & Stone (2003); Oishi & Mac Low (2009) who used 64 or fewer.

Curiously, there is a second and independent reason that numerical codes need high resolution to create zombie instabilities. The rms Mach number of the initial noise needed to trigger the instability depends on the length of inertial range of the turbulence, which is the ratio of the largest to smallest length scales in the turbulence. If that ratio, which in a numerical experiment is approximately the number of grid points in any spatial direction (and which in a real fluid
is the ratio of the largest length scales to the small dissipative lengthscales) is not large, then an exceptionally large initial rms Mach number is required to trigger the zombie instability. For Kolmogorov turbulence in a protoplanetary disk or laboratory experiment, this ratio dependence is what leads to the Mach number required to trigger instability scaling as $Re^{-1/2}$.

Another reason that the zombie instability might not be been seen in initial-value simulations in which the equilibrium is perturbed with noise is that the calculation is not run long enough. The instability begins at lengthscales with large Rossby numbers and for noise with a spectral index greater than unity, the smallest lengthscales have the largest Rossby numbers. The energy at the smallest scales initially decreases because the numerical dissipation is largest at those scales, and it takes time for the energy to inverse cascade to larger scales where the flow is efficient at drawing energy from the Keplerian shear. In our studies (c.f., Figure 3), it takes at least 50 years for the energy of the non-Keplerian component to move away from the smallest lengthscales and begin growing, so if the calculation is terminated early (after 8 years in Figure 1), then the instability would not be observed.

The zombie instability would not have been seen in previous disk studies that were initiated with flows that were far from equilibrium. These types of initial-value experiments cannot be used to determine whether an equilibrium is stable but are designed to determine whether there are multiple statistically-steady states of the flow. For example, an initial disk flow and magnetic field that are not in equilibrium might evolve to a steady disk with Keplerian velocity, indicating that laminar Keplerian flow is a strong, possibly unique, attractor. On the other hand, the flow could evolve to a statistically-steady, non-Keplerian flow with sustained turbulence, indicating that there are multiple solutions to the equations of motion and that a state of sustained turbulence is a robust attractor. However, that type of initial-value experiment cannot determine whether an equilibrium Keplerian flow with an accompanying magnetic field is unstable to the zombie instability, streaming instability, or the magneto-rotational instability (MRI). Constructing numerical experiments to test whether the MRI instability overtakes a competing zombie instability or vice versa, or whether the zombie instability overtakes a competing streaming instability is more complicated then comparing the linear growth rates of eigenmodes and requires a careful consideration of the initial perturbations not only because because the zombie instability is a finite (albeit small) amplitude instability but also because the winner of such a competition also depends upon the nonlinear dynamics. There are situations where it is important to know which instability dominates. For example, if the zombie instability grows and creates turbulence so that the streaming instability cannot occur, then it would be impossible to create planetesimals via clumping from the streaming instability. On the other hand, the concept that there is a competition among the various instabilities of a Keplerian disk is possibly very misleading. Numerical simulations by Krumholz et al. (2007) show that when a prestellar core collapses and a protoplanetary disk begins to form, the disk is highly turbulent. Currently, there is no evidence that the late-time disk formed from collapse does not remain in a state of sustained turbulence or that the flow ever evolves to a steady Keplerian disk. It is also possible that while a disk is still forming from a collapsing core that large spatial fluctuations in its
velocity, temperature, and density evolve into features that resemble an MRI instability, a zombie vortex, or the signature of some other instability of a steady Keplerian disk. Furthermore, there is no reason to believe that instability that overtakes its competition in a forming turbulent disk is the same instability that overtakes its competition in a steady disk.

**Zombie** turbulence is dominated by large, turbulent vortices. So an important question to be addressed in future studies is whether highly turbulent, three-dimensional vortices accumulate dust like the laminar ones examined by Barge & Sommeria (1995) and therefore can be locations of planetesimal formation. Our original motivation (BM06) for the studying protoplanetary disks was the final stages of star formation. In the “dead” zones of protoplanetary disks, a purely hydrodynamic instability is needed to create fluid motions that transport angular momentum and energy radially outward so that protostars can finish their accretion of mass. In BM05, we showed that zombie vortices transport angular momentum radially outward from the protostar but found that the rate of transport, as parameterized by \( \alpha \equiv \langle \rho v_x v_y \rangle / \left( C_s^2 \langle \rho \rangle \right) \), was a factor of 10-100 times smaller than the values needed for the observed rate of star formation from a protostar, where \( \langle Q \rangle \) means the volume-averaged value of \( Q \) over the entire domain. The value of \( \alpha \) is approximately \( \text{Ma}_{\text{rms}}^2 f_{Ma} f c \), where \( c \) is the correlation between \( v_x \) and \( v_y \), \( f_{Ma} \) is the time-averaged spatial fill factor of the flow with \( \text{Ma}_{\text{rms}} \) (c.f., Figure 12), and \( f \) is the ratio of the Mach-number-weighted average density of the gas to the density of the gas in the mid-plane of the protoplanetary disk. In BM05, the small values of \( \alpha \) were due to the small values of the \( c \), and the small values of \( c \) were due to the inherent symmetry of a vortex. Vortices in the disk are approximately mirror-symmetric with respect to the \( x-z \) plane that passes through the vortex center; thus, the values of \( \langle \rho v_x v_y \rangle \) on either side of the symmetry plane nearly cancel. We therefore believe that zombie turbulence may not be able to produce a sufficiently large \( \alpha \) in anelastic fluids to allow star formation. However, we hypothesize that zombie turbulence in a fully compressible fluid will produce much larger values of \( \alpha \). A fully compressible fluid has three orthogonal velocity components (Chandrasekhar 1981): two are rotational and divergence-free, i.e., the poloidal and toroidal components, which anelastic flows also have; and one component is irrotational, i.e., it is the curl-free potential flow, which anelastic flows do not have. Acoustic waves are a potential flow and have correlations \( c \) of order unity. Johnson & Gammie (2005); Shen et al. (2006); Lesur & Papaloizou (2010); Lyra & Klahr (2011); Raettig et al. (2013) have shown that acoustic waves in a protoplanetary disk transport angular momentum radially outward with values of \( \alpha \) of order \( \text{Ma}_{\text{rms}}^2 \). In a turbulent fully compressible gas there should be an equi-partition of energy. In particular, Kritsuk et al. (2007, 2009); Lemaster & Stone (2009) showed numerically that when a fully compressible fluid in a disk is forced by sustained rotational turbulence, the potential flow becomes turbulent and its energy comes into equi-partition with the two rotational components of the turbulence. Therefore, sustained zombie turbulence in a fully compressible flow could, by energy equi-partition, create irrotational turbulence with acoustic waves with Mach numbers of \( \sim 0.2-0.3 \). \(^{(n.b. \text{ We cannot test these ideas in the simulation of the zombie instability in the fully compressible flow in Figure 4 because that simulation ended at 180 years, well before the flow reached a statistical equilibrium.})} \) Estimating the value of \( \alpha \approx \text{Ma}_{\text{rms}}^2 f_{Ma} f c \), due to the acoustic waves created from zombie turbulence, with \( c \approx 1, \text{Ma}_{\text{rms}} \approx 0.2, f_{Ma} \approx 0.2, \)
and \( f \simeq 0.3 \), gives \( \alpha \simeq 2 \times 10^{-3} \), which is consistent with values inferred from observed rates of star formation. Our estimate for \( f \) is based on based on BM05 and Lesur & Papaloizou (2009) who were unable to find stable steady vortices at the disk’s mid-plane with Rossby numbers greater 0.1. However, _zombie_ turbulence with coherent structures as in Figures 5 – 7 is not the same as steady vortices, and we have no evidence that _zombie_ turbulence does not exist at the mid-plane. Furthermore, if _zombie_ turbulence does not exist at the mid-plane, it may still be possible that the potential flow’s turbulence and acoustic waves that is spawns will advect to and fill in the mid-plane region. Therefore it is possible that \( f \) is close to unity.

Regardless of the values that \( \alpha \) that the _zombie_ instability creates at late times in a proto-planetary disk made of a fully compressible gas, our calculations show that Keplerian disks have a purely hydrodynamic instability that creates space-filling turbulence with rms Mach numbers between 0.2 and 0.3. Whatever mechanisms a disk has that radially transport angular momentum and that accumulate dust will certainly be different in a turbulent disk than they are in a steady laminar disk.

We thank the academy for ...

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